BOUNDS FOR THE ENTROPIC TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA KANTOROVICH'S CONSTANT

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp[-\operatorname{tr}(PA \ln A)].$$

In this paper we show, among others that, if A is an operator satisfying the condition $0 < m \leq A \leq M$, then

$$\begin{aligned}
1 &\geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}\operatorname{tr}(PA) - \frac{1}{(M-m)}\operatorname{tr}\left(PA\left|A - \frac{1}{2}(m+M)\right|\right)\right]} \\
&\geq \frac{\eta_P(A)}{m^{\frac{\operatorname{tr}\left(PA^2\right) - M\operatorname{tr}(PA)}{M-m}} M^{\frac{m\operatorname{tr}(PA) - \operatorname{tr}\left(PA^2\right)}{M-m}}} \\
&\geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}\operatorname{tr}(PA) + \frac{1}{(M-m)}\operatorname{tr}\left(PA\left|A - \frac{1}{2}(m+M)\right|\right)\right]} \\
&\geq \left[K\left(\frac{M}{m}\right)\right]^{-\operatorname{tr}(PA)} \geq \left[K\left(\frac{M}{m}\right)\right]^{-M},
\end{aligned}$$

where $K(\cdot)$ is *Kantorovich's constant.*

1. Introduction

In 1952, in the paper [5], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

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If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [8].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{and} \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P\ge0$ and $P\in\mathcal{B}_1(H)$, then for all $T\in\mathcal{B}(H)$, PT, $TP\in\mathcal{B}_1(H)$ and $\operatorname{tr}(PT)=\operatorname{tr}(TP)$. Also, since $P^{1/2}\in\mathcal{B}_2(H)$, $TP^{1/2}\in\mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2}=TP\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)=\operatorname{tr}(TP)$. Therefore, if $P\ge0$ and $P\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(PT)=\operatorname{tr}(TP)=\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T\in\mathcal{B}(H)$.

If $T\ge0$, then $P^{1/2}TP^{1/2}\ge0$, which implies that $\operatorname{tr}(PT)\ge0$ that shows that the functional $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n\to T$ for $n\to\infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty}\operatorname{tr}(PT_n)=\operatorname{tr}(PT)$, namely $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [3] and the references therein.

Now, for a given $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A)=\exp\operatorname{tr}((\ln A)P)=\exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. We observe that we have the following elementary properties [4]:

(i) *continuity*: the map $A\to\Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t)=\Delta_P(A)^t$ for all $t>0$;
(iii) *homogeneity*: $\Delta_P(tA)=t\Delta_P(A)$ and $\Delta_P(tI)=t$ for all $t>0$;
(iv) *monotonicity*: $0<A\le B$ implies $\Delta_P(A)\le\Delta_P(B)$.

In [4], we presented some fundamental properties of this determinant. Among others we showed that

$$1\le\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\le\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]$$

and

$$1\le\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\le\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right],$$

for $A>0$ and $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

For the entropy function $\eta(t)=-t\ln t$, $t>0$, the operator entropy has the following expression:

$$\eta(A)=-A\ln A$$

for positive A.

Now, for a given $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A):=\exp\left[-\operatorname{tr}(PA\ln A)\right]=\exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\}=\exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_P(I)=1 \text{ and } \eta_P(tI)=t^{-t} \tag{1.14}$$

for $t>0$.

We consider the *Kantorovich's constant* defined by

$$K\left(h\right):=\frac{\left(h+1\right)^2}{4h},\ h>0. \tag{1.15}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu}\leq\right)K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq\left(1-\nu\right)a+\nu b\leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.16}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.16) was obtained by Zou et al. in [10] while the second by Liao et al. [9].

Motivated by the above results, in this paper we provided several upper and lower bounds for the quantity

$$\frac{\eta_P\left(A\right)}{m^{\frac{\operatorname{tr}\left(PA^2\right)-M\operatorname{tr}(PA)}{M-m}}M^{\frac{m\operatorname{tr}(PA)-\operatorname{tr}\left(PA^2\right)}{M-m}}}$$

where $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$ and $0<m\leq A\leq M$ for some constants m, M.

