INEQUALITIES FOR TRACE CLASS ENTROPIC P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA ČEBYŠEV'S TYPE RESULTS

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* and the *entropic P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp[\operatorname{tr}(P \ln A)]$$

and

$$\eta_P(A) := \exp[-\operatorname{tr}(PA\ln A)],$$

respectively.

In this paper we show among others that, if $A,\ B > 0,\ P,\ Q \in \mathcal{B}_1(H) \setminus \{0\}$ with $P, Q \geq 0$ and $\operatorname{tr}(P) = \operatorname{tr}(Q) = 1$, then for $r > 0$

$$[\eta_P(A^r)]^{1/r} [\eta_Q(B^r)]^{1/r} \leq [\Delta_Q(B)]^{-\operatorname{tr}(PA^r)} [\Delta_P(A)]^{-\operatorname{tr}(QB^r)}.$$

In particular, we have

$$\eta_P(A)\,\eta_Q(B) \leq [\Delta_Q(B)]^{-\operatorname{tr}(PA)} [\Delta_P(A)]^{-\operatorname{tr}(QB)}.$$

1. Introduction

In 1952, in the paper [8], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

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Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [11].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I}\langle Ae_i, Be_i\rangle = \sum_{i\in I}\langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{and} \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \ | \ B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i \rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \text{and} \ |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [6] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [7]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In [7], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA\ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln(tA)\right]\right\}\right)\\
&= \exp\left(-\operatorname{tr}\left\{P\left[tA(\ln t + \ln A)\right]\right\}\right) = \exp\left(-\operatorname{tr}\left\{P\left(tA\ln t + tA\ln A\right)\right\}\right)\\
&= \exp\left(-t\ln t\operatorname{tr}(PA)\right)\exp\left(-t\operatorname{tr}(PA\ln A)\right)\\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}(PA\ln A)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\,\mathrm{tr}(PA)} [\eta_P(A)]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(I) = 1 \text{ and } \eta_P(tI) = t^{-t} \tag{1.14}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that, if A, $B > 0$, P, $Q \in \mathcal{B}_1(H) \setminus \{0\}$ with $P, Q \geq 0$ and $\mathrm{tr}(P) = \mathrm{tr}(Q) = 1$, then for $r > 0$

$$[\eta_P(A^r)]^{1/r} [\eta_Q(B^r)]^{1/r} \leq [\Delta_Q(B)]^{-\mathrm{tr}(PA^r)} [\Delta_P(A)]^{-\mathrm{tr}(QB^r)}.$$

In particular, we have

$$\eta_P(A)\,\eta_Q(B) \leq [\Delta_Q(B)]^{-\mathrm{tr}(PA)} [\Delta_P(A)]^{-\mathrm{tr}(QB)}.$$

2. Main Results

We say that the functions f, $g : [a,b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \text{ for each } t,\ s \in [a,b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a,b]$, then they are synchronous on $[a,b]$ while if they have opposite monotonicity, they are asynchronous.

In [4] we obtained the following Čebyšev's type result:

Lemma 1. *Let A be a selfadjoint operator on the Hilbert space H with $\mathrm{Sp}(A) \subseteq J$ and assume that the continuous functions f, $g : J \to \mathbb{R}$ are synchronous on J. If $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$, then*

$$\begin{aligned} &\frac{\mathrm{tr}[Pf(A)g(A)]}{\mathrm{tr}(P)} - \frac{\mathrm{tr}[Pf(A)]}{\mathrm{tr}(P)} \frac{\mathrm{tr}[Pg(A)]}{\mathrm{tr}(P)} \\ &\geq \left(\frac{\mathrm{tr}[Pf(A)]}{\mathrm{tr}(P)} - f\left(\frac{\mathrm{tr}(PA)}{\mathrm{tr}(P)} \right) \right) \left(g\left(\frac{\mathrm{tr}(PA)}{\mathrm{tr}(P)} \right) - \frac{\mathrm{tr}[Pg(A)]}{\mathrm{tr}(P)} \right). \end{aligned} \tag{2.1}$$

In particular, we have:

