SOME INEQUALITIES FOR TRACE CLASS ENTROPIC P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp[-\operatorname{tr}(PA\ln A)].$$

In this paper we show among others that

$$1 \leq \frac{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}}{\eta_P(A)} \leq \exp\left[\frac{\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right]$$

and

$$1 \leq \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}\left[\frac{\operatorname{tr}(PA^2)}{[\operatorname{tr}(PA)]^2}\right]^{-\operatorname{tr}(PA)}} \leq \exp\left[\frac{\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right].$$

1. Introduction

In 1952, in the paper [7], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

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Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [8], [9], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [10].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \ | \ B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \text{ and } \ |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [5] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [6]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In [6], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.13}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right], \tag{1.14}$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA \ln A)\right] = \exp\{\operatorname{tr}[P\eta(A)]\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A) P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp(-\operatorname{tr}\{P[tA\ln(tA)]\}) \\
&= \exp(-\operatorname{tr}\{P[tA(\ln t + \ln A)]\}) = \exp(-\operatorname{tr}\{P(tA\ln t + tA\ln A)\}) \\
&= \exp(-t\ln t \operatorname{tr}(PA)) \exp(-t\operatorname{tr}(PA\ln A)) \\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)[\exp(-\operatorname{tr}(PA\ln A))]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\,\mathrm{tr}(PA)}\,[\eta_P(A)]^{-t} \tag{1.15}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(I) = 1 \text{ and } \eta_P(tI) = t^{-t} \tag{1.16}$$

for $t > 0$.

Motivated by the above results, in this paper we provide various upper bounds for the quantities

$$\frac{[\mathrm{tr}\,(PA)]^{-\mathrm{tr}(PA)}}{\eta_P\,(A)}$$

and

$$\frac{\eta_P\,(A)}{[\mathrm{tr}\,(PA)]^{-\mathrm{tr}(PA)}\left[\frac{\mathrm{tr}(PA^2)}{[\mathrm{tr}(PA)]^2}\right]^{-\mathrm{tr}(PA)}}$$

under various assumptions for the operators $A > 0$, where $P \geq 0$ with $P \in \mathcal{B}_1\,(H)$ and $\mathrm{tr}\,(P) = 1$.

2. Main Results

We start with the following main result:

Theorem 4. *Assume that* $A > 0$, $P \geq 0$ *with* $P \in \mathcal{B}_1\,(H)$ *and* $\mathrm{tr}\,(P) = 1$, *then*

$$1 \leq \frac{[\mathrm{tr}\,(PA)]^{-\mathrm{tr}(PA)}}{\eta_P\,(A)} \leq \exp\left[\frac{\mathrm{tr}\left(PA^2\right) - [\mathrm{tr}\,(PA)]^2}{\mathrm{tr}\,(PA)}\right] \tag{2.1}$$

and

$$1 \leq \frac{\eta_P\,(A)}{[\mathrm{tr}\,(PA)]^{-\mathrm{tr}(PA)}\left[\frac{\mathrm{tr}(PA^2)}{[\mathrm{tr}(PA)]^2}\right]^{-\mathrm{tr}(PA)}} \leq \exp\left[\frac{\mathrm{tr}\left(PA^2\right) - [\mathrm{tr}\,(PA)]^2}{\mathrm{tr}\,(PA)}\right]. \tag{2.2}$$

Proof. Consider $Q := \frac{A^{1/2}PA^{1/2}}{\mathrm{tr}(PA)}$, then $Q \geq 0$, $\mathrm{tr}\,(Q) = 1$ and for $B = A^{-1}$,

$$\begin{aligned}
\Delta_Q\,(B) &= \Delta_{\frac{A^{1/2}PA^{1/2}}{\mathrm{tr}(PA)}}\left(A^{-1}\right) = \exp\mathrm{tr}\left(\frac{A^{1/2}PA^{1/2}}{\mathrm{tr}\,(PA)}\ln A^{-1}\right) \\
&= \exp\left[\frac{1}{\mathrm{tr}\,(PA)}\,\mathrm{tr}\left(A^{1/2}PA^{1/2}\ln A^{-1}\right)\right] \\
&= \exp\left[\frac{1}{\mathrm{tr}\,(PA)}\,\mathrm{tr}\left(PA^{1/2}\left(\ln A^{-1}\right)A^{1/2}\right)\right] \\
&= \exp\left[\frac{-1}{\mathrm{tr}\,(PA)}\,\mathrm{tr}\,(PA\ln A)\right] = \left(\exp\left[\mathrm{tr}\,(-PA\ln A)\right]\right)^{\frac{1}{\mathrm{tr}(PA)}} \\
&= [\eta_P\,(A)]^{\frac{1}{\mathrm{tr}(PA)}},
\end{aligned}$$

which gives the follow identity of interest

$$\eta_P\,(A) = \left[\Delta_{\frac{A^{1/2}PA^{1/2}}{\mathrm{tr}(PA)}}\left(A^{-1}\right)\right]^{\mathrm{tr}(PA)}, \tag{2.3}$$

which holds for $A > 0$, $P \geq 0$ with $P \in \mathcal{B}_1\,(H)$ and $\mathrm{tr}\,(P) = 1$.

From (1.13) written for $\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}$ and A^{-1} we derive

$$\begin{aligned}1 &\leq \frac{\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\right)}{\Delta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)} \\ &\leq \exp\left[\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A\right)\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\right)-1\right],\end{aligned}$$

namely

$$1 \leq \frac{\frac{1}{\operatorname{tr}(PA)}}{\Delta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)} \leq \exp\left[\frac{\operatorname{tr}\left(PA^{2}\right)-\left[\operatorname{tr}(PA)\right]^{2}}{\left[\operatorname{tr}(PA)\right]^{2}}\right].$$

If we take the power $\operatorname{tr}(PA)>0$, then we get

$$1 \leq \frac{\left[\operatorname{tr}(PA)\right]^{-\operatorname{tr}(PA)}}{\left[\Delta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)\right]^{\operatorname{tr}(PA)}} \leq \exp\left[\frac{\operatorname{tr}\left(PA^{2}\right)-\left[\operatorname{tr}(PA)\right]^{2}}{\operatorname{tr}(PA)}\right].$$

By utilizing (2.3) we derive (2.1).

From (1.14) for $\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}$ and A^{-1} we derive

$$1 \leq \frac{\Delta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)}{\left[\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A\right)\right]^{-1}} \leq \exp\left[\frac{\operatorname{tr}\left(PA^{2}\right)-\left[\operatorname{tr}(PA)\right]^{2}}{\left[\operatorname{tr}(PA)\right]^{2}}\right],$$

namely

$$1 \leq \frac{\Delta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)}{\left[\operatorname{tr}(PA)\right]^{-1}\left[\frac{\operatorname{tr}(PA^{2})}{[\operatorname{tr}(PA)]^{2}}\right]^{-1}} \leq \exp\left[\frac{\operatorname{tr}\left(PA^{2}\right)-\left[\operatorname{tr}(PA)\right]^{2}}{\left[\operatorname{tr}(PA)\right]^{2}}\right].$$

By taking the power $\operatorname{tr}(PA)>0$, we obtain

$$1 \leq \frac{\left[\Delta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)\right]^{\operatorname{tr}(PA)}}{\left[\operatorname{tr}(PA)\right]^{-\operatorname{tr}(PA)}\left[\frac{\operatorname{tr}(PA^{2})}{[\operatorname{tr}(PA)]^{2}}\right]^{-\operatorname{tr}(PA)}} \leq \exp\left[\frac{\operatorname{tr}\left(PA^{2}\right)-\left[\operatorname{tr}(PA)\right]^{2}}{\operatorname{tr}(PA)}\right]$$

and the inequality (2.2) is proved. □

Corollary 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If A satisfies the condition*

$$0 < m \leq A \leq M \tag{2.4}$$

for some constants $0 < m < M$, *then*

$$1 \leq \frac{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}}{\eta_P(A)} \leq \exp\left[\frac{\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right] \tag{2.5}$$
$$\leq \exp\left[\frac{1}{2}(M-m)\operatorname{tr}\left(P\left|\frac{A}{\operatorname{tr}(PA)} - 1\right|\right)\right]$$
$$\leq \exp\left(\frac{1}{2}(M-m)\left[\frac{\operatorname{tr}(PA^2)}{(\operatorname{tr}(PA))^2} - 1\right]^{1/2}\right)$$
$$\leq \exp\left[\frac{1}{4\operatorname{tr}(PA)}(M-m)^2\right] \leq \exp\left[\frac{1}{4m}(M-m)^2\right]$$