2. Main Results

We start to the following main result:

Theorem 4. *Assume that $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If $0<m\leq A\leq M$ for some constants m, M, then*

$$1\geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}\operatorname{tr}(PA)-\frac{1}{(M-m)}\operatorname{tr}\left(PA\left|A-\frac{1}{2}(m+M)\right|\right)\right]} \tag{2.1}$$

$$\geq \frac{\eta_P(A)}{m^{\frac{\operatorname{tr}(PA^2)-M\operatorname{tr}(PA)}{M-m}}M^{\frac{m\operatorname{tr}(PA)-\operatorname{tr}(PA^2)}{M-m}}}$$
$$\geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{(M-m)}\operatorname{tr}\left(P A\left|A-\frac{1}{2}(m+M)\right|\right)\right]}$$
$$\geq \left[K\left(\frac{M}{m}\right)\right]^{-\operatorname{tr}(PA)} \geq \left[K\left(\frac{M}{m}\right)\right]^{-M}.$$

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0,1]$. Then

$$\min\{1-\nu,\nu\} = \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right| = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$\max\{1-\nu,\nu\} = \frac{1}{2} + \left|\nu - \frac{1}{2}\right| = \frac{1}{2} + \left|\frac{t-m}{M-m} - \frac{1}{2}\right| = \frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t$$

and

$$m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.$$

By using (1.16) we get

$$m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \leq t \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \tag{2.2}$$

for $t \in [m, M]$.

By taking the log in (2.2) we get

$$\begin{aligned} &\frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\leq \left[\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\ &+ \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\leq \ln t \leq \left[\frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\ &+ \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\leq \ln K\left(\frac{M}{m}\right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \end{aligned} \tag{2.3}$$

for $t \in [m, M]$.

If $0 < m \le A \le M$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$\begin{aligned} & \ln m \frac{M-A}{M-m} + \ln M \frac{A-m}{M-m} \\ & \le \left[\frac{1}{2} - \frac{1}{M-m}\left|A - \frac{1}{2}(m+M)\right|\right] \ln K\left(\frac{M}{m}\right) \\ & + \ln m \frac{M-A}{M-m} + \ln M \frac{A-m}{M-m} \\ & \le \ln A \le \left[\frac{1}{2} + \frac{1}{M-m}\left|A - \frac{1}{2}(m+M)\right|\right] \ln K\left(\frac{M}{m}\right) \\ & + \ln m \frac{M-A}{M-m} + \ln M \frac{A-m}{M-m} \\ & \le \ln K\left(\frac{M}{m}\right) + \ln m \frac{M-A}{M-m} + \ln M \frac{A-m}{M-m}. \end{aligned} \tag{2.4}$$

Assume that $Q \ge 0$ with $Q \in \mathcal{B}_1(H)$ and $\operatorname{tr}(Q) = 1$. If we multiply the inequality (2.4) both sides by $Q^{1/2}$ we get

$$\begin{aligned} & \ln m \frac{MQ - Q^{1/2}AQ^{1/2}}{M-m} + \ln M \frac{Q^{1/2}AQ^{1/2} - mQ}{M-m} \\ & \le \left[\frac{1}{2}Q - \frac{1}{M-m} Q^{1/2}\left|A - \frac{1}{2}(m+M)\right| Q^{1/2}\right] \ln K\left(\frac{M}{m}\right) \\ & + \ln m \frac{MQ - Q^{1/2}AQ^{1/2}}{M-m} + \ln M \frac{Q^{1/2}AQ^{1/2} - mQ}{M-m} \end{aligned} \tag{2.5}$$

$$\begin{aligned} & \le Q^{1/2}(\ln A) Q^{1/2} \\ & \le \left[\frac{1}{2}Q + \frac{1}{M-m} Q^{1/2}\left|A - \frac{1}{2}(m+M)\right| Q^{1/2}\right] \ln K\left(\frac{M}{m}\right) \\ & + \ln m \frac{MQ - Q^{1/2}AQ^{1/2}}{M-m} + \ln M \frac{Q^{1/2}AQ^{1/2} - mQ}{M-m} \\ & \le \ln K\left(\frac{M}{m}\right) Q + \ln m \frac{MQ - Q^{1/2}AQ^{1/2}}{M-m} + \ln M \frac{Q^{1/2}AQ^{1/2} - mQ}{M-m}. \end{aligned}$$