Corollary 1. *With the assumptions of Lemma 1 and if one of the functions f and g is convex while the other is concave, then we have*

$$\begin{aligned} &\frac{\mathrm{tr}[Pf(A)g(A)]}{\mathrm{tr}(P)} - \frac{\mathrm{tr}[Pf(A)]}{\mathrm{tr}(P)} \frac{\mathrm{tr}[Pg(A)]}{\mathrm{tr}(P)} \\ &\geq \left(\frac{\mathrm{tr}[Pf(A)]}{\mathrm{tr}(P)} - f\left(\frac{\mathrm{tr}(PA)}{\mathrm{tr}(P)} \right) \right) \left(g\left(\frac{\mathrm{tr}(PA)}{\mathrm{tr}(P)} \right) - \frac{\mathrm{tr}[Pg(A)]}{\mathrm{tr}(P)} \right) \\ &\geq 0. \end{aligned} \tag{2.2}$$

By utilizing these inequalities, we can state the following main result:

Theorem 4. *Let $A > 0$, $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$ and $\mathrm{tr}(P) = 1$, then for $p \in (-\infty, 0) \cup [1, \infty)$ we have*

$$\begin{aligned} [\eta_P(A^p)]^{1/p} &\leq [\Delta_P(A)]^{-[\mathrm{tr}(PA)]^p} [\mathrm{tr}(PA)]^{[\mathrm{tr}(PA)]^p - \mathrm{tr}(PA^p)} \\ &\leq [\Delta_P(A)]^{-\mathrm{tr}(PA^p)}. \end{aligned} \tag{2.3}$$

In particular, we have

$$\eta_P(A) \leq [\Delta_P(A)]^{-\operatorname{tr}(PA)} \tag{2.4}$$

and

$$\begin{aligned} \left[\eta_P(A^2)\right]^{1/2} &\leq [\Delta_P(A)]^{-[\operatorname{tr}(PA)]^2} [\operatorname{tr}(PA)]^{[\operatorname{tr}(PA)]^2 - \operatorname{tr}\left(PA^2\right)} \\ &\leq [\Delta_P(A)]^{-\operatorname{tr}\left(PA^2\right)}. \end{aligned} \tag{2.5}$$

Also

$$\begin{aligned} \eta_P(A^{-1}) &\geq [\Delta_P(A)]^{[\operatorname{tr}(PA)]^{-1}} [\operatorname{tr}(PA)]^{\operatorname{tr}\left(PA^{-1}\right) - [\operatorname{tr}(PA)]^{-1}} \\ &\geq [\Delta_P(A)]^{\operatorname{tr}\left(PA^{-1}\right)}. \end{aligned} \tag{2.6}$$

Proof. If we take $f(t) = t^p$, $p \in (-\infty, 0) \cup [1, \infty)$ and $g(t) = \ln t$, $t > 0$ in (2.2), then we get

$$\begin{aligned} &\operatorname{tr}(PA^p \ln A) - \operatorname{tr}(PA^p)\operatorname{tr}(P\ln A) \\ &\geq (\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p)(\ln(\operatorname{tr}(PA)) - \operatorname{tr}(P\ln A)) \\ &\geq 0, \end{aligned}$$

namely

$$\begin{aligned} &\frac{1}{p}\operatorname{tr}(PA^p \ln A^p) - \operatorname{tr}(PA^p)\operatorname{tr}(P\ln A) \\ &\geq (\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p)(\ln(\operatorname{tr}(PA)) - \operatorname{tr}(P\ln A)) \\ &\geq 0. \end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned} &\frac{\exp[-\operatorname{tr}(PA^p)\operatorname{tr}(P\ln A)]}{\exp\left[-\frac{1}{p}\operatorname{tr}(PA^p \ln A^p)\right]} \\ &\geq \exp[(\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p)(\ln(\operatorname{tr}(PA)) - \operatorname{tr}(P\ln A))] \\ &\geq 1. \end{aligned} \tag{2.7}$$