and

$$1 \leq \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}\left[\frac{\operatorname{tr}(PA^2)}{[\operatorname{tr}(PA)]^2}\right]^{-\operatorname{tr}(PA)}} \tag{2.6}$$
$$\leq \exp\left[\frac{\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right]$$
$$\leq \exp\left[\frac{1}{2}(M-m)\operatorname{tr}\left(P\left|\frac{A}{\operatorname{tr}(PA)} - 1\right|\right)\right]$$
$$\leq \exp\left(\frac{1}{2}(M-m)\left[\frac{\operatorname{tr}(PA^2)}{(\operatorname{tr}(PA))^2} - 1\right]^{1/2}\right)$$
$$\leq \exp\left[\frac{1}{4\operatorname{tr}(PA)}(M-m)^2\right] \leq \exp\left[\frac{1}{4m}(M-m)^2\right].$$

Proof. Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ and $\operatorname{tr}(P) = 1$, then we have [4]

$$0 \leq \operatorname{tr}(Pf(A)) - f(\operatorname{tr}(PA)) \tag{2.7}$$
$$\leq \operatorname{tr}(Pf'(A)A) - \operatorname{tr}(PA)\operatorname{tr}(Pf'(A))$$

$$\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)]\operatorname{tr}(P|A - \operatorname{tr}(PA)|) \\ \\ \frac{1}{2}(M-m)\operatorname{tr}(P|f'(A) - \operatorname{tr}(Pf'(A))|) \end{cases}$$
$$\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)]\left[\operatorname{tr}(PA^2) - (\operatorname{tr}(PA))^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(P[f'(A)]^2\right) - (\operatorname{tr}(Pf'(A)))^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}[f'(M) - f'(m)](M-m).$$

From (2.7) we have for $f(t)=t^2$ that

$$\begin{aligned}
&\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2\\
&\le \frac{1}{2}(M-m)\operatorname{tr}(P\left|A-\operatorname{tr}(PA)\right|)\\
&\le \frac{1}{2}(M-m)\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}\le \frac{1}{4}(M-m)^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\\
&\le \frac{1}{2}(M-m)\operatorname{tr}\left(P\left|\frac{A}{\operatorname{tr}(PA)}-1\right|\right)\\
&\le \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(PA^2\right)}{(\operatorname{tr}(PA))^2}-1\right]^{1/2}\le \frac{1}{4\operatorname{tr}(PA)}(M-m)^2\\
&\le \frac{1}{4m}(M-m)^2.
\end{aligned}$$

By utilizing Theorem 4 we derive the desired inequalities (2.5) and (2.6). □

Corollary 2. *With the assumptions of Corollary 1 we have*

$$\begin{aligned}
1&\le \frac{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}}{\eta_P(A)}\le \exp\left[\frac{\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right]\\
&\le \exp\left[\frac{1}{4M}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\operatorname{tr}\left(PA^2\right)\right]\\
&\le \exp\left[\frac{1}{4}(M+m)\left(\frac{M}{m}-1\right)^2\right]
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
1&\le \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}\left[\frac{\operatorname{tr}(PA^2)}{[\operatorname{tr}(PA)]^2}\right]^{-\operatorname{tr}(PA)}}\\
&\le \exp\left[\frac{\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right]\\
&\le \exp\left[\frac{1}{4M}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\operatorname{tr}\left(PA^2\right)\right]\\
&\le \exp\left[\frac{1}{4}(M+m)\left(\frac{M}{m}-1\right)^2\right].
\end{aligned}\tag{2.9}$$

