SEVERAL BOUNDS FOR THE ENTROPIC TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA JENSEN'S TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

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ABSTRACT. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic trace P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\left[-\operatorname{tr}(PA \ln A)\right].$$

In this paper we show among others that, if $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$, $p \in (-\infty, 0) \cup (1, \infty)$ and that A_j are operators such that $0 < m \leq A_j \leq M$, for $i \in \{1, ..., n\}$, then

$$1 \leq \exp\left(\gamma_p \left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^p\right]\right)$$
$$\leq \frac{\left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^{-\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)}}{\prod_{i=1}^n \left[\eta_{P_i}(A_i)\right]^{p_i}}$$
$$\leq \exp\left(\Gamma_p \left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^p\right]\right),$$

where

$$\gamma_p := \begin{cases} \frac{M^{1-p}}{p(p-1)} & \text{for } p \in (1, \infty), \\ \frac{m^{1-p}}{p(p-1)} & \text{for } p \in (-\infty, 0), \end{cases} \qquad \Gamma_p := \begin{cases} \frac{m^{1-p}}{p(p-1)} & \text{for } p \in (1, \infty), \\ \frac{M^{1-p}}{p(p-1)} & \text{for } p \in (-\infty, 0). \end{cases}$$

1. INTRODUCTION

In 1952, in the paper [3], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

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For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [6].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*, then* AT*,* $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}(P^{1/2}TP^{1/2}) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [1] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [2]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In [2], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA \ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A) P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp(-\operatorname{tr}\{P[tA\ln(tA)]\}) \\
&= \exp(-\operatorname{tr}\{P[tA(\ln t + \ln A)]\}) = \exp(-\operatorname{tr}\{P(tA\ln t + tA\ln A)\}) \\
&= \exp(-t\ln t\operatorname{tr}(PA))\exp(-t\operatorname{tr}(PA\ln A)) \\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)[\exp(-\operatorname{tr}(PA\ln A))]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\operatorname{tr}(PA)}[\eta_P(A)]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(I) = 1 \text{ and } \eta_P(tI) = t^{-t} \tag{1.14}$$

for $t > 0$.

2. The Case for $p \in (-\infty, 0) \cup (1, \infty)$

The following result is of interest in itself as well:

Lemma 1. *Assume that f is twice differentiable on the interior $\mathring{I}$ of the interval $I \subset (0,\infty)$ and the second derivative f'' is continuous on $\mathring{I}$ and for $p \in (-\infty,0)\cup(1,\infty)$ satisfies the condition*

$$\gamma \le \frac{t^{2-p}}{p(p-1)} f''(t) \le \Gamma \text{ for any } t \in \mathring{I}, \tag{2.1}$$

where $\gamma < \Gamma$ are constants. If $Q_i \ge 0$ with $Q_i \in \mathcal{B}_1(H)$ for $i \in \{1,...,n\}$ and $\sum_{i=1}^n \operatorname{tr}(Q_i) > 0$, then for all B_i with the spectra $\operatorname{Sp}(B_i) \subset \mathring{I}$ for $i \in \{1,...,n\}$ and $a \in \mathring{I}$,

$$\begin{aligned}
&\gamma\left[\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a^p - pa^{p-1}\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right)\right] \\
&\le \frac{\sum_{i=1}^n \operatorname{tr}[Q_i f(B_i)]}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - f(a) - f'(a)\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right) \\
&\le \Gamma\left[\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a^p - pa^{p-1}\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right)\right].
\end{aligned} \tag{2.2}$$

Proof. We use the Taylor's expansion for twice differentiable functions

$$f(x) = f(a) + (x-a) f'(a) + (x-a)^2 \int_0^1 f''(sa + (1-s)x)\, sds \tag{2.3}$$

that holds for all x, $a \in \mathring{I}$.

Since

$$\gamma p(p-1)t^{p-2} \le f''(t) \le p(p-1)\Gamma t^{p-2} \text{ for any } t \in \mathring{I},$$

hence

$$\begin{aligned}
p(p-1)\gamma \int_0^1 (sa+(1-s)x)^{p-2} sds &\le \int_0^1 f''(sa+(1-s)x)\, sds \\
&\le p(p-1)\Gamma \int_0^1 (sa+(1-s)x)^{p-2} sds,
\end{aligned}$$

which, by (2.3) gives that

$$\begin{aligned}
&p\left(p-1\right)\gamma\left(x-a\right)^{2}\int_{0}^{1}\left(sa+\left(1-s\right)x\right)^{p-2}sds \\
&\leq f\left(x\right)-f\left(a\right)-\left(x-a\right)f'\left(a\right) \\
&\leq p\left(p-1\right)\Gamma\left(x-a\right)^{2}\int_{0}^{1}\left(sa+\left(1-s\right)x\right)^{p-2}sds
\end{aligned} \tag{2.4}$$

for all $x,\ a\in\mathring{I}$.

Using integration by parts, we get

$$\begin{aligned}
&\int_{0}^{1}\left(sa+\left(1-s\right)x\right)^{p-2}sds \\
&=\frac{1}{\left(p-1\right)\left(a-x\right)}\int_{0}^{1}sd\left[\left(sa+\left(1-s\right)x\right)^{p-1}\right] \\
&=\frac{1}{\left(p-1\right)\left(a-x\right)}\left[a^{p-1}-\int_{0}^{1}\left(sa+\left(1-s\right)x\right)^{p-1}ds\right]
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{\left(p-1\right)\left(a-x\right)}\left[a^{p-1}-\frac{1}{p\left(a-x\right)}\int_{0}^{1}d\left(sa+\left(1-s\right)x\right)^{p}\right] \\
&=\frac{1}{\left(p-1\right)\left(a-x\right)}\left[a^{p-1}-\frac{1}{p\left(a-x\right)}\left(a^{p}-x^{p}\right)\right] \\
&=\frac{1}{p\left(p-1\right)\left(a-x\right)^{2}}\left[x^{p}-a^{p}-p\left(x-a\right)a^{p-1}\right]
\end{aligned}$$

and by (2.4) we get

$$\begin{aligned}
\gamma\left[x^{p}-a^{p}-p\left(x-a\right)a^{p-1}\right]&\leq f\left(x\right)-f\left(a\right)-\left(x-a\right)f'\left(a\right) \\
&\leq\Gamma\left[x^{p}-a^{p}-p\left(x-a\right)a^{p-1}\right]
\end{aligned} \tag{2.5}$$

for all $x,\ a\in\mathring{I}$.

Now, by using the continuous functional calculus for the selfadjoint operators , we get from (2.5) that

$$\begin{aligned}
\gamma\left[B_{i}^{p}-a^{p}I-pa^{p-1}\left(B_{i}-aI\right)\right]&\leq f\left(B_{i}\right)-f\left(a\right)I-f'\left(a\right)\left(B_{i}-aI\right) \\
&\leq\Gamma\left[B_{i}^{p}-a^{p}I-pa^{p-1}\left(B_{i}-aI\right)\right]
\end{aligned} \tag{2.6}$$

for B_i with the spectra $\operatorname{Sp}\left(B_i\right)\subset\mathring{I}$ for $i\in\{1,...,n\}$ and $a\in\mathring{I}$.

If we multiply both sides by $Q_i^{1/2}$ we get

$$\begin{aligned}
&\gamma\left[Q_{i}^{1/2}B_{i}^{p}Q_{i}^{1/2}-a^{p}Q_{i}-pa^{p-1}\left(Q_{i}^{1/2}B_{i}Q_{i}^{1/2}-aQ_{i}\right)\right] \\
&\leq Q_{i}^{1/2}f\left(B_{i}\right)Q_{i}^{1/2}-f\left(a\right)Q_{i}-f'\left(a\right)\left(Q_{i}^{1/2}B_{i}Q_{i}^{1/2}-aQ_{i}\right) \\
&\leq\Gamma\left[Q_{i}^{1/2}B_{i}^{p}Q_{i}^{1/2}-a^{p}Q_{i}-pa^{p-1}\left(Q_{i}^{1/2}B_{i}Q_{i}^{1/2}-aQ_{i}\right)\right]
\end{aligned}$$

for $i\in\{1,...,n\}$ and $a\in\mathring{I}$.

Now, if we take the trace and use its properties, we derive

$$\begin{aligned}&\gamma\left[\operatorname{tr}\left(Q_iB_i^p\right)-a^p\operatorname{tr}\left(Q_i\right)-pa^{p-1}\left(\operatorname{tr}\left(Q_iB_i\right)-a\operatorname{tr}\left(Q_i\right)\right)\right]\\&\le\operatorname{tr}\left[Q_if\left(B_i\right)\right]-f\left(a\right)\operatorname{tr}\left(Q_i\right)-f'\left(a\right)\left(\operatorname{tr}\left(Q_iB_i\right)-a\operatorname{tr}\left(Q_i\right)\right)\\&\le\Gamma\left[\operatorname{tr}\left(Q_iB_i^p\right)-a^p\operatorname{tr}\left(Q_i\right)-pa^{p-1}\left(\operatorname{tr}\left(Q_iB_i\right)-a\operatorname{tr}\left(Q_i\right)\right)\right]\end{aligned}$$

for $i\in\{1,...,n\}$ and $a\in\mathring{I}$.

If we sum over $i\in\{1,...,n\}$ and divide by $\sum_{i=1}^n\operatorname{tr}(Q_i)>0$, we get (2.2). □

Remark 1. *Assume that f is twice differentiable on the interior $\mathring{I}$ of the interval $I\subset(0,\infty)$ and the second derivative f'' is continuous on $\mathring{I}$ and satisfies the condition*

$$\varphi\le f''(t)\le\Phi \text{ for any } t\in\mathring{I}, \tag{2.7}$$

where $\varphi<\Phi$ are constants. If $Q_i\ge 0$ with $Q_i\in\mathcal{B}_1(H)$ for $i\in\{1,...,n\}$ and $\sum_{i=1}^n\operatorname{tr}(Q_i)>0$, then for all B_i with the spectra $\operatorname{Sp}(B_i)\subset\mathring{I}$ for $i\in\{1,...,n\}$ and $a\in\mathring{I}$,

$$\begin{aligned}&\frac{1}{2}\varphi\left[\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i^2\right)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}\right)^2\right.\\&\left.+\left(a-\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}\right)^2\right]\\&\le\frac{\sum_{i=1}^n\operatorname{tr}\left[Q_if(B_i)\right]}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-f(a)-f'(a)\left(\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-a\right)\\&\le\frac{1}{2}\Phi\left[\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i^2\right)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}\right)^2\right.\\&\left.+\left(a-\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}\right)^2\right].\end{aligned}\tag{2.8}$$

The proof follows by Lemma 1 for $p=2$ and $\gamma=\frac{1}{2}\varphi$, $\Gamma=\frac{1}{2}\Phi$.

If

$$\frac{\psi}{t^3}\le f''(t)\le\frac{\Psi}{t^3} \text{ for any } t\in\mathring{I}, \tag{2.9}$$

then

$$\begin{aligned}&\frac{1}{2}\psi\left[\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}+a^{-2}\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-2a^{-1}\right]\\&\le\frac{\sum_{i=1}^n\operatorname{tr}\left[Q_if(B_i)\right]}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-f(a)-f'(a)\left(\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-a\right)\\&\le\frac{1}{2}\Psi\left[\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}+a^{-2}\frac{\sum_{i=1}^n\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^n\operatorname{tr}(Q_i)}-2a^{-1}\right].\end{aligned}\tag{2.10}$$

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}(2.11)\quad &\gamma\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^p\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)^p\right]\\ &\le\frac{\sum_{i=1}^{n}\operatorname{tr}\left[Q_if\left(B_i\right)\right]}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)\\ &\le\Gamma\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^p\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)^p\right].\end{aligned}$$

In particular, if f satisfies the condition (2.7), then

$$\begin{aligned}(2.12)\quad &\frac{1}{2}\varphi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^2\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)^2\right]\\ &\le\frac{\sum_{i=1}^{n}\operatorname{tr}\left[Q_if\left(B_i\right)\right]}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)\\ &\le\frac{1}{2}\Phi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^2\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)^2\right].\end{aligned}$$

If f satisfies the condition (2.9), then

$$\begin{aligned}(2.13)\quad &\frac{1}{2}\psi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)^{-1}\right]\\ &\le\frac{\sum_{i=1}^{n}\operatorname{tr}\left[Q_if\left(B_i\right)\right]}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)\\ &\le\frac{1}{2}\Psi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right)^{-1}\right].\end{aligned}$$

We have the following main result:

Theorem 4. *If $P_i\ge 0$ with $P_i\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$, $p_i\ge 0$ with $\sum_{i=1}^{n}p_i=1$, $p\in(-\infty,0)\cup(1,\infty)$ and that A_j are operators such that $0<m\le A_j\le M$, for $i\in\{1,...,n\}$, then for all $a>0$ we have the lower and upper bounds*

$$\begin{aligned}(2.14)\quad 1&\le\exp\left(\gamma_p\left[\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^p\right)-a^p-pa^{p-1}\left(\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i\right)-a\right)\right]\right)\\ &\le\frac{a^{-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)}\exp\left(a-\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i\right)\right)}{\prod_{i=1}^{n}\left[\eta_{P_i}\left(A_i\right)\right]^{p_i}}\\ &\le\exp\left(\Gamma_p\left[\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^p\right)-a^p-pa^{p-1}\left(\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i\right)-a\right)\right]\right),\end{aligned}$$

where

$$\gamma_p:=\begin{cases}\frac{M^{1-p}}{p(p-1)} & \text{for } p\in(1,\infty),\\[2ex] \frac{m^{1-p}}{p(p-1)} & \text{for } p\in(-\infty,0)\end{cases}$$

and

$$\Gamma_p := \begin{cases} \frac{m^{1-p}}{p(p-1)} \text{ for } p \in (1,\infty), \\ \\ \frac{M^{1-p}}{p(p-1)} \text{ for } p \in (-\infty,0). \end{cases}$$

Proof. We consider the convex function $f(t) = t\ln t$, $t \in [m,M] \subset (0,\infty)$. Then

$$g(t) = \frac{t^{2-p}}{p(p-1)} f''(t) = \frac{t^{2-p}}{p(p-1)} \frac{1}{t} = \frac{t^{1-p}}{p(p-1)}.$$

For $p \in (1,\infty)$, we have

$$\sup_{t\in[m,M]} g(t) = \frac{m^{1-p}}{p(p-1)} \text{ and } \inf_{t\in[m,M]} g(t) = \frac{M^{1-p}}{p(p-1)}$$

and for $p \in (-\infty,0)$

$$\sup_{t\in[m,M]} g(t) = \sup_{t\in[m,M]} \frac{t^{1-p}}{p(p-1)} = \frac{M^{1-p}}{p(p-1)}$$

and

$$\inf_{t\in[m,M]} g(t) = \inf_{t\in[m,M]} \frac{t^{1-p}}{p(p-1)} = \frac{m^{1-p}}{p(p-1)}.$$

From (2.2) applied for $f(t) = t\ln t$, $t \in [m,M] \subset (0,\infty)$, we get for $Q_i = p_i P_i$ and $B_i = A_i$ that

$$\begin{aligned} 0 &\le \gamma_p \left[\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i^p) - a^p - pa^{p-1} \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i) - a \right) \right] \\ &\le \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i \ln A_i) - a\ln a - (\ln a + 1) \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i) - a \right) \\ &\le \Gamma_p \left[\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i^p) - a^p - pa^{p-1} \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i) - a \right) \right], \end{aligned}$$

namely

$$\begin{aligned} 0 &\le \gamma_p \left[\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i^p) - a^p - pa^{p-1} \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i) - a \right) \right] \\ &\le \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i \ln A_i) - a\ln a - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i) - a \right) \ln(ea) \\ &\le \Gamma_p \left[\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i^p) - a^p - pa^{p-1} \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i) - a \right) \right], \end{aligned} \tag{2.15}$$

for all $a > 0$.

If we take the exponential in (2.15), then we get

$$\exp\left(\gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right)$$
$$\leq\exp\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i\ln A_i)-\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\ln(ea)\right.$$
$$\left.-a\ln a\right.$$
$$\leq\exp\left(\Gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right),$$

namely

$$0\leq\exp\left(\gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right)\tag{2.16}$$
$$\leq\frac{\exp\left(\left(a-\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)\ln(ea)\right)}{a^a\exp\left(\sum_{i=1}^n p_i\operatorname{tr}(-P_iA_i\ln A_i)\right)}$$
$$\leq\exp\left(\Gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right).$$

Since

$$\exp\left(\left(a-\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)\ln(ea)\right)$$
$$=\exp\left(\ln(ea)^{\left(a-\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)}\right)=(ea)^{a-\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)}$$

and

$$\exp\left(\sum_{i=1}^n p_i\operatorname{tr}(-P_iA_i\ln A_i)\right)=\prod_{i=1}^n\exp\left(\operatorname{tr}(-P_iA_i\ln A_i)\right)^{p_i}$$
$$=\prod_{i=1}^n\left[\eta_{P_i}(A_i)\right]^{p_i},$$

hence by (2.16) we get the desired result (2.14). □

Corollary 2. *With the assumptions of Theorem 4,*

$$1\leq\exp\left(\gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)^p\right]\right)\tag{2.17}$$
$$\leq\frac{\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)^{-\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)}}{\prod_{i=1}^n\left[\eta_{P_i}(A_i)\right]^{p_i}}$$
$$\leq\exp\left(\Gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)^p\right]\right),$$

for $p\in(-\infty,0)\cup(1,\infty)$.

Remark 2. *With the assumptions of Theorem 4, we have for $p=2$ that*

$$1 \leq \exp\left(\frac{1}{2M}\left[\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^2\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^2\right]\right) \tag{2.18}$$
$$\leq \frac{\left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)}}{\prod_{i=1}^{n}\left[\eta_{P_i}\left(A_i\right)\right]^{p_i}}$$
$$\leq \exp\left(\frac{1}{2m}\left[\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^2\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^2\right]\right),$$

while for $p=-1$, that

$$1 \leq \exp\left(\frac{m^2}{2}\left[\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1}\right]\right) \tag{2.19}$$
$$\leq \frac{\left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)}}{\prod_{i=1}^{n}\left[\eta_{P_i}\left(A_i\right)\right]^{p_i}}$$
$$\leq \exp\left(\frac{M^2}{2}\left[\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1}\right]\right).$$

Corollary 3. *With the assumptions of Theorem 4, we also have*

$$1 \leq \exp\left(\tilde{\gamma}_p\left[\sum_{i=1}^{n} \frac{p_i \operatorname{tr}\left(P_i A_i^{2-p}\right)}{\operatorname{tr}\left(P_i A_i^2\right)} - \left(\sum_{i=1}^{n} \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}\right)^p\right]\right) \tag{2.20}$$
$$\leq \frac{\prod_{i=1}^{n}\left[\eta_{P_i}\left(A_i\right)\right]^{\frac{p_i}{\operatorname{tr}\left(P_i A_i^2\right)}}}{\left(\sum_{i=1}^{n} \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}\right)^{\sum_{i=1}^{n} \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}}}$$
$$\leq \exp\left(\tilde{\Gamma}_p\left[\sum_{i=1}^{n} \frac{p_i \operatorname{tr}\left(P_i A_i^{2-p}\right)}{\operatorname{tr}\left(P_i A_i^2\right)} - \left(\sum_{i=1}^{n} \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}\right)^p\right]\right),$$

where

$$\tilde{\gamma}_p := \begin{cases} \frac{m^{p-1}}{p(p-1)} \text{ for } p \in (1,\infty), \\ \\ \frac{M^{p-1}}{p(p-1)} \text{ for } p \in (-\infty,0) \end{cases}$$

and

$$\tilde{\Gamma}_p := \begin{cases} \frac{M^{p-1}}{p(p-1)} \text{ for } p \in (1,\infty), \\ \\ \frac{m^{p-1}}{p(p-1)} \text{ for } p \in (-\infty,0). \end{cases}$$

Proof. If $0 < m \leq A_i \leq M$, $i \in \{1, ..., n\}$ then $0 < M^{-1} \leq A_i^{-1} \leq m^{-1}$, $i \in \{1, ..., n\}$ and by writing (2.17) for $\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} > 0$ and A_i^{-1}, $i \in \{1, ..., n\}$ we get

(2.21)

$$1 \leq \exp\left(\tilde{\gamma}_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-p}\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)\right)^p\right]\right)$$
$$\leq \frac{\left(\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)\right)^{-\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)}}{\prod_{i=1}^n \left[\eta_{\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)}}\left(A_i^{-1}\right)\right]^{p_i}}$$
$$\leq \exp\left(\tilde{\Gamma}_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-p}\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)\right)^p\right]\right).$$

Observe that

$$\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-p}\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)\right)^p$$
$$= \sum_{i=1}^n p_i \frac{\operatorname{tr}\left(P_i A_i^{2-p}\right)}{\operatorname{tr}\left(P_i A_i^2\right)} - \left(\sum_{i=1}^n \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}\right)^p,$$

$$\left(\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)\right)^{-\sum_{i=1}^n p_i \operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1}\right)}$$
$$= \left(\sum_{i=1}^n \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}\right)^{-\sum_{i=1}^n \frac{p_i \operatorname{tr}\left(P_i A_i\right)}{\operatorname{tr}\left(P_i A_i^2\right)}}$$

and

$$\prod_{i=1}^n \left[\eta_{\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)}}\left(A_i^{-1}\right)\right]^{p_i} = \prod_{i=1}^n \left(\exp\left[-\operatorname{tr}\left(\frac{A_i P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} A_i^{-1} \ln A_i^{-1}\right)\right]\right)^{p_i}$$
$$= \prod_{i=1}^n \left(\exp\left[\operatorname{tr}\left(\frac{P_i A_i}{\operatorname{tr}\left(P_i A_i^2\right)} \ln A_i\right)\right]\right)^{p_i}$$
$$= \prod_{i=1}^n \left(\exp\left[-\operatorname{tr}\left(P_i A_i \ln A_i\right)\right]\right)^{\frac{-p_i}{\operatorname{tr}\left(P_i A_i^2\right)}}$$
$$= \prod_{i=1}^n \left[\eta_{P_i}\left(A_i\right)\right]^{\frac{-p_i}{\operatorname{tr}\left(P_i A_i^2\right)}}.$$

By making use of (2.21), we derive the desired result (2.20). □

Remark 3. *For $p=2$ in (2.20) we obtain*

$$1 \leq \exp\left(\frac{m}{2}\left[\sum_{i=1}^{n}\frac{p_i}{\operatorname{tr}(P_iA_i^2)}-\left(\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^2\right]\right) \tag{2.22}$$
$$\leq \frac{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{\frac{p_i}{\operatorname{tr}(P_iA_i^2)}}}{\left(\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^{\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}}}$$
$$\leq \exp\left(\frac{M}{2}\left[\sum_{i=1}^{n}\frac{p_i}{\operatorname{tr}(P_iA_i^2)}-\left(\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^2\right]\right),$$

while for $p=-1$ we get

$$1 \leq \exp\left(\frac{1}{2m^2}\left[\sum_{i=1}^{n}\frac{p_i\operatorname{tr}\left(P_iA_i^3\right)}{\operatorname{tr}(P_iA_i^2)}-\left(\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^{-1}\right]\right) \tag{2.23}$$
$$\leq \frac{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{\frac{p_i}{\operatorname{tr}(P_iA_i^2)}}}{\left(\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^{\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}}}$$
$$\leq \exp\left(\frac{1}{2M^2}\left[\sum_{i=1}^{n}\frac{p_i\operatorname{tr}\left(P_iA_i^3\right)}{\operatorname{tr}(P_iA_i^2)}-\left(\sum_{i=1}^{n}\frac{p_i\operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^{-1}\right]\right).$$

3. The Case for $p \in (0,1)$

We also have:

Lemma 2. *Assume that f is twice differentiable on the interior $\mathring{I}$ of the interval $I \subset (0,\infty)$ with the second derivative f'' is continuous on $\mathring{I}$ and for $p \in (0,1)$ satisfies the condition*

$$\delta \leq \frac{t^{2-p}}{p(1-p)}f''(t) \leq \Delta \text{ for any } t \in \mathring{I} \tag{3.1}$$

for some $\delta < \Delta$. If $Q_i \geq 0$ with $Q_i \in \mathcal{B}_1(H)$ for $i \in \{1,...,n\}$ and $\sum_{i=1}^{n}\operatorname{tr}(Q_i) > 0$, then for all B_i with the spectra $\operatorname{Sp}(B_i) \subset \mathring{I}$ for $i \in \{1,...,n\}$ and $a \in \mathring{I}$,

$$\delta\left[pa^{p-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-a\right)+a^p-\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i^p)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right] \tag{3.2}$$
$$\leq \frac{\sum_{i=1}^{n}\operatorname{tr}\left[Q_if(B_i)\right]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f(a)-f'(a)\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-a\right)$$
$$\leq \Delta\left[pa^{p-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-a\right)+a^p-\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i^p)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right].$$

In particular,

$$\delta\left[\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^p-\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i^p)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right] \le \frac{\sum_{i=1}^{n}\operatorname{tr}[Q_if(B_i)]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right) \le \Delta\left[\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^p-\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i^p)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right]. \tag{3.3}$$

Proof. As above, from (3.1) we derive

$$\gamma\left[p(x-a)a^{p-1}+a^p-x^p\right]\le f(x)-f(a)-(x-a)f'(a) \le \Delta\left[p(x-a)a^{p-1}+a^p-x^p\right]$$

for all $x,\ a\in\mathring{I}$.

By making use of a similar argument as in the proof of Lemma 1 we derive the desired result (3.2). □

Remark 4. *If*

$$\frac{\varphi}{t^{3/2}}\le f''(t)\le\frac{F}{t^{3/2}} \text{ for any } t\in\mathring{I}, \tag{3.4}$$

then

$$4\varphi\left[\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{1/2}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right] \le \frac{\sum_{i=1}^{n}\operatorname{tr}[Q_if(B_i)]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right) \le 4F\left[\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{1/2}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right]. \tag{3.5}$$

We also have the following bounds:

Theorem 5. *If $P_i\ge 0$ with $P_i\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$, $p_i\ge 0$ with $\sum_{i=1}^{n}p_i=1$, $p\in(0,1)$ and that A_j are operators such that $0<m\le A_j\le M$, for $i\in\{1,...,n\}$, then for all $a>0$ we have the lower and upper bounds*

$$1\le\exp\left(\frac{m^{1-p}}{p(1-p)}\left[pa^{p-1}\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-a\right)+a^p-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i^p)\right]\right) \le\frac{a^{-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)}\exp\left(a-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)}{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{p_i}} \le\exp\left(\frac{M^{1-p}}{p(1-p)}\left[pa^{p-1}\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-a\right)+a^p-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i^p)\right]\right). \tag{3.6}$$

Proof. If we take $f(t) = t\ln t$, then

$$\begin{aligned} h(t) &= \frac{t^{2-p}}{p(1-p)}\frac{1}{t} \\ &= \frac{t^{1-p}}{p(1-p)} \in \left[\frac{m^{1-p}}{p(1-p)}, \frac{M^{1-p}}{p(1-p)}\right]. \end{aligned}$$

From (3.2) applied for $f(t) = t\ln t$, $t \in [m, M] \subset (0,\infty)$, we get for $Q_i = p_i P_i$ and $B_i = A_i$

$$\begin{aligned} &\frac{m^{1-p}}{p(1-p)}\left[pa^{p-1}\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - a\right) + a^p - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i^p)\right] \\ &\leq \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i \ln A_i) - a\ln a - (\ln a + 1)\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - a\right) \\ &\leq \frac{M^{1-p}}{p(1-p)}\left[pa^{p-1}\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - a\right) + a^p - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i^p)\right], \end{aligned}$$

which produces the desired inequality (3.6). □

Corollary 4. *With the assumptions of Theorem 5, we have for all $p \in (0,1)$ that*

$$\begin{aligned} 1 &\leq \exp\left(\frac{m^{1-p}}{p(1-p)}\left[\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^p - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i^p)\right]\right) \\ &\leq \frac{\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^{-\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)}}{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{p_i}} \\ &\leq \exp\left(\frac{M^{1-p}}{p(1-p)}\left[\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^p - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i^p)\right]\right). \end{aligned} \tag{3.7}$$

Remark 5. *For $p = 1/2$ we get*

$$\begin{aligned} 1 &\leq \exp\left(4m^{1/2}\left[\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{1/2}\right)\right]\right) \\ &\leq \frac{\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^{-\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)}}{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{p_i}} \\ &\leq \exp\left(4M^{1/2}\left[\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{1/2}\right)\right]\right). \end{aligned} \tag{3.8}$$

We also have:

Corollary 5. *With the assumptions of Theorem 5, we have for all $p \in (0,1)$ that*

$$1 \le \exp\left(\frac{1}{p(1-p)M^{1-p}}\left[\left(\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^p - \sum_{i=1}^{n}\frac{p_i \operatorname{tr}\left(P_iA_i^{2-p}\right)}{\operatorname{tr}(P_iA_i^2)}\right]\right) \tag{3.9}$$
$$\le \frac{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{\frac{p_i}{\operatorname{tr}\left(P_iA_i^2\right)}}}{\left(\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}\left(P_iA_i^2\right)}\right)^{\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}\left(P_iA_i^2\right)}}}$$
$$\le \exp\left(\frac{1}{p(1-p)m^{1-p}}\left[\left(\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^p - \sum_{i=1}^{n}\frac{p_i \operatorname{tr}\left(P_iA_i^{2-p}\right)}{\operatorname{tr}(P_iA_i^2)}\right]\right).$$

For $p = 1/2$, we also have

$$1 \le \exp\left(\frac{4}{M^{1/2}}\left[\left(\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^{1/2} - \sum_{i=1}^{n}\frac{p_i \operatorname{tr}\left(P_iA_i^{3/2}\right)}{\operatorname{tr}(P_iA_i^2)}\right]\right) \tag{3.10}$$
$$\le \frac{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{\frac{p_i}{\operatorname{tr}\left(P_iA_i^2\right)}}}{\left(\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}\left(P_iA_i^2\right)}\right)^{\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}\left(P_iA_i^2\right)}}}$$
$$\le \exp\left(\frac{4}{m^{1/2}}\left[\left(\sum_{i=1}^{n}\frac{p_i \operatorname{tr}(P_iA_i)}{\operatorname{tr}(P_iA_i^2)}\right)^{1/2} - \sum_{i=1}^{n}\frac{p_i \operatorname{tr}\left(P_iA_i^{3/2}\right)}{\operatorname{tr}(P_iA_i^2)}\right]\right).$$

4. The Case of One Operator

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$ and $0 < m \le A \le M$ for some constants m, M. From (2.17) we then get

$$1 \le \exp\left(\gamma_p\left[\operatorname{tr}(PA^p) - (\operatorname{tr}(PA))^p\right]\right) \le \frac{(\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\eta_P(A)} \tag{4.1}$$
$$\le \exp\left(\Gamma_p\left[\operatorname{tr}(PA^p) - (\operatorname{tr}(PA))^p\right]\right),$$

for $p \in (-\infty, 0) \cup (1, \infty)$, where γ_p and Γ_p are defined in Theorem 4.

For $p = 2$ in (4.1) we get

$$1 \le \exp\left(\frac{1}{2M}\left[\operatorname{tr}\left(PA^2\right) - (\operatorname{tr}(PA))^2\right]\right) \le \frac{(\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\eta_P(A)} \tag{4.2}$$
$$\le \exp\left(\frac{1}{2m}\left[\operatorname{tr}\left(PA^2\right) - (\operatorname{tr}(PA))^2\right]\right),$$

while for $p=1$, we derive

$$1\le\exp\left(\frac{m^2}{2}\left[\operatorname{tr}\left(PA^{-1}\right)-\left(\operatorname{tr}\left(PA\right)\right)^{-1}\right]\right)\le\frac{\left(\operatorname{tr}\left(PA\right)\right)^{-\operatorname{tr}(PA)}}{\eta_P\left(A\right)}\le\exp\left(\frac{M^2}{2}\left[\operatorname{tr}\left(PA^{-1}\right)-\left(\operatorname{tr}\left(PA\right)\right)^{-1}\right]\right).\tag{4.3}$$

From (2.20) we obtain

$$\begin{aligned}1&\le\exp\left(\tilde{\gamma}_p\left[\frac{\operatorname{tr}\left(PA^{2-p}\right)}{\operatorname{tr}\left(PA^2\right)}-\left(\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(PA^2\right)}\right)^p\right]\right)\\&\le\frac{\left[\eta_P\left(A\right)\right]^{\frac{1}{\operatorname{tr}(PA^2)}}}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}}}\\&\le\exp\left(\tilde{\Gamma}_p\left[\frac{\operatorname{tr}\left(PA^{2-p}\right)}{\operatorname{tr}\left(PA^2\right)}-\left(\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(PA^2\right)}\right)^p\right]\right),\end{aligned}$$

and by taking the power $\operatorname{tr}\left(PA^2\right)>0$,

$$\begin{aligned}1&\le\exp\left(\tilde{\gamma}_p\left[\operatorname{tr}\left(PA^{2-p}\right)-\left[\operatorname{tr}\left(PA^2\right)\right]^{1-p}\left(\operatorname{tr}\left(PA\right)\right)^p\right]\right)\\&\le\frac{\eta_P\left(A\right)}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)^{\operatorname{tr}(PA)}}\\&\le\exp\left(\tilde{\Gamma}_p\left[\operatorname{tr}\left(PA^{2-p}\right)-\left[\operatorname{tr}\left(PA^2\right)\right]^{1-p}\left(\operatorname{tr}\left(PA\right)\right)^p\right]\right),\end{aligned}\tag{4.4}$$

where $\tilde{\gamma}_p$ and $\tilde{\Gamma}_p$ are defined in Corollary 3 for $p\in(-\infty,0)\cup(1,\infty)$.

Now, if we take $p=2$ in (4.4) we derive

$$1\le\exp\left(\frac{m}{2}\left[1-\frac{\left(\operatorname{tr}\left(PA\right)\right)^2}{\operatorname{tr}\left(PA^2\right)}\right]\right)\le\frac{\eta_P\left(A\right)}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)^{\operatorname{tr}(PA)}}\le\exp\left(\frac{M}{2}\left[1-\frac{\left(\operatorname{tr}\left(PA\right)\right)^2}{\operatorname{tr}\left(PA^2\right)}\right]\right),\tag{4.5}$$

while for $p=-1$,

$$\begin{aligned}1&\le\exp\left(\frac{1}{2m^2}\left[\operatorname{tr}\left(PA^3\right)-\frac{\left[\operatorname{tr}\left(PA^2\right)\right]^2}{\operatorname{tr}\left(PA\right)}\right]\right)\\&\le\frac{\eta_P\left(A\right)}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)^{\operatorname{tr}(PA)}}\\&\le\exp\left(\frac{1}{2M^2}\left[\operatorname{tr}\left(PA^3\right)-\frac{\left[\operatorname{tr}\left(PA^2\right)\right]^2}{\operatorname{tr}\left(PA\right)}\right]\right).\end{aligned}\tag{4.6}$$

From (3.7) we get for $p\in(0,1)$ that

$$1\leq\exp\left(\frac{m^{1-p}}{p(1-p)}\left[(\operatorname{tr}(PA))^{p}-\operatorname{tr}(PA^{p})\right]\right)\leq\frac{(\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\eta_{P}(A)}\leq\exp\left(\frac{M^{1-p}}{p(1-p)}\left[(\operatorname{tr}(PA))^{p}-\operatorname{tr}(PA^{p})\right]\right)\tag{4.7}$$

and for $p=1/2$, that

$$1\leq\exp\left(4m^{1/2}\left[(\operatorname{tr}(PA))^{1/2}-\operatorname{tr}\left(PA^{1/2}\right)\right]\right)\leq\frac{(\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\eta_{P}(A)}\leq\exp\left(4M^{1/2}\left[(\operatorname{tr}(PA))^{1/2}-\operatorname{tr}\left(PA^{1/2}\right)\right]\right).\tag{4.8}$$

From (3.9) we get

$$\begin{aligned}1&\leq\exp\left(\frac{1}{p(1-p)M^{1-p}}\left[\left[\operatorname{tr}\left(PA^{2}\right)\right]^{1-p}(\operatorname{tr}(PA))^{p}-\operatorname{tr}\left(PA^{2-p}\right)\right]\right)\\&\leq\frac{\eta_{P}(A)}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^{2})}\right)^{\operatorname{tr}(PA)}}\\&\leq\exp\left(\frac{1}{p(1-p)m^{1-p}}\left[\left[\operatorname{tr}\left(PA^{2}\right)\right]^{1-p}(\operatorname{tr}(PA))^{p}-\operatorname{tr}\left(PA^{2-p}\right)\right]\right)\end{aligned}\tag{4.9}$$

for $p\in(0,1)$ and for $p=1/2$, that

$$\begin{aligned}1&\leq\exp\left(\frac{4}{M^{1/2}}\left[\left[\operatorname{tr}\left(PA^{2}\right)\right]^{1/2}(\operatorname{tr}(PA))^{1/2}-\operatorname{tr}\left(PA^{3/2}\right)\right]\right)\\&\leq\frac{\eta_{P}(A)}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^{2})}\right)^{\operatorname{tr}(PA)}}\\&\leq\exp\left(\frac{4}{m^{1/2}}\left[\left[\operatorname{tr}\left(PA^{2}\right)\right]^{1/2}(\operatorname{tr}(PA))^{1/2}-\operatorname{tr}\left(PA^{3/2}\right)\right]\right).\end{aligned}\tag{4.10}$$

5. Further Bounds

We also have some simpler upper bounds as follows:

Proposition 2. *If $P_i\geq0$ with $P_i\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$, $p_i\geq0$ with $\sum_{i=1}^{n}p_i=1$ and that A_j are operators such that $0<m\leq A_j\leq M$, for*

$i\in\{1,...,n\}$, *then*

$$\frac{\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)^{-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)}}{\prod_{i=1}^{n}\left[\eta_{P_i}(A_i)\right]^{p_i}} \tag{5.1}$$
$$\leq\exp\left(\frac{1}{2m}\left[\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^2\right)-\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)^2\right]\right),$$
$$\leq\exp\left[\frac{1}{2m}\left(M-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-m\right)\right]$$
$$\leq\exp\left[\frac{1}{8}m\left(\frac{M}{m}-1\right)^2\right].$$

Proof. We observe that

$$\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^2\right)-\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)^2 \tag{5.2}$$
$$=\left(M-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-m\right)$$
$$-\sum_{i=1}^{n}p_i\operatorname{tr}\left[P_i(MI-A_i)(A_i-mI)\right].$$

Since $(M-t)(m-t)\geq 0$ for all $t\in[m,M]$, then by the continuous functional calculus for selfadjoint operators we get that

$$(MI-A_i)(A_i-mI)\geq 0,\ i\in\{1,...,n\}.$$

If we multiply this inequality both sides by $P_i^{1/2}\geq 0$ we get

$$P_i^{1/2}(MI-A_i)(A_i-mI)P_i^{1/2}\geq 0,\ i\in\{1,...,n\},$$

and by taking the trace, we derive

$$\operatorname{tr}\left[P_i(MI-A_i)(A_i-mI)\right]\geq 0,\ i\in\{1,...,n\},$$

which implies that

$$\sum_{i=1}^{n}p_i\operatorname{tr}\left[P_i(MI-A_i)(A_i-mI)\right]\geq 0$$

and by (5.2) we obtain

$$\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^2\right)-\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)^2$$
$$\leq\left(M-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-m\right)$$
$$\leq\frac{1}{4}(M-m)^2.$$

By utilizing (2.18) we derive the desired result (5.1). □

We also have:

Proposition 3. *If* $P_i \geq 0$ *with* $P_i \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P_i) = 1$ *for* $i \in \{1, ..., n\}$, $p_i \geq 0$ *with* $\sum_{i=1}^n p_i = 1$ *and that* A_j *are operators such that* $0 < m \leq A_j \leq M$, *for* $i \in \{1, ..., n\}$, *then*

$$\frac{\left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^{-\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)}}{\prod_{i=1}^n [\eta_{P_i}(A_i)]^{p_i}} \tag{5.3}$$
$$\leq \exp\left(\frac{M^2}{2}\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^{-1}\right]\right)$$
$$\leq \exp\left(\frac{M}{2}\left(\sqrt{\frac{M}{m}} - 1\right)^2\right).$$

Proof. If $t \in [m, M] \subset (0, \infty)$, then $(M - t)\left(m^{-1} - t^{-1}\right) \geq 0$. Since $0 < mI \leq A_i \leq MI$, $i \in \{1, ..., n\}$ hence by using the functional calculus for selfadjoint operators we get

$$(M - A_i)\left(m^{-1} - A_i^{-1}\right) \geq 0$$

for all $i \in \{1, ..., n\}$, which is equivalent to

$$(M + m) \geq MmA_i^{-1} + A_i \tag{5.4}$$

for all $i \in \{1, ..., n\}$.

If we multiply (5.4) both sides by $P_i^{1/2}$ we get

$$(M + m) P_i \geq MmP_i^{1/2} A_i^{-1} P_i^{1/2} + P_i^{1/2} A_i P_i^{1/2}$$

for all $i \in \{1, ..., n\}$.

If we take the trace and use its properties, we get

$$M + m \geq Mm \operatorname{tr}\left(P_i A_i^{-1}\right) + \operatorname{tr}(P_i A_i)$$

for all $i \in \{1, ..., n\}$.

If we multiply by $p_i \geq 0$ and summing over i from 1 to n, we get

$$M + m \geq Mm \sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right) + \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i). \tag{5.5}$$

From (5.5) we get

$$\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \leq \frac{1}{m} + \frac{1}{M} - \frac{1}{mM} \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i),$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1} \\
&\leq \frac{1}{m} + \frac{1}{M} - \frac{1}{mM} \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1} \\
&= \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 \\
&- \left(\frac{1}{\sqrt{mM}} \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{1/2} - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1/2}\right)^2 \\
&\leq \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 .
\end{aligned}$$

By making use of (2.19) we derive (5.3). □

Remark 6. *If $0 < m \leq A \leq M$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then we have the one operator inequalities*

$$\begin{aligned}
\frac{(\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\eta_P(A)} &\leq \exp\left\{\frac{1}{2m}\left[\operatorname{tr}\left(PA^2\right) - [\operatorname{tr}(PA)]^2\right]\right\} \\
&\leq \exp\left[\frac{1}{2m}\left(M - [\operatorname{tr}(PA)]\right)\left([\operatorname{tr}(PA)] - m\right)\right] \\
&\leq \exp\left[\frac{1}{8} m \left(\frac{M}{m} - 1\right)^2\right]
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
\frac{(\operatorname{tr}(PA))^{-\operatorname{tr}(PA)}}{\eta_P(A)} &\leq \exp\left\{\frac{M^2}{2}\left[\operatorname{tr}\left(PA^{-1}\right) - [\operatorname{tr}(PA)]^{-1}\right]\right\} \\
&\leq \exp\left[\frac{1}{2} M \left(\sqrt{\frac{M}{m}} - 1\right)^2\right].
\end{aligned} \tag{5.7}$$

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