Observe that

$$\begin{aligned} \exp[-\operatorname{tr}(PA^p)\operatorname{tr}(P\ln A)] &= (\exp[\operatorname{tr}(P\ln A)])^{-\operatorname{tr}(PA^p)} \\ &= [\Delta_P(A)]^{-\operatorname{tr}(PA^p)}, \end{aligned}$$

$$\begin{aligned} \exp\left[-\frac{1}{p}\operatorname{tr}(PA^p \ln A^p)\right] &= (\exp[-\operatorname{tr}(PA^p \ln A^p)])^{1/p} \\ &= [\eta_P(A^p)]^{1/p} \end{aligned}$$

and

$$\begin{aligned} &\exp[(\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p)(\ln(\operatorname{tr}(PA)) - \operatorname{tr}[P\ln A])] \\ &= [\exp(\ln(\operatorname{tr}(PA)) - \operatorname{tr}(P\ln A))]^{\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p} \\ &= \left[\frac{\exp\ln(\operatorname{tr}(PA))}{\exp\operatorname{tr}(P\ln A)}\right]^{\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p} \\ &= \left[\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\right]^{\operatorname{tr}(PA^p) - [\operatorname{tr}(PA)]^p}. \end{aligned}$$

Then by (2.7) we get

$$\frac{[\Delta_P(A)]^{-\operatorname{tr}(PA^p)}}{[\eta_P(A^p)]^{1/p}} \geq \left[\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\right]^{\operatorname{tr}(PA^p)-[\operatorname{tr}(PA)]^p} \geq 1.$$

Now, if we multiply this inequality by $[\Delta_P(A)]^{\operatorname{tr}(PA^p)}$, then we get (2.3). □

Corollary 2. *With the assumptions of Theorem 4 and $p \neq 2$, we have*

$$\left[\eta_P\left(A^{2-p}\right)\right]^{\frac{1}{2-p}} \geq \left[\eta_P\left(A^2\right)\right]^{\frac{\operatorname{tr}\left(PA^{2-p}\right)}{2\operatorname{tr}(PA)}}. \tag{2.8}$$

In particular, for $p=1$, we get

$$[\eta_P(A)]^2 \geq \eta_P\left(A^2\right). \tag{2.9}$$

Proof. If we write the inequality (2.4) for $\frac{APA}{\operatorname{tr}(PA)}$ instead of P and A^{-1} instead of A, then we get

$$\frac{1}{\left[\eta_{\frac{APA}{\operatorname{tr}(PA)}}(A^{-p})\right]^{1/p}} \geq \left[\Delta_{\frac{APA}{\operatorname{tr}(PA)}}(A)\right]^{\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA)}A^{-p}\right)}. \tag{2.10}$$

Observe that

$$\begin{aligned}
\eta_{\frac{APA}{\operatorname{tr}(PA)}}(A^{-p}) &= \exp\left(-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA)}A^{-p}\ln A^{-p}\right)\right)\\
&= \exp\left(-\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA^{2-p}\ln A^{-p}\right)\right)\\
&= \exp\left(\frac{p}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA^{2-p}\ln A\right)\right)\\
&= \exp\left(\frac{p}{(2-p)\operatorname{tr}(PA)}\operatorname{tr}\left(PA^{2-p}\ln A^{2-p}\right)\right)\\
&= \exp\left(\frac{p}{(p-2)\operatorname{tr}(PA)}\operatorname{tr}\left(-PA^{2-p}\ln A^{2-p}\right)\right)\\
&= \left[\eta_P\left(A^{2-p}\right)\right]^{\frac{p}{(p-2)\operatorname{tr}(PA)}}
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\frac{APA}{\operatorname{tr}(PA)}}\left(A^{-1}\right) &= \exp\left(\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA)}\ln A^{-1}\right)\right)\\
&= \exp\left(\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(PA^2\ln A^{-1}\right)\right)\\
&= \exp\left(\frac{1}{\operatorname{tr}(PA)}\operatorname{tr}\left(-\frac{1}{2}PA^2\ln A^2\right)\right)\\
&= \left[\eta_P\left(A^2\right)\right]^{\frac{1}{2\operatorname{tr}(PA)}}
\end{aligned}$$

and by (2.10) we get

$$\frac{1}{\left[\left[\eta_P\left(A^{2-p}\right)\right]^{\frac{p}{(p-2)\operatorname{tr}(PA)}}\right]^{1/p}} \geq \left[\left[\eta_P\left(A^2\right)\right]^{\frac{1}{2\operatorname{tr}(PA)}}\right]^{\frac{\operatorname{tr}\left(PA^{2-p}\right)}{\operatorname{tr}(PA)}},$$

namely

$$\frac{1}{\left[\eta_P\left(A^{2-p}\right)\right]^{\frac{1}{(p-2)\operatorname{tr}(PA)}}} \geq \left[\eta_P\left(A^2\right)\right]^{\frac{\operatorname{tr}\left(PA^{2-p}\right)}{2[\operatorname{tr}(PA)]^2}}.$$

If we take the power $\operatorname{tr}(PA)$, then we get (2.8). □

Theorem 5. *Let $A>0$, $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$ and $\operatorname{tr}(P)=1$, then for $q \in (0,1)$ we have*

$$\left[\eta_P\left(A^{q+1}\right)\right]^{\frac{1}{q+1}} \leq \frac{(\operatorname{tr}(PA))^{(\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)}}{[\eta_P(A)]^{-(\operatorname{tr}(PA))^q}} \leq [\eta_P(A)]^{\operatorname{tr}(PA^q)}. \tag{2.11}$$

Proof. If we take $f(t)=t\ln t$, $g(t)=t^q$, $q\in(0,1)$, $t>0$ in (2.2), then we get

$$\begin{aligned}&\operatorname{tr}\left(PA^{q+1}\ln A\right)-\operatorname{tr}(PA\ln A)\operatorname{tr}(PA^q)\\ &\geq (\operatorname{tr}(PA\ln A)-\ln(\operatorname{tr}(PA)))\left((\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)\right)\\ &\geq 0\end{aligned}$$

namely

$$\begin{aligned}&\frac{1}{q+1}\operatorname{tr}\left(PA^{q+1}\ln A^{q+1}\right)-\operatorname{tr}(PA\ln A)\operatorname{tr}(PA^q)\\ &\geq (\operatorname{tr}(PA\ln A)-\ln(\operatorname{tr}(PA)))\left((\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)\right)\\ &\geq 0.\end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned}&\exp\left[\frac{1}{q+1}\operatorname{tr}\left(PA^{q+1}\ln A^{q+1}\right)-\operatorname{tr}(PA\ln A)\operatorname{tr}(PA^q)\right]\\ &\geq \exp\left[(\operatorname{tr}(PA\ln A)-\ln(\operatorname{tr}(PA)))\left((\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)\right)\right]\\ &\geq 1.\end{aligned} \tag{2.12}$$

Observe that

$$\begin{aligned}&\exp\left[\frac{1}{q+1}\operatorname{tr}\left(PA^{q+1}\ln A^{q+1}\right)-\operatorname{tr}(PA\ln A)\operatorname{tr}(PA^q)\right]\\ &=\frac{\exp\left[-\operatorname{tr}(PA\ln A)\operatorname{tr}(PA^q)\right]}{\exp\left[-\frac{1}{q+1}\operatorname{tr}\left(PA^{q+1}\ln A^{q+1}\right)\right]}=\frac{\left(\exp\left[-\operatorname{tr}(PA\ln A)\right]\right)^{\operatorname{tr}(PA^q)}}{\left(\exp\left[-\operatorname{tr}\left(PA^{q+1}\ln A^{q+1}\right)\right]\right)^{\frac{1}{q+1}}}\\ &=\frac{[\eta_P(A)]^{\operatorname{tr}(PA^q)}}{\left[\eta_P\left(A^{q+1}\right)\right]^{\frac{1}{q+1}}}\end{aligned}$$

and

$$\begin{aligned}&\exp\left[(\operatorname{tr}(PA\ln A)-\ln(\operatorname{tr}(PA)))\left((\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)\right)\right]\\ &=\left[\exp\left[\operatorname{tr}(PA\ln A)-\ln(\operatorname{tr}(PA))\right]\right]^{(\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)}\\ &=\frac{\left(\exp\left[\ln(\operatorname{tr}(PA))^{-1}\right]\right)^{(\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)}}{\left[\exp\left(-\operatorname{tr}(PA\ln A)\right)\right]^{(\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)}}\\ &=\frac{(\operatorname{tr}(PA))^{\operatorname{tr}(PA^q)-(\operatorname{tr}(PA))^q}}{[\eta_P(A)]^{(\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)}}.\end{aligned}$$