Proof. If we write the inequality (2.7) for $f(t) = t^{-1}$, $t > 0$, then we get for A with $\mathrm{Sp}(A) \subseteq [m, M]$ and $Q \in \mathcal{B}_1(H) \setminus \{0\}$, $Q \geq 0$ with $\mathrm{tr}(Q) = 1$, that

$$\begin{aligned}
0 &\leq \mathrm{tr}\left(QA^{-1}\right) - \frac{1}{\mathrm{tr}(QA)} \\
&\leq \mathrm{tr}(QA)\,\mathrm{tr}\left(QA^{-2}\right) - \mathrm{tr}\left(QA^{-1}\right) \\
&\leq \begin{cases} \frac{1}{2}\frac{M^2-m^2}{m^2M^2}\,\mathrm{tr}\left(Q\left|A - \mathrm{tr}(QA)\right|\right) \\[2ex] \frac{1}{2}(M-m)\,\mathrm{tr}\left(Q\left|A^{-2} - \mathrm{tr}\left(QA^{-2}\right)\right|\right) \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\mathrm{tr}\left(QA^2\right) - \left(\mathrm{tr}(QA)\right)^2\right]^{1/2} \\[4ex] \frac{1}{2}(M-m)\left[\mathrm{tr}\left(QA^{-4}\right) - \left(\mathrm{tr}\left(QA^{-2}\right)\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\frac{(M+m)(M-m)^2}{m^2M^2},
\end{aligned}$$

which gives that

$$\begin{aligned}
0 &\leq \mathrm{tr}\left(QA^{-1}\right)\mathrm{tr}(QA) - 1 \\
&\leq \mathrm{tr}(QA)\left[\mathrm{tr}(QA)\,\mathrm{tr}\left(QA^{-2}\right) - \mathrm{tr}\left(QA^{-1}\right)\right]
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\leq \mathrm{tr}(QA) \times \begin{cases} \frac{1}{2}\frac{M^2-m^2}{m^2M^2}\,\mathrm{tr}\left(Q\left|A - \mathrm{tr}(QA)\right|\right) \\[2ex] \frac{1}{2}(M-m)\,\mathrm{tr}\left(Q\left|A^{-2} - \mathrm{tr}\left(QA^{-2}\right)\right|\right) \end{cases} \\
&\leq \mathrm{tr}(QA) \times \begin{cases} \frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\mathrm{tr}\left(QA^2\right) - \left(\mathrm{tr}(QA)\right)^2\right]^{1/2} \\[4ex] \frac{1}{2}(M-m)\left[\mathrm{tr}\left(QA^{-4}\right) - \left(\mathrm{tr}\left(QA^{-2}\right)\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\left(1 + \frac{m}{M}\right)\left(\frac{M}{m} - 1\right)^2 \frac{\mathrm{tr}(QA)}{M}.
\end{aligned}$$

Now, if we take $Q = \frac{A^{1/2}PA^{1/2}}{\mathrm{tr}(PA)}$ in (2.10), then we get

$$\begin{aligned}
0 &\leq \frac{\mathrm{tr}\left(PA^2\right)}{\left[\mathrm{tr}(PA)\right]^2} - 1 \\
&\leq \frac{\mathrm{tr}\left(PA^2\right)}{\mathrm{tr}(PA)}\left[\frac{\mathrm{tr}\left(PA^2\right)\mathrm{tr}\left(PA^{-1}\right) - \mathrm{tr}(PA)}{\left[\mathrm{tr}(PA)\right]^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)} \\
&\times \begin{cases}
\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA\left|A-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right|\right) \\
\\
\frac{1}{2}(M-m)\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA\left|A^{-2}-\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(PA)}\right|\right)
\end{cases} \\
&\leq \frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)} \\
&\times \begin{cases}
\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\frac{\operatorname{tr}\left(PA^3\right)}{\operatorname{tr}(PA)}-\left(\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right)^2\right]^{1/2} \\
\\
\frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(PA^{-3}\right)}{\operatorname{tr}(PA)}-\left(\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(PA)}\right)^2\right]^{1/2}
\end{cases} \\
&\leq \frac{1}{4M}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}.
\end{aligned}
$$

If we multiply this by $\operatorname{tr}(PA)>0$ we get

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}\left(PA^2\right)-\left[\operatorname{tr}(PA)\right]^2}{\operatorname{tr}(PA)} \\
&\leq \operatorname{tr}\left(PA^2\right)\left[\frac{\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PA^{-1}\right)-\operatorname{tr}(PA)}{\left[\operatorname{tr}(PA)\right]^2}\right]
\end{aligned}
$$

$$
\leq \operatorname{tr}\left(PA^2\right)\times\begin{cases}
\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA\left|A-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right|\right) \\
\\
\frac{1}{2}(M-m)\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA\left|A^{-2}-\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(PA)}\right|\right)
\end{cases}
$$

$$
\begin{aligned}
&\leq \operatorname{tr}\left(PA^2\right)\times\begin{cases}
\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\frac{\operatorname{tr}\left(PA^3\right)}{\operatorname{tr}(PA)}-\left(\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right)^2\right]^{1/2} \\
\\
\frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(PA^{-3}\right)}{\operatorname{tr}(PA)}-\left(\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(PA)}\right)^2\right]^{1/2}
\end{cases} \\
&\leq \frac{1}{4M}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\operatorname{tr}\left(PA^2\right)\leq\frac{1}{4}(M+m)\left(\frac{M}{m}-1\right)^2.
\end{aligned}
$$

By utilizing Theorem 4 we derive the desired inequalities (2.8) and (2.9). □

Corollary 3. *With the assumptions of Corollary 1 we have*

$$
\begin{aligned}
(2.11)\qquad 1 &\leq \frac{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}}{\eta_P(A)} \leq \exp\left[\frac{\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right] \\
&\leq \exp\left[\frac{1}{2}\frac{M-m}{mM}\operatorname{tr}\left(PA\left|A-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right|\right)\right] \\
&\leq \exp\left[\frac{1}{2}\frac{M-m}{mM}\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^3\right)-\left(\operatorname{tr}\left(PA^2\right)\right)^2\right]^{1/2}\right] \\
&\leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\operatorname{tr}(PA)\right] \leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{m}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(2.12)\qquad 1 &\leq \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}\left[\frac{\operatorname{tr}(PA^2)}{[\operatorname{tr}(PA)]^2}\right]^{-\operatorname{tr}(PA)}} \\
&\leq \exp\left[\frac{\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\right] \\
&\leq \exp\left[\frac{1}{2}\frac{M-m}{mM}\operatorname{tr}\left(PA\left|A-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right|\right)\right] \\
&\leq \exp\left[\frac{1}{2}\frac{M-m}{mM}\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^3\right)-\left(\operatorname{tr}\left(PA^2\right)\right)^2\right]^{1/2}\right] \\
&\leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\operatorname{tr}(PA)\right] \leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{m}\right].
\end{aligned}
$$

Proof. In [3] we proved among others that, if $\operatorname{Sp}(S) \subseteq [m, M] \subset (0, \infty)$ and $Q \in \mathcal{B}_1(H)$ and $Q > 0$, then

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(QS)\operatorname{tr}\left(QS^{-1}\right)}{[\operatorname{tr}(Q)]^2}-1 \\
&\leq \frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}(Q)}\operatorname{tr}\left(\left|S-\frac{\operatorname{tr}(QS)}{\operatorname{tr}(Q)}\right|Q\right) \\
&\leq \frac{1}{2}\frac{M-m}{mM}\left[\frac{\operatorname{tr}\left(QS^2\right)}{\operatorname{tr}(Q)}-\left(\frac{\operatorname{tr}(QS)}{\operatorname{tr}(Q)}\right)^2\right]^{1/2} \\
&\leq \frac{1}{4}\frac{(M-m)^2}{mM}.
\end{aligned}
$$

By taking $S=A$ and $Q=\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}$ we obtain

$$\begin{aligned}
0 &\le \frac{\operatorname{tr}\left(PA^2\right)}{\left[\operatorname{tr}\left(PA\right)\right]^2}-1\\
&\le \frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}\left(PA\right)}\operatorname{tr}\left(PA\left|A-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}\left(PA\right)}\right|\right)\\
&\le \frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}\left(PA\right)}\left[\operatorname{tr}\left(PA\right)\operatorname{tr}\left(PA^3\right)-\left(\operatorname{tr}\left(PA^2\right)\right)^2\right]^{1/2}\\
&\le \frac{1}{4}\frac{\left(M-m\right)^2}{mM},
\end{aligned}$$