Now, if we take the trace and use the fact that $\operatorname{tr}(Q) = 1$, then we get

$$\begin{aligned} & \ln m \frac{M - \operatorname{tr}(QA)}{M-m} + \ln M \frac{\operatorname{tr}(QA) - m}{M-m} \\ & \le \left[\frac{1}{2} - \frac{1}{M-m} \operatorname{tr}\left(Q\left|A - \frac{1}{2}(m+M)\right|\right)\right] \ln K\left(\frac{M}{m}\right) \\ & + \ln m \frac{M - \operatorname{tr}(QA)}{M-m} + \ln M \frac{\operatorname{tr}(QA) - m}{M-m} \\ & \le \operatorname{tr}[Q(\ln A)] \end{aligned}$$

$$\leq \left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(Q\left|A-\frac{1}{2}(m+M)\right|\right)\right]\ln K\left(\frac{M}{m}\right)$$
$$+\ln m\frac{M-\operatorname{tr}(QA)}{M-m}+\ln M\frac{\operatorname{tr}(QA)-m}{M-m}$$
$$\leq \ln K\left(\frac{M}{m}\right)+\ln m\frac{M-Q\operatorname{tr}(QA)}{M-m}+\ln M\frac{\operatorname{tr}(QA)-m}{M-m},$$

namely

$$\ln\left(m^{\frac{M-\operatorname{tr}(QA)}{M-m}}M^{\frac{\operatorname{tr}(QA)-m}{M-m}}\right) \tag{2.6}$$
$$\leq \ln\left(m^{\frac{M-\operatorname{tr}(QA)}{M-m}}M^{\frac{\operatorname{tr}(QA)-m}{M-m}}K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\operatorname{tr}\left(Q\left|A-\frac{1}{2}(m+M)\right|\right)\right]}\right)$$
$$\leq \operatorname{tr}\left[Q(\ln A)\right]$$
$$\leq \ln\left(m^{\frac{M-\operatorname{tr}(QA)}{M-m}}M^{\frac{\operatorname{tr}(QA)-m}{M-m}}K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(Q\left|A-\frac{1}{2}(m+M)\right|\right)\right]}\right)$$
$$\leq \ln\left[m^{\frac{M-\operatorname{tr}(QA)}{M-m}}M^{\frac{\operatorname{tr}(QA)-m}{M-m}}K\left(\frac{M}{m}\right)\right].$$

If we take $Q=\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}\geq 0$, then $\operatorname{tr}(Q)=1$ and by (2.6) we derive, by taking the exponential, that

$$m^{\frac{M-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}}{M-m}}M^{\frac{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}-m}{M-m}}$$
$$\leq m^{\frac{M-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}}{M-m}}M^{\frac{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}-m}{M-m}}K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}\left|A-\frac{1}{2}(m+M)\right|\right)\right]}$$
$$\leq \exp\left(\operatorname{tr}\left[\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}(\ln A)\right]\right)$$
$$\leq m^{\frac{M-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}}{M-m}}M^{\frac{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}-m}{M-m}}K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(\frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)}\left|A-\frac{1}{2}(m+M)\right|\right)\right]}$$
$$\leq m^{\frac{M-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}}{M-m}}M^{\frac{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}-m}{M-m}}K\left(\frac{M}{m}\right),$$

namely

$$1\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{(M-m)\operatorname{tr}(PA)}\operatorname{tr}\left(PA\left|A-\frac{1}{2}(m+M)\right|\right)\right]}$$
$$\leq \frac{\left[\eta_P(A)\right]^{-1/\operatorname{tr}(PA)}}{m^{\frac{M-\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}}{M-m}}M^{\frac{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}-m}{M-m}}}$$
$$\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{(M-m)\operatorname{tr}(PA)}\operatorname{tr}\left(PA\left|A-\frac{1}{2}(m+M)\right|\right)\right]}$$
$$\leq K\left(\frac{M}{m}\right).$$

Now, by taking the power $-\operatorname{tr}(PA)<0$, we get the desired result (2.1). □

The second result is as follows:

Theorem 5. *Assume that $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If $0<m\leq A\leq M$ for some constants m, M, then*

$$
\begin{aligned}
(2.7)\qquad 1 &\geq \frac{\eta_P(A)}{m^{\frac{\operatorname{tr}(PA^2)-M\operatorname{tr}(PA)}{M-m}}M^{\frac{m\operatorname{tr}(PA)-\operatorname{tr}(PA^2)}{M-m}}}\\
&\geq \exp\left[-\frac{1}{Mm}\operatorname{tr}\left[PA(M-A)(A-m)\right]\right]\\
&\geq \exp\left[-\frac{\left(\left(M\operatorname{tr}(PA)-\operatorname{tr}\left(PA^2\right)\right)\left(\operatorname{tr}\left(PA^2\right)-m\operatorname{tr}(PA)\right)\right)}{Mm\left[\operatorname{tr}(PA)\right]}\right]\\
&\geq \exp\left[-\frac{1}{4Mm}(M-m)^2\operatorname{tr}(PA)\right]\geq\exp\left[-\frac{1}{4m}(M-m)^2\right].
\end{aligned}
$$

Proof. In [1] we obtained the following reverses of Young's inequality:

$$1\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right],$$

where $a,\,b>0$, $\nu\in[0,1]$.

This is equivalent, by taking the logarithm, with

$$0\leq\ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b\leq\nu(1-\nu)\frac{(b-a)^2}{ba}$$

where $a,\,b>0$, $\nu\in[0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$
\begin{aligned}
0&\leq\ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M\leq\frac{(M-t)(t-m)}{(M-m)^2}\frac{(M-m)^2}{Mm}\\
&=\frac{(M-t)(t-m)}{Mm}.
\end{aligned}
$$

Using the continuous functional calculus for selfadjoint operators, we have

$$0\leq\ln A-\frac{M-A}{M-m}\ln m-\frac{AQ^{1/2}-m}{M-m}\ln M\leq\frac{(M-A)(A-m)}{Mm}.$$

If we multiply both sides by $Q^{1/2}$ we get

$$
\begin{aligned}
0&\leq Q^{1/2}(\ln A)Q^{1/2}-\frac{MQ-Q^{1/2}AQ^{1/2}}{M-m}\ln m-\frac{Q^{1/2}AQ^{1/2}-mQ}{M-m}\ln M\\
&\leq\frac{Q^{1/2}(M-A)(A-m)Q^{1/2}}{Mm}.
\end{aligned}
$$

If we take the trace and use the fact that $\operatorname{tr}(Q)=1$, then we obtain

$$
\begin{aligned}
0&\leq\operatorname{tr}(Q\ln A)-\frac{M-\operatorname{tr}(QA)}{M-m}\ln m-\frac{\operatorname{tr}(QA)-m}{M-m}\ln M\\
&\leq\frac{1}{Mm}\operatorname{tr}\left[Q(M-A)(A-m)\right].
\end{aligned}
$$

The function $g(t)=(M-t)(t-m)$ is concave on $[m,M]$ and by Jensen's inequality for trace

$$\operatorname{tr}(Qg(A))\leq g(\operatorname{tr}(QA)),$$

for $Q \geq 0$ with $Q \in \mathcal{B}_1(H)$ and $\operatorname{tr}(Q) = 1$, we have

$$\operatorname{tr}[(M-A)(A-m)] \leq ((M - \operatorname{tr}(QA))(\operatorname{tr}(QA) - m)).$$

If we take the exponential, then we get

$$\begin{aligned}
(2.8) \qquad 1 &\leq \frac{\exp[\operatorname{tr}(Q \ln A)]}{\exp\left[\frac{M-\operatorname{tr}(QA)}{M-m} \ln m + \frac{\operatorname{tr}(QA)-m}{M-m} \ln M\right]} \\
&\leq \exp\left[\frac{1}{Mm} \operatorname{tr}[Q(M-A)(A-m)]\right] \\
&\leq \exp\left[\frac{1}{Mm}((M - \operatorname{tr}(QA))(\operatorname{tr}(QA) - m))\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
\exp\left[\frac{M - \operatorname{tr}(QA)}{M-m} \ln m + \frac{\operatorname{tr}(QA) - m}{M-m} \ln M\right] &= \exp\left[\ln\left(m^{\frac{M-\operatorname{tr}(QA)}{M-m}} M^{\frac{\operatorname{tr}(QA)-m}{M-m}}\right)\right] \\
&= m^{\frac{M-\operatorname{tr}(QA)}{M-m}} M^{\frac{\operatorname{tr}(QA)-m}{M-m}}
\end{aligned}$$

and by (2.8) we obtain

$$\begin{aligned}
(2.9) \qquad 1 &\leq \frac{\exp[\operatorname{tr}(Q \ln A)]}{m^{\frac{M-\operatorname{tr}(QA)}{M-m}} M^{\frac{\operatorname{tr}(QA)-m}{M-m}}} \\
&\leq \exp\left[\frac{1}{Mm} \operatorname{tr}[Q(M-A)(A-m)]\right] \\
&\leq \exp\left[\frac{1}{Mm}((M - \operatorname{tr}(QA))(\operatorname{tr}(QA) - m))\right].
\end{aligned}$$

If we take $Q = \frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)} \geq 0$, then $\operatorname{tr}(Q) = 1$ and by (2.9) we derive

$$\begin{aligned}
1 &\leq \frac{(\exp[\operatorname{tr}(PA \ln A)])^{1/\operatorname{tr}(PA)}}{m^{\frac{M - \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(PA)}}{M-m}} M^{\frac{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(PA)} - m}{M-m}}} \\
&\leq \exp\left[\frac{1}{Mm \operatorname{tr}(PA)} \operatorname{tr}[PA(M-A)(A-m)]\right] \\
&\leq \exp\left[\frac{1}{Mm[\operatorname{tr}(PA)]^2}\left(\left(M \operatorname{tr}(PA) - \operatorname{tr}\left(PA^2\right)\right)\left(\operatorname{tr}\left(PA^2\right) - m \operatorname{tr}(PA)\right)\right)\right] \\
&\leq \exp\left[\frac{1}{4Mm}(M-m)^2\right].
\end{aligned}$$

Now, by taking the power $-\operatorname{tr}(PA) < 0$, we derive the desired result (2.7). □

3. Related Results

In [2] we obtained the following refinement and reverse of Young's inequality:

$$
\begin{aligned}
1 &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\
&\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \\
&\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right],
\end{aligned} \tag{3.1}
$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

Theorem 6. *With the assumptions of Theorem ,*

$$
\begin{aligned}
1 &\ge \frac{\eta_P(A)}{m^{\frac{\operatorname{tr}(PA^2)-M\operatorname{tr}(PA)}{M-m}}M^{\frac{m\operatorname{tr}(PA)-\operatorname{tr}(PA^2)}{M-m}}} \\
&\ge \exp\left[\frac{-1}{2m^2}\operatorname{tr}\left[PA(M-A)(A-m)\right]\right] \\
&\ge \exp\left[-\frac{\left(\left(M\operatorname{tr}(PA)-\operatorname{tr}\left(PA^2\right)\right)\left(\operatorname{tr}\left(PA^2\right)-m\operatorname{tr}(PA)\right)\right)}{2m^2\operatorname{tr}(PA)}\right] \\
&\ge \exp\left[-\frac{1}{8}\left(\frac{M}{m}-1\right)^2\operatorname{tr}(PA)\right]\ge \exp\left[-\frac{1}{8}M\left(\frac{M}{m}-1\right)^2\right].
\end{aligned} \tag{3.2}
$$

Proof. From (3.1) we have

$$
\begin{aligned}
1 &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\right] \\
&\le \frac{(1-\nu)m+\nu M}{m^{1-\nu}M^{\nu}} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2\right],
\end{aligned}
$$

for $\nu\in[0,1]$.

By taking the logarithm, we obtain

$$
\begin{aligned}
0 &\le \frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2 \\
&\le \ln((1-\nu)m+\nu M)-(1-\nu)\ln m-\nu\ln M \\
&\le \frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2,
\end{aligned} \tag{3.3}
$$

for $\nu\in[0,1]$.

If we take $a=m,\ b=M,\ t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$
\begin{aligned}
0 &\le \frac{(M-t)(t-m)}{2M^2}\le \ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M \\
&\le \frac{(M-t)(t-m)}{2m^2}
\end{aligned}
$$

for $t\in[m,M]$.

As above, we get the trace inequality

$$
\begin{aligned}
0 &\leq \frac{1}{2M^2} \operatorname{tr}\left[Q\left(M-A\right)\left(A-m\right)\right] \\
&\leq \operatorname{tr}\left(Q \ln A\right) - \frac{M - \operatorname{tr}\left(QA\right)}{M-m} \ln m - \frac{\operatorname{tr}\left(QA\right) - m}{M-m} \ln M \\
&\leq \frac{1}{2m^2} \operatorname{tr}\left[Q\left(M-A\right)\left(A-m\right)\right].
\end{aligned}
$$

If we take the exponential, then we derive

$$
\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M^2} \operatorname{tr}\left[Q\left(M-A\right)\left(A-m\right)\right]\right] \\
&\leq \frac{\exp\left[\operatorname{tr}\left(Q \ln A\right)\right]}{\exp\left[\frac{M-\operatorname{tr}(QA)}{M-m} \ln m + \frac{\operatorname{tr}(QA)-m}{M-m} \ln M\right]} \\
&\leq \exp\left[\frac{1}{2m^2} \operatorname{tr}\left[Q\left(M-A\right)\left(A-m\right)\right]\right] \\
&\leq \exp\left[\frac{1}{2m^2} \left(\left(M-\operatorname{tr}\left(QA\right)\right)\left(\operatorname{tr}\left(QA\right)-m\right)\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right].
\end{aligned} \tag{3.4}
$$

If we take $Q = \frac{A^{1/2}PA^{1/2}}{\operatorname{tr}(PA)} \geq 0$, then $\operatorname{tr}\left(Q\right) = 1$ and by (3.4) we derive

$$
\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M^2 \operatorname{tr}\left(PA\right)} \operatorname{tr}\left[PA\left(M-A\right)\left(A-m\right)\right]\right] \\
&\leq \frac{\left(\exp\left[\operatorname{tr}\left(PA \ln A\right)\right]\right)^{1/\operatorname{tr}(PA)}}{m^{\frac{M-\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(PA)}}{M-m}} M^{\frac{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(PA)}-m}{M-m}}} \\
&\leq \exp\left[\frac{1}{2m^2 \operatorname{tr}\left(PA\right)} \operatorname{tr}\left[PA\left(M-A\right)\left(A-m\right)\right]\right] \\
&\leq \exp\left[\frac{1}{2m^2\left[\operatorname{tr}\left(PA\right)\right]^2} \left(\left(M \operatorname{tr}\left(PA\right) - \operatorname{tr}\left(PA^2\right)\right)\left(\operatorname{tr}\left(PA^2\right) - m \operatorname{tr}\left(PA\right)\right)\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right],
\end{aligned}
$$

which is equivalent to (3.2). □

Remark 1. *Consider the quantities*

$$
B_1\left(m,M\right) := -\frac{1}{4m}\left(M-m\right)^2 \text{ and } B_2\left(m,M\right) := -\frac{1}{8} M \left(\frac{M}{m}-1\right)^2
$$

defined for $0 < m < M$.

Observe that

$$
\begin{aligned}
B_1\left(m,M\right) - B_2\left(m,M\right) &= -\frac{1}{4m}\left(M-m\right)^2 + \frac{1}{8}\frac{M}{m^2}\left(M-m\right)^2 \\
&= \frac{1}{4m}\left(M-m\right)^2 \left(\frac{M}{2m}-1\right),
\end{aligned}
$$

which shows that $B_1(m,M) < B_2(m,M)$ *for* $m < M < 2m$ *and* $B_1(m,M) > B_2(m,M)$ *for* $M > 2m$.

Therefore the lower bound from (2.7) is better than the one from (3.2) for $M > 2m$, *while for* $m < M < 2m$ *the conclusion is the other way around.*

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