By (2.12) we get

$$\frac{[\eta_P(A)]^{\operatorname{tr}(PA^q)}}{[\eta_P(A^{q+1})]^{\frac{1}{q+1}}} \geq \frac{(\operatorname{tr}(PA))^{\operatorname{tr}(PA^q)-(\operatorname{tr}(PA))^q}}{[\eta_P(A)]^{(\operatorname{tr}(PA))^q-\operatorname{tr}(PA^q)}} \geq 1.$$

If we divide by $[\eta_P(A)]^{\operatorname{tr}(PA^q)}$ we get the desired result (2.11). □

Corollary 3. *With the assumptions of Theorem 5, we have*

$$[\eta_P(A^{1-q})]^{\frac{1}{1-q}} \geq [\eta_P(A^2)]^{-\frac{\operatorname{tr}\left(PA^{2-q}\right)}{2\operatorname{tr}(PA)}}. \tag{2.13}$$

In particular, for $q = 1/2$, we get

$$\left[\eta_P\left(A^{1/2}\right)\right]^2 \geq [\eta_P(A^2)]^{-\frac{\operatorname{tr}\left(PA^{3/2}\right)}{2\operatorname{tr}(PA)}}. \tag{2.14}$$

Proof. If we write the inequality (2.11) for $\frac{APA}{\operatorname{tr}(PA)}$ instead of P and A^{-1} instead of A, then we get

$$\frac{1}{\left[\eta_{\frac{APA}{\operatorname{tr}(PA)}}\left(A^{-(q+1)}\right)\right]^{\frac{1}{q+1}}} \geq \left[\eta_{\frac{APA}{\operatorname{tr}(PA)}}\left(A^{-1}\right)\right]^{-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA)}A^{-q}\right)}. \tag{2.15}$$

As above,

$$\eta_{\frac{APA}{\operatorname{tr}(PA)}}(A^{-(q+1)}) = [\eta_P(A^{1-q})]^{\frac{q+1}{(q-1)\operatorname{tr}(PA)}}$$

and

$$\Delta_{\frac{APA}{\operatorname{tr}(PA)}}\left(A^{-1}\right) = [\eta_P(A^2)]^{\frac{1}{2\operatorname{tr}(PA)}}$$

and by (2.15) we obtain

$$\frac{1}{\left[[\eta_P(A^{1-q})]^{\frac{q+1}{(q-1)\operatorname{tr}(PA)}}\right]^{\frac{1}{q+1}}} \geq \left[[\eta_P(A^2)]^{\frac{1}{2\operatorname{tr}(PA)}}\right]^{-\frac{\operatorname{tr}\left(PA^{2-q}\right)}{\operatorname{tr}(PA)}},$$

namely

$$[\eta_P(A^{1-q})]^{\frac{1}{(1-q)\operatorname{tr}(PA)}} \geq [\eta_P(A^2)]^{-\frac{\operatorname{tr}\left(PA^{2-q}\right)}{2[\operatorname{tr}(PA)]^2}}.$$

If we take the power $\operatorname{tr}(PA)$, then we get (2.13). □

3. Related Results

In [4] we also obtained the following Čebyšev's type inequalities:

Lemma 2. *Let A and B be two selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on J. If $P, Q \in \mathcal{B}_1(H) \setminus \{0\}$ with $P, Q \geq 0$, then*

$$\begin{aligned} &\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} + \frac{\operatorname{tr}[Qf(B)g(B)]}{\operatorname{tr}(Q)} \\ &\geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qg(B)]}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qf(B)]}{\operatorname{tr}(Q)} \end{aligned} \tag{3.1}$$

and, in particular,

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)}+\frac{\operatorname{tr}[Pf(B)g(B)]}{\operatorname{tr}(P)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(B)]}{\operatorname{tr}(P)}+\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pf(B)]}{\operatorname{tr}(P)}. \tag{3.2}$$

We also have:

Corollary 4. *Let A be a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A)\subseteq J$ and assume that the continuous functions f, $g : J\to\mathbb{R}$ are synchronous on J. If P, $Q\in\mathcal{B}_1(H)\setminus\{0\}$ with P, $Q\geq 0$, then*

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)}+\frac{\operatorname{tr}[Qf(A)g(A)]}{\operatorname{tr}(Q)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qg(A)]}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qf(A)]}{\operatorname{tr}(Q)} \tag{3.3}$$

and, in particular,

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)}\geq\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}. \tag{3.4}$$

Using the above inequalities we can also state:

Theorem 6. *Let A, $B>0$, P, $Q\in\mathcal{B}_1(H)\setminus\{0\}$ with P, $Q\geq 0$ and $\operatorname{tr}(P)=\operatorname{tr}(Q)=1$, then for $r>0$*

$$[\eta_P(A^r)]^{1/r}\left[\eta_Q(B^r)\right]^{1/r}\leq[\Delta_Q(B)]^{-\operatorname{tr}(PA^r)}[\Delta_P(A)]^{-\operatorname{tr}(QB^r)}. \tag{3.5}$$

In particular, we have

$$\eta_P(A)\,\eta_Q(B)\leq[\Delta_Q(B)]^{-\operatorname{tr}(PA)}[\Delta_P(A)]^{-\operatorname{tr}(QB)} \tag{3.6}$$

and

$$\left[\eta_P\left(A^2\right)\right]^{1/2}\left[\eta_Q\left(B^2\right)\right]^{1/2}\leq[\Delta_Q(B)]^{-\operatorname{tr}\left(PA^2\right)}[\Delta_P(A)]^{-\operatorname{tr}\left(QB^2\right)}. \tag{3.7}$$

For $r<0$, the inequality in (3.5) reverses.

Proof. If we write the inequality (3.1) for the functions $f(t)=t^r$, $r>0$ and $g(t)=\ln t$, $t>0$ then we get

$$\operatorname{tr}(PA^r\ln A)+\operatorname{tr}(QB^r\ln B)\geq\operatorname{tr}(PA^r)\operatorname{tr}(Q\ln B)+\operatorname{tr}(P\ln A)\operatorname{tr}(QB^r),$$

namely

$$\frac{1}{r}[\operatorname{tr}(-PA^r\ln A^r)+\operatorname{tr}(-QB^r\ln B^r)] \leq-\operatorname{tr}(PA^r)\operatorname{tr}(Q\ln B)-\operatorname{tr}(P\ln A)\operatorname{tr}(QB^r).$$

If we take the exponential, then we get

$$(\exp[\operatorname{tr}(-PA^r\ln A^r)])^{\frac{1}{r}}(\exp\operatorname{tr}(-QB^r\ln B^r))^{\frac{1}{r}} \leq[\exp(\operatorname{tr}(Q\ln B))]^{-\operatorname{tr}(PA^r)}[\exp\operatorname{tr}(P\ln A)]^{-\operatorname{tr}(QB^r)},$$

namely

$$(\eta_P(A^r))^{1/r}\left(\eta_Q(B^r)\right)^{1/r}\leq[\Delta_Q(B)]^{-\operatorname{tr}(PA^r)}[\Delta_P(A)]^{-\operatorname{tr}(QB^r)}$$

and the inequality (3.5) is proved. □

Corollary 5. *With the assumptions of Theorem 6 we have*

$$[\eta_P(A^r)]^{1/r}[\eta_P(B^r)]^{1/r} \le [\Delta_P(B)]^{-\operatorname{tr}(PA^r)}[\Delta_P(A)]^{-\operatorname{tr}(PB^r)}, \tag{3.8}$$

$$[\eta_P(A^r)]^{1/r}\left[\eta_Q(A^r)\right]^{1/r} \le [\Delta_Q(A)]^{-\operatorname{tr}(PA^r)}[\Delta_P(A)]^{-\operatorname{tr}(QA^r)} \tag{3.9}$$

and

$$[\eta_P(A^r)]^{1/r} \le [\Delta_P(A)]^{-\operatorname{tr}(PA^r)}. \tag{3.10}$$

Remark 1. *For $r = 1$ we derive that*

$$\eta_P(A)\,\eta_P(B) \le [\Delta_P(B)]^{-\operatorname{tr}(PA)}[\Delta_P(A)]^{-\operatorname{tr}(PB)}, \tag{3.11}$$

$$\eta_P(A)\,\eta_Q(A) \le [\Delta_Q(A)]^{-\operatorname{tr}(PA)}[\Delta_P(A)]^{-\operatorname{tr}(QA)} \tag{3.12}$$

and

$$\eta_P(A) \le [\Delta_P(A)]^{-\operatorname{tr}(PA)}. \tag{3.13}$$

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