namely

$$\begin{aligned}
0 &\le \frac{\operatorname{tr}\left(PA^2\right)-\left[\operatorname{tr}\left(PA\right)\right]^2}{\operatorname{tr}\left(PA\right)}\\
&\le \frac{1}{2}\frac{M-m}{mM}\operatorname{tr}\left(PA\left|A-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}\left(PA\right)}\right|\right)\\
&\le \frac{1}{2}\frac{M-m}{mM}\left[\operatorname{tr}\left(PA\right)\operatorname{tr}\left(PA^3\right)-\left(\operatorname{tr}\left(PA^2\right)\right)^2\right]^{1/2}\\
&\le \frac{1}{4}\frac{\left(M-m\right)^2}{mM}\operatorname{tr}\left(PA\right).
\end{aligned}$$

By utilizing Theorem 4 we derive the desired inequalities (2.11) and (2.12). $\square$

Finally, we also have:

Corollary 4. *With the assumptions of Corollary 1 we have*

$$\begin{aligned}
1 &\le \frac{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}{\eta_P\left(A\right)}\le \exp\left[\frac{\operatorname{tr}\left(PA^2\right)-\left[\operatorname{tr}\left(PA\right)\right]^2}{\operatorname{tr}\left(PA\right)}\right]\\
&\le \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}\left(PA^2\right)\right]\le \exp\left[\frac{M}{m}\left(\sqrt{M}-\sqrt{m}\right)^2\right]
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
1 &\le \frac{\eta_P\left(A\right)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}\left[\frac{\operatorname{tr}(PA^2)}{\left[\operatorname{tr}(PA)\right]^2}\right]^{-\operatorname{tr}(PA)}}\\
&\le \exp\left[\frac{\operatorname{tr}\left(PA^2\right)-\left[\operatorname{tr}\left(PA\right)\right]^2}{\operatorname{tr}\left(PA\right)}\right]\\
&\le \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}\left(PA^2\right)\right]\le \exp\left[\frac{M}{m}\left(\sqrt{M}-\sqrt{m}\right)^2\right].
\end{aligned}\tag{2.14}$$

Proof. If $t\in[m,M]\subset(0,\infty)$, then obviously

$$\left(M-t\right)\left(m^{-1}-t^{-1}\right)\ge 0,$$

which is equivalent to

$$m+M\geq mMt^{-1}+t$$

for all $t\in[m,M]$.

Using the functional calculus for selfadjoint operators, we then get

$$(m+M)\,I\geq mMA^{-1}+A$$

for $0<mI\leq A\leq MI$.

If we multiply both sides with $Q^{1/2}$ we get

$$(m+M)\,Q\geq mMQ^{1/2}A^{-1}Q^{1/2}+Q^{1/2}AQ^{1/2}$$

and is we take the trace, then we get

$$m+M\geq mM\operatorname{tr}\left(QA^{-1}\right)+\operatorname{tr}(QA),$$

namely

$$\frac{m+M}{mM}\geq\operatorname{tr}\left(QA^{-1}\right)+\frac{\operatorname{tr}(QA)}{mM}.$$

This gives

$$\begin{aligned}
&\operatorname{tr}\left(QA^{-1}\right)-[\operatorname{tr}(QA)]^{-1}\\
&\leq\frac{1}{m}+\frac{1}{M}-\frac{1}{mM}\operatorname{tr}(QA)-[\operatorname{tr}(QA)]^{-1}\\
&=\left(\frac{1}{\sqrt{m}}-\frac{1}{\sqrt{M}}\right)^2-\left(\frac{1}{\sqrt{mM}}[\operatorname{tr}(QA)]^{1/2}-[\operatorname{tr}(QA)]^{-1/2}\right)^2\\
&\leq\left(\frac{1}{\sqrt{m}}-\frac{1}{\sqrt{M}}\right)^2,
\end{aligned}$$

which implies, by multiplying with $\operatorname{tr}(QA)$ that

$$\operatorname{tr}\left(QA^{-1}\right)\operatorname{tr}(QA)-1\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}(QA)\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}.\tag{2.15}$$

If we take in (2.15) $Q=\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}$, then we get

$$\frac{\operatorname{tr}\left(PA^2\right)}{[\operatorname{tr}(PA)]^2}-1\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m},$$

which gives that

$$\frac{\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2}{\operatorname{tr}(PA)}\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}\left(PA^2\right)\leq\frac{M}{m}\left(\sqrt{M}-\sqrt{m}\right)^2.$$

By utilizing Theorem 4 we derive the desired inequalities (2.11) and (2.12). □

3. Related Results

We also have the related results:

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A satisfies the condition (2.4), then*

$$
\begin{aligned}
(3.1) \qquad 1 &\leq \frac{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}}{\eta_P(A)} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}
\end{aligned}
$$

Proof. Now, if we use the inequality (2.7) for the function $f(t) = t\ln t$ and A a selfadjoint operator on the Hilbert space H such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. then for $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ with $\operatorname{tr}(P) = 1$,

$$
\begin{aligned}
0 &\leq \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln(\operatorname{tr}(PA)) \\
&\leq \operatorname{tr}(PA\ln(eA)) - \operatorname{tr}(PA)\operatorname{tr}(P\ln(eA)) \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\operatorname{tr}(P|A-\operatorname{tr}(PA)|) \\ \\ \frac{1}{2}(M-m)\operatorname{tr}(P|\ln(eA)-\operatorname{tr}(P\ln(eA))|) \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(P[\ln(eA)]^2\right)-(\operatorname{tr}(P\ln(eA)))^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(3.2) \qquad 0 &\leq \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln(\operatorname{tr}(PA)) \\
&\leq \ln\left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \leq \ln\left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}} \\
&\leq \ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.
\end{aligned}
$$

If we take the exponential, then we get

$$
\begin{aligned}
1 &\leq \exp\left[\operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln(\operatorname{tr}(PA))\right] \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)},
\end{aligned}
$$

namely

$$
\begin{aligned}
1 &\leq \frac{\exp \ln (\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\exp(-\operatorname{tr}(PA\ln A))} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}
\end{aligned}
$$

and the inequality (3.1) is proved. □

From (3.1) we can obtain the following inequalities for the P-*determinant* of the positive invertible operator A :

Corollary 5. *With the assumptions of Theorem 5, we have*

$$
\begin{aligned}
1 &\leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2\operatorname{tr}(PA)}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]^{1/2}} \leq \left(\frac{M}{m}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)\operatorname{tr}(PA)} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{4}\left(\frac{M}{m}-1\right)}.
\end{aligned} \tag{3.3}
$$

Proof. If $0<m\leq A\leq M$, then $0<M^{-1}\leq A^{-1}\leq m^{-1}$. From (1.14) for $\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}$ and A^{-1} we derive

$$
\begin{aligned}
1 &\leq \frac{\left[\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\right)\right]^{-\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\right)}}{\eta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}\left|A^{-1}-\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\right)\right|\right)} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-2}\right)-\left(\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\right)\right)^2\right]^{1/2}} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)},
\end{aligned}
$$

namely

$$
\begin{aligned}
1 &\leq \frac{[\operatorname{tr}(PA)]^{[\operatorname{tr}(PA)]^{-1}}}{\eta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2[\operatorname{tr}(PA)]^2}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2\operatorname{tr}(PA)}\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]^{1/2}} \leq \left(\frac{M}{m}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)}.
\end{aligned}
$$

Observe that

$$\eta_{\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}}\left(A^{-1}\right) = \exp\left[-\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}A^{-1}\ln A^{-1}\right)\right]$$
$$= \exp\left(\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(PA)}\right) = [\Delta_P(A)]^{[\operatorname{tr}(PA)]^{-1}}.$$

If we take the power $\operatorname{tr}(PA) > 0$, then we get

$$1 \leq \frac{\operatorname{tr}(PA)}{\left([\Delta_P(A)]^{[\operatorname{tr}(PA)]^{-1}}\right)^{\operatorname{tr}(PA)}} \leq \left(\frac{M}{m}\right)^{\frac{1}{2\operatorname{tr}(PA)}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)}$$
$$\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]^{1/2}} \leq \left(\frac{M}{m}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)\operatorname{tr}(PA)}$$
$$\leq \left(\frac{M}{m}\right)^{\frac{1}{4m}(M-m)}$$

and the the inequality (3.3) is proved. □

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa