BOUNDS FOR THE GEOMETRIC MEAN OF TRACE CLASS ENTROPIC P-DETERMINANTS OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the entropic P-*determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp[-\operatorname{tr}(PA\ln A)].$$

Assume that $P_j \in \mathcal{B}_1(H)$ with $P_j \geq 0$ and $\operatorname{tr}(P_j) = 1$ for $j \in \{1, ..., n\}$. In this paper we show, among others, that, if $0 < m \leq A_j \leq M$, $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$\begin{aligned} 1 &\leq \frac{\left(\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)\right)^{-\left(\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)\right)}}{\prod_{j=1}^n [\eta_{P_j}(A_j)]^{p_j}} \\ &\leq \left(\frac{M}{m}\right)^{\frac{1}{M-m}\left(M - \sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j) - m\right)} \\ &\leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}. \end{aligned}$$

1. Introduction

In 1952, in the paper [9], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

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If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [10], [11], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x \rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [12].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A| x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^* Ae_i, e_i \rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*, then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \ | \ B \in \mathcal{B}_2(H), \ \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}(P^{1/2}TP^{1/2}) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [7] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [8]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In [8], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA) \operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}(PA^{-1})\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right) \operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA \ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A) P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_P(I)=1 \text{ and } \eta_P(tI)=t^{-t} \tag{1.14}$$

for $t>0$.

Assume that $P_j\in\mathcal{B}_1(H)$ with $P_j\geq 0$ and $\operatorname{tr}(P_j)=1$ for $j\in\{1,...,n\}$. In this paper we show, among others, that, if $0<m\leq A_j\leq M$, $p_j\geq 0$ for $j\in\{1,...,n\}$ with $\sum_{j=1}^n p_j=1$, then

$$\begin{aligned}
1&\leq\frac{\left(\sum_{j=1}^n p_j\operatorname{tr}\left(P_jA_j\right)\right)^{-\left(\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)\right)}}{\prod_{j=1}^n\left[\eta_{P_j}\left(A_j\right)\right]^{p_j}}\\
&\leq\left(\frac{M}{m}\right)^{\frac{1}{M-m}\left(M-\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)\right)\left(\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)-m\right)}\\
&\leq\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.
\end{aligned}$$

2. Main Results

We use the following result that was obtained in [1]:

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then*

$$\begin{aligned}
0&\leq\frac{(b-t)f(a)+(t-a)f(b)}{b-a}-f(t)\\
&\leq(b-t)(t-a)\frac{f'_-(b)-f'_+(a)}{b-a}\leq\frac{1}{4}(b-a)\left[f'_-(b)-f'_+(a)\right]
\end{aligned} \tag{2.1}$$

for any $t\in[a,b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have the following reverse for the Jensen's trace inequality:

Lemma 2. *Assume that f is differentiable convex on the interior $\mathring{I}$ of an interval. Let $Q_j\geq 0$ with $Q_j\in\mathcal{B}_1(H)$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^n\operatorname{tr}(Q_j)>0$, then for all*

B_j with the spectra $\operatorname{Sp}(B_j) \subseteq [m, M] \subset \mathring{I}$ *for* $j \in \{1, ..., n\}$, *we have*

$$\begin{aligned}
0 &\leq \frac{\sum_{j=1}^{n} \operatorname{tr}[Q_j f(B_j)]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) \\
&\leq \frac{f'_{-}(M) - f'_{+}(m)}{M - m} \\
&\times \left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} - m\right) \\
&\leq \frac{1}{4}(M - m)\left[f'_{-}(M) - f'_{+}(m)\right].
\end{aligned} \tag{2.2}$$

Proof. Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1$ and the convexity of f on $[m, M]$, we have

$$f(m(1 - T) + MT) \leq f(m)(1 - T) + f(M)T \tag{2.3}$$

in the operator order.

If we take in (2.3)

$$0 \leq T = \frac{B_j - m}{M - m} \leq 1,$$

then we get

$$\begin{aligned}
&f\left(m\left(1 - \frac{B_j - m}{M - m}\right) + M\frac{B_j - m}{M - m}\right) \\
&\leq f(m)\left(1 - \frac{B_j - m}{M - m}\right) + f(M)\frac{B_j - m}{M - m}.
\end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned}
&m\left(1 - \frac{B_j - m}{M - m}\right) + M\frac{B_j - m}{M - m} \\
&= \frac{m(M - B_j) + M(B_j - m)}{M - m} = B_j
\end{aligned}$$

and

$$\begin{aligned}
&f(m)\left(1 - \frac{B_j - m}{M - m}\right) + f(M)\frac{B_j - m}{M - m} \\
&= \frac{f(m)(M - B_j) + f(M)(B_j - m)}{M - m}
\end{aligned}$$

and by (2.4) we get the following inequality of interest

$$f(B_j) \leq \frac{f(m)(M - B_j) + f(M)(B_j - m)}{M - m} \tag{2.5}$$

for all $j \in \{1, ..., n\}$.

If we multiply (2.5) both sides with $Q_j^{1/2}$ we get

$$\begin{aligned}
&\sum_{j=1}^{n} Q_j^{1/2} f(B_j) Q_j^{1/2} \\
&\leq \sum_{j=1}^{n} Q_j^{1/2} \left[\frac{f(m)(M-B_j)+f(M)(B_j-m)}{M-m}\right] Q_j^{1/2} \\
&= \frac{f(m)\sum_{j=1}^{n} Q_j^{1/2}(M-B_j)Q_j^{1/2}+f(M)\sum_{j=1}^{n} Q_j^{1/2}(B_j-m)Q_j^{1/2}}{M-m} \\
&= \frac{1}{M-m}\left[f(m)\left(M\sum_{j=1}^{n} Q_j-\sum_{j=1}^{n} Q_j^{1/2}B_jQ_j^{1/2}\right)\right. \\
&\left.+f(M)\left(\sum_{j=1}^{n} Q_j^{1/2}B_jQ_j^{1/2}-m\sum_{j=1}^{n} Q_j\right)\right],
\end{aligned}$$

which implies, by taking the trace and using its properties, that

$$\begin{aligned}
&\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f(B_j)\right] \\
&\leq \frac{1}{M-m}\left[f(m)\left(M\sum_{j=1}^{n} \operatorname{tr}(Q_j)-\sum_{j=1}^{n} \operatorname{tr}(Q_jB_j)\right)\right. \\
&\left.+f(M)\left(\sum_{j=1}^{n} \operatorname{tr}(Q_jB_j)-m\sum_{j=1}^{n} \operatorname{tr}(Q_j)\right)\right],
\end{aligned}$$

which gives that

$$\begin{aligned}
&\frac{\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f(B_j)\right]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} \\
&\leq \frac{f(m)\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)+f(M)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}-m\right)}{M-m},
\end{aligned}$$

namely

$$\begin{aligned}
0 &\leq \frac{\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f(B_j)\right]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}-f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) \\
&\leq \frac{f(m)\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)+f(M)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}-m\right)}{M-m} \\
&-f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right).
\end{aligned} \tag{2.6}$$

Here the first inequality is Jensen's inequality.

Using the inequality (2.1) for

$$t=\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} \in [m, M],$$

$a=m$ and $b=M$ we have

$$\begin{aligned}(2.7)\qquad & \frac{f(m)\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)+f(M)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}-m\right)}{M-m}\\ & -f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)\\ & \le\frac{f'_-(M)-f'_+(m)}{M-m}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}-m\right)\\ & \le\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right].\end{aligned}$$

By making use of (2.6) and (2.7) we derive (2.2). □

Theorem 4. *Assume that $P_j\in\mathcal{B}_1(H)$ with $P_j\ge 0$ and $\operatorname{tr}(P_j)=1$ for $j\in\{1,...,n\}$. If $0<m\le A_j\le M$, $p_j\ge 0$ for $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$, then*

$$\begin{aligned}(2.8)\qquad 1 & \le\frac{\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)^{-\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)}}{\prod_{j=1}^{n}\left[\eta_{P_j}(A_j)\right]^{p_j}}\\ & \le\left(\frac{M}{m}\right)^{\frac{1}{M-m}\left(M-\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)-m\right)}\\ & \le\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.\end{aligned}$$

Proof. If we write the inequality (2.2) for the convex function $f(t)=t\ln t$, $t>0$, $Q_j=p_jP_j$ and $B_j=A_j$, $j\in\{1,...,n\}$, then we get

$$\begin{aligned}(2.9)\qquad 0 & \le\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j\ln A_j)-\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\ln\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\\ & \le\ln\left(\frac{M}{m}\right)^{\frac{1}{M-m}\left(M-\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)-m\right)}\\ & \le\ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.\end{aligned}$$

By taking the exponential in (2.9), we get

(2.10)

$$\begin{aligned} 1 & \le\exp\left[\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j\ln A_j)-\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\ln\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\right]\\ & \le\left(\frac{M}{m}\right)^{\frac{1}{M-m}\left(M-\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)\right)\left(\sum_{j=1}^{n}p_j\operatorname{tr}(P_jA_j)-m\right)}\le\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.\end{aligned}$$

Observe that

$$\begin{aligned}
&\exp\left[\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j \ln A_j\right) - \left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right) \ln\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)\right] \\
&= \frac{\exp\left[-\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right) \ln\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)\right]}{\exp \sum_{j=1}^{n} p_j \left[-\operatorname{tr}\left(P_j A_j \ln A_j\right)\right]} \\
&= \frac{\exp\left[\ln\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)^{-\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)}\right]}{\exp \sum_{j=1}^{n} p_j \left[-\operatorname{tr}\left(P_j A_j \ln A_j\right)\right]} \\
&= \frac{\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)^{-\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)}}{\prod_{j=1}^{n} \left(\exp\left[-\operatorname{tr}\left(P_j A_j \ln A_j\right)\right]\right)^{p_j}} \\
&= \frac{\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)^{-\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right)}}{\prod_{j=1}^{n} \left[\eta_{P_j}\left(A_j\right)\right]^{p_j}}
\end{aligned}$$

and by (2.10) we deduce (2.8). □

Remark 1. *The case of one operator is as follows:*

$$\begin{aligned}
1 &\le \frac{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}}{\eta_P(A)} \le \left(\frac{M}{m}\right)^{\frac{1}{M-m}(M-\operatorname{tr}(PA))(\operatorname{tr}(PA)-m)} \\
&\le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)},
\end{aligned} \tag{2.11}$$

provided that $P \in \mathcal{B}_1(H)$ *with* $P \ge 0$ *and* $\operatorname{tr}(P) = 1$ *while* $0 < m \le A \le M$.

If $0 < m \le A,\ B \le M$ *and* $t \in [0,1]$, *then for* $P \in \mathcal{B}_1(H)$ *with* $P \ge 0$ *and* $\operatorname{tr}(P) = 1$,

$$\begin{aligned}
1 &\le \frac{\left(\operatorname{tr}\left(P\left[(1-t)A + tB\right]\right)\right)^{-\operatorname{tr}(P[(1-t)A+tB])}}{\left[\eta_P(A)\right]^{(1-t)}\left[\eta_P(B)\right]^{t}} \\
&\le \left(\frac{M}{m}\right)^{\frac{1}{M-m}(M-\operatorname{tr}(P[(1-t)A+tB]))(\operatorname{tr}(P[(1-t)A+tB])-m)} \\
&\le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.
\end{aligned} \tag{2.12}$$

We also have [1]:

Lemma 3. *Assume that* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$. *If* f' *is* K*-Lipschitzian on* $[a,b]$, *then*

$$\begin{aligned}
\left|(1-t)f(a) + tf(b) - f((1-t)a + tb)\right| &\le \frac{1}{2}K(b-t)(t-a) \\
&\le \frac{1}{8}K(b-a)^2
\end{aligned} \tag{2.13}$$

for all $t\in[0,1]$.

The constants $1/2$ *and* $1/8$ *are the best possible in (2.13).*

Remark 2. *If* $f:[a,b]\to\mathbb{R}$ *is twice differentiable and* $f''\in L_\infty[a,b]$, *then*

$$\begin{aligned}(2.14)\quad |(1-t)f(a)+tf(b)-f((1-t)a+tb)| &\le \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a)\\ &\le \frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2,\end{aligned}$$

where $\|f''\|_{[a,b],\infty}:=\operatorname{essup}_{t\in[a,b]}|f''(t)|<\infty$. *The constants* $1/2$ *and* $1/8$ *are the best possible in (2.14).*

Lemma 4. *Assume that* f *is twice differentiable convex on the interior* $\mathring{I}$ *of the interval* I *and the derivative* f'' *is bounded on* $\mathring{I}$. *Let* $Q_j\ge 0$ *with* $Q_j\in\mathcal{B}_1(H)$ *for* $j\in\{1,...,n\}$ *and* $\sum_{j=1}^n\operatorname{tr}(Q_j)>0$, *then for all* B_j *with the spectra* $\operatorname{Sp}(B_j)\subseteq[m,M]\subset\mathring{I}$ *for* $j\in\{1,...,n\}$, *we have*

$$\begin{aligned}(2.15)\quad 0&\le\frac{\sum_{j=1}^n\operatorname{tr}[Q_jf(B_j)]}{\sum_{j=1}^n\operatorname{tr}(Q_j)}-f\left(\frac{\sum_{j=1}^n\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\right)\\ &\le\frac{1}{2}\|f''\|_{[m,M],\infty}\left(M-\frac{\sum_{j=1}^n\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\right)\left(\frac{\sum_{j=1}^n\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^n\operatorname{tr}(Q_j)}-m\right)\\ &\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2.\end{aligned}$$

Proof. From (2.14) and the continuous functional calculus, we get

$$\begin{aligned}(2.16)\quad 0&\le\frac{f(m)(M-B_j)+f(M)(B_j-m)}{M-m}-f(B_j)\\ &\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M-B_j)(B_j-m)\\ &\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2\end{aligned}$$

where B_j are selfadjoint operators with the spectra $\operatorname{Sp}(B_j)\subset[m,M]$, $j\in\{1,...,n\}$.

Now, by employing a similar argument to the one in the proof of Lemma 2 we derive the desired result (2.15). □

We also have:

Theorem 5. *Assume that* $P_j\in\mathcal{B}_1(H)$ *with* $P_j\ge 0$ *and* $\operatorname{tr}(P_j)=1$ *for* $j\in\{1,...,n\}$. *If* $0<m\le A_j\le M$, $p_j\ge 0$ *for* $j\in\{1,...,n\}$ *with* $\sum_{j=1}^n p_j=1$, *then*

$$\begin{aligned}(2.17)\quad 1&\le\frac{\left(\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)\right)^{-\left(\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)\right)}}{\prod_{j=1}^n\left[\eta_{P_j}(A_j)\right]^{p_j}}\\ &\le\exp\left[\frac{1}{2m}\left(M-\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)\right)\left(\sum_{j=1}^n p_j\operatorname{tr}(P_jA_j)-m\right)\right]\\ &\le\exp\left[\frac{1}{8m}(M-m)^2\right].\end{aligned}$$

Proof. If we write the inequality (2.2) for the convex function $f(t) = t \ln t$, $t > 0$, $Q_j = p_j P_j$ and $B_j = A_j$, $j \in \{1, ..., n\}$, then we get

$$
\begin{aligned}
0 &\leq \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j \ln A_j) - \left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) \right) \ln \left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) \right) \\
&\leq \frac{1}{2m} \left(M - \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) \right) \left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m \right) \\
&\leq \frac{1}{8m} (M - m)^2 .
\end{aligned}
$$

If we take the exponential, then we get the desired result (2.17). $\square$

Remark 3. *The case of one operator is as follows:*

$$
\begin{aligned}
1 &\leq \frac{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}}{\eta_P(A)} \leq \exp\left[\frac{1}{2m} (M - \operatorname{tr}(PA)) (\operatorname{tr}(PA) - m) \right] \\
&\leq \exp\left[\frac{1}{8m} (M - m)^2 \right],
\end{aligned} \tag{2.18}
$$

provided that $P \in \mathcal{B}_1(H)$ with $P \geq 0$ and $\operatorname{tr}(P) = 1$ while $0 < m \leq A \leq M$.

If $0 < m \leq A,\ B \leq M$ and $t \in [0,1]$, then for $P \in \mathcal{B}_1(H)$ with $P \geq 0$ and $\operatorname{tr}(P) = 1$,

$$
\begin{aligned}
1 &\leq \frac{(\operatorname{tr}(P[(1-t)A + tB]))^{-\operatorname{tr}(P[(1-t)A+tB])}}{[\eta_P(A)]^{(1-t)} [\eta_P(B)]^t} \\
&\leq \exp\left[\frac{1}{2m} (M - \operatorname{tr}(P[(1-t)A + tB])) (\operatorname{tr}(P[(1-t)A + tB]) - m) \right] \\
&\leq \exp\left[\frac{1}{8m} (M - m)^2 \right].
\end{aligned} \tag{2.19}
$$

3. Related Results

We also have the following scalar inequality of interest:

Lemma 5. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and $t \in [0,1]$, then*

$$
\begin{aligned}
&2 \min\{t, 1-t\} \left[\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2} \right) \right] \\
&\leq (1-t) f(a) + t f(b) - f((1-t)a + tb) \\
&\leq 2 \max\{t, 1-t\} \left[\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2} \right) \right].
\end{aligned} \tag{3.1}
$$

The proof follows, for instance, by Corollary 1 from [2] for $n = 2$, $p_1 = 1 - t$, $p_2 = t$, $t \in [0,1]$ and $x_1 = a$, $x_2 = b$.

Lemma 6. *Assume that f is convex on the interior $\mathring{I}$ of an interval I. Let $Q_j \geq 0$ with $Q_j \in \mathcal{B}_1(H)$ for $j \in \{1, ..., n\}$ and $\sum_{j=1}^{n} \operatorname{tr}(Q_j) > 0$, then for all B_j with the* *spectra* $\operatorname{Sp}(B_j) \subseteq [m, M] \subset \mathring{I}$ *for* $j \in \{1, ..., n\}$, *we have*

$$0 \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \tag{3.2}$$
$$\times\left(\frac{1}{2}(M-m) - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j \left|B_j - \frac{1}{2}(m+M)\right|\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right)$$
$$\leq \frac{f(m)\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) + f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} - m\right)}{M-m}$$
$$- \frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j f(B_j))}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m) + \frac{1}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} \sum_{j=1}^{n} \operatorname{tr}\left(Q_j \left|B_j - \frac{1}{2}(m+M)\right|\right)\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].$$

Proof. We have from (3.1) that

$$0 \leq 2\left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \tag{3.3}$$
$$\leq (1-t) f(m) + t f(M) - f((1-t) m + tM)$$
$$\leq 2\left(\frac{1}{2} + \left|t - \frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right],$$

for all $t \in [0,1]$.

Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1$ we get from (3.3) that

$$0 \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2} - \left|T - \frac{1}{2}\right|\right) \tag{3.4}$$
$$\leq (1-T) f(m) + T f(M) - f((1-T) m + TM)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2} + \left|T - \frac{1}{2}\right|\right),$$

in the operator order.

If we take in (3.4)

$$0 \leq T = \frac{B_j - m}{M-m} \leq 1,$$

then we get

$$
\begin{aligned}
(3.5)\qquad 0 &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)-\left|B_j-\frac{1}{2}(m+M)\right|\right) \\
&\leq \frac{f(m)(M-B_j)+f(M)(B_j-m)}{M-m}-f(B_j) \\
&\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)+\left|B_j-\frac{1}{2}(m+M)\right|\right).
\end{aligned}
$$

If we multiply both sides by $Q_j^{1/2}$ we derive

$$
\begin{aligned}
0 &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)Q_j-Q_j^{1/2}\left|B_j-\frac{1}{2}(m+M)\right|Q_j^{1/2}\right) \\
&\leq \frac{f(m)\left(M-Q_j^{1/2}B_jQ_j^{1/2}\right)+f(M)\left(Q_j^{1/2}B_jQ_j^{1/2}-m\right)}{M-m} \\
&\quad-Q_j^{1/2}f(B_j)Q_j^{1/2} \\
&\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)Q_j+Q_j^{1/2}\left|B_j-\frac{1}{2}(m+M)\right|Q_j^{1/2}\right).
\end{aligned}
$$

Now, by taking the trace and summing over j from 1 to n, we derive

$$
\begin{aligned}
0 &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)\sum_{j=1}^{n}\operatorname{tr}(Q_j)-\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\left|B_j-\frac{1}{2}(m+M)\right|\right)\right) \\
&\leq \frac{1}{M-m}\left[f(m)\left(M\sum_{j=1}^{n}\operatorname{tr}(Q_j)-\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)\right)\right. \\
&\quad\left.+f(M)\left(\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)-m\sum_{j=1}^{n}\operatorname{tr}(Q_j)\right)\right]-\sum_{j=1}^{n}\operatorname{tr}(Q_jf(B_j)) \\
&\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad\times\left(\frac{1}{2}(M-m)\sum_{j=1}^{n}\operatorname{tr}(Q_j)+\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\left|B_j-\frac{1}{2}(m+M)\right|\right)\right).
\end{aligned}
$$

This proves (3.2). □

Theorem 6. *Assume that* $P_j \in \mathcal{B}_1(H)$ *with* $P_j \geq 0$ *and* $\operatorname{tr}(P_j) = 1$ *for* $j \in \{1, ..., n\}$. *If* $0 < m \leq A_j \leq M$, $p_j \geq 0$ *for* $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n p_j = 1$, *then*

$$1 \leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m) - \sum_{j=1}^n p_j \operatorname{tr}\left(P_j\left|A_j - \frac{1}{2}(m+M)\right|\right)\right)} \tag{3.6}$$
$$\leq \frac{\prod_{j=1}^n \left[\eta_{P_j}(A_j)\right]^{p_j}}{m^{-m\left(\frac{M - \sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)}{M-m}\right)} M^{-M\left(\frac{\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j) - m}{M-m}\right)}}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m) + \sum_{j=1}^n p_j \operatorname{tr}\left(P_j\left|A_j - \frac{1}{2}(m+M)\right|\right)\right)}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^2.$$

Proof. If we write the inequality (3.2) for the convex function $f(t) = t \ln t$, $t > 0$, $Q_j = p_j P_j$ and $B_j = A_j$, $j \in \{1, ..., n\}$, then we get

$$0 \leq \ln \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}}$$
$$\times \left(\frac{1}{2}(M-m) - \sum_{j=1}^n p_j \operatorname{tr}\left(P_j \left|A_j - \frac{1}{2}(m+M)\right|\right) \right)$$
$$\leq \frac{\ln m^{m\left(M - \sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)\right)} + \ln M^{M\left(\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j) - m\right)}}{M-m}$$
$$- \sum_{j=1}^n p_j \operatorname{tr}(P_j A_j \ln A_j)$$
$$\leq \ln \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}}$$
$$\times \left(\frac{1}{2}(M-m) + \sum_{j=1}^n p_j \operatorname{tr}\left(P_j \left|A_j - \frac{1}{2}(m+M)\right|\right) \right)$$
$$\leq \ln \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^2,$$

namely

$$\begin{aligned}
0 &\le \ln \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)-\sum_{j=1}^n p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M)\right|\right)\right)} \\
&\le \ln \left[m^{m\left(\frac{M-\sum_{j=1}^n p_j \operatorname{tr}\left(P_j A_j\right)}{M-m}\right)} M^{M\left(\frac{\sum_{j=1}^n p_j \operatorname{tr}\left(P_j A_j\right)-m}{M-m}\right)} \right] \\
&- \sum_{j=1}^n p_j \operatorname{tr}\left(P_j A_j \ln A_j\right) \\
&\le \ln \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\sum_{j=1}^n p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M)\right|\right)\right)} \\
&\le \ln \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^2 .
\end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned}
1 &\le \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)-\sum_{j=1}^n p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M)\right|\right)\right)} \\
&\le \frac{m^{m\left(\frac{M-\sum_{j=1}^n p_j \operatorname{tr}\left(P_j A_j\right)}{M-m}\right)} M^{M\left(\frac{\sum_{j=1}^n p_j \operatorname{tr}\left(P_j A_j\right)-m}{M-m}\right)}}{\exp\left(\sum_{j=1}^n p_j \operatorname{tr}\left(P_j A_j \ln A_j\right)\right)} \\
&\le \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\sum_{j=1}^n p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M)\right|\right)\right)} \\
&\le \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^2 ,
\end{aligned}$$

which is equivalent to the desired result (3.6). □

Remark 4. *The case of one operator is as follows:*

$$\begin{aligned}
1 &\le \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)-\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)\right|\right)\right)} \\
&\le \frac{\eta_P(A)}{m^{-m\left(\frac{M-\operatorname{tr}(PA)}{M-m}\right)} M^{-M\left(\frac{\operatorname{tr}(PA)-m}{M-m}\right)}} \\
&\le \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)\right|\right)\right)} \le \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}} \right)^2 ,
\end{aligned} \tag{3.7}$$

provided that $P \in \mathcal{B}_1(H)$ *with* $P \ge 0$ *and* $\operatorname{tr}(P) = 1$ *while* $0 < m \le A \le M$.

We also have:

Lemma 7. *With the assumptions of Lemma 6 we have*

$$0 \leq \frac{\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f\left(B_j\right)\right]}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right) \tag{3.8}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]$$
$$\times \left(\frac{1}{2}(M-m) + \left|\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)} - \frac{1}{2}(m+M)\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].$$

Proof. From (2.6) we derive

$$0 \leq \frac{\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f\left(B_j\right)\right]}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right) \tag{3.9}$$
$$\leq \frac{f(m)\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) + f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} - m\right)}{M-m}$$
$$- f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right).$$

From the second part of the scalar version of (3.5) we also have the scalar inequality

$$\frac{f(m)\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) + f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)} - m\right)}{M-m} \tag{3.10}$$
$$- f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right)$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]$$
$$\times \left(\frac{1}{2}(M-m) + \left|\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)} - \frac{1}{2}(m+M)\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].$$

By utilizing (3.9) and (3.10) we obtain the desired result (3.8). □

Finally, by the use of Lemma 7 we have

Theorem 7. *With the assumptions of Theorem 6,*

$$1 \leq \frac{\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)^{-\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)}}{\prod_{j=1}^{n} \left[\eta_{P_j}(A_j)\right]^{p_j}} \tag{3.11}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M)\right|\right)\right)}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{2}.$$

Remark 5. *If $P \in \mathcal{B}_1(H)$ with $P \geq 0$ and $\operatorname{tr}(P) = 1$ while $0 < m \leq A \leq M$, then*

$$1 \leq \frac{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}}{\eta_P(A)} \tag{3.12}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)\right|\right)\right)}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{2}.$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa