FUNCTIONAL PROPERTIES FOR THE ENTROPIC TRACE CLASS P-DETERMINANT OF SEQUENCES OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic trace P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp[-\operatorname{tr}(PA \ln A)].$$

We define the entropic determinant functional

$$E_n(\mathbf{q}; \mathbf{A}, Q) . = \frac{\left(\frac{1}{Q_n} \sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{-\sum_{j=1}^n q_j \operatorname{tr}(QA_j)}}{\prod_{i=1}^n [\eta_Q(A_j)]^{q_i}}$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators, $\mathbf{q} \in \mathcal{P}_n^+$, the set of positive n-tuple and $Q \in \mathcal{B}_1(H)$, $Q > 0$ with $\operatorname{tr} Q = 1$.

In this paper we show among others that, if p, $\mathbf{q} \in \mathcal{P}_n^+$, then we have

$$E_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, Q) \geq E_n(\mathbf{p}; \mathbf{A}, Q) E_n(\mathbf{q}; \mathbf{A}, Q) \geq 1$$

i.e., the functional $E_n(\cdot, Q, \mathbf{A})$ is super-multiplicative on $\mathcal{P}_n^+$. For p, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$,

$$E_n(\mathbf{p}; Q, \mathbf{A}) \geq E_n(\mathbf{q}; Q, \mathbf{A}) \geq 1,$$

i.e., the functional $E_n(\cdot, Q, \mathbf{A})$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

1. Introduction

In 1952, in the paper [5], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

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If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [8].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \ \textit{and} \ \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \ | \ B \in \mathcal{B}_2(H), \ \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT,\ TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [3] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [4]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In [4], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA\ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(I)=1 \text{ and } \eta_P(tI)=t^{-t} \tag{1.14}$$

for $t > 0$.

2. Preliminary Facts

Consider a convex function f on the interval I. We define

$$\mathcal{B}_1^{++}\left(H\right):=\left\{Q\in\mathcal{B}_1\left(H\right)\mid Q>0\right\}$$

and consider the n-tuples

$$\mathbf{Q}:=\left(Q_1,...,Q_n\right)\in\left[\mathcal{B}_1^{++}\left(H\right)\right]^n:=\mathcal{B}_1^{++}\left(H\right)\times...\times\mathcal{B}_1^{++}\left(H\right)$$

and $\mathbf{A} := (A_1, ..., A_n)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \{1, ..., n\}$. We have the following Jensen type trace inequality for convex function f,

$$f\left(\frac{\sum_{j=1}^n\operatorname{tr}\left(Q_jA_j\right)}{\sum_{j=1}^n\operatorname{tr}\left(Q_j\right)}\right)\le\frac{\sum_{j=1}^n\operatorname{tr}\left[Q_jf\left(A_j\right)\right]}{\sum_{j=1}^n\operatorname{tr}\left(Q_j\right)}, \tag{2.1}$$

and can introduce the *Jensen's gap functional*

$$J_n\left(\mathbf{Q},\mathbf{A},f\right):=\sum_{j=1}^n\operatorname{tr}\left[Q_jf\left(A_j\right)\right]-\sum_{j=1}^n\operatorname{tr}\left(Q_j\right)f\left(\frac{\sum_{j=1}^n\operatorname{tr}\left(Q_jA_j\right)}{\sum_{j=1}^n\operatorname{tr}\left(Q_j\right)}\right).$$

We have the following functional properties:

Theorem 4. *Assume that f is convex on the interval I and $\mathbf{A} := (A_1, ..., A_n)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \{1, ..., n\}$.*

(i) *For all $\mathbf{P}$, $\mathbf{Q} \in \left[\mathcal{B}_1^{+}(H)\right]^n$ we have*

$$J_n\left(\mathbf{P}+\mathbf{Q},\mathbf{A},f\right)\ge J_n\left(\mathbf{P},\mathbf{A},f\right)+J_n\left(\mathbf{Q},\mathbf{A},f\right)\ge 0, \tag{2.2}$$

i.e., the functional $J_n(\cdot, \mathbf{A}, f)$ is superadditive on $\left[\mathcal{B}_1^{++}(H)\right]^n$;

(ii) *For all $\mathbf{P}$, $\mathbf{Q} \in \left[\mathcal{B}_1^{+}(H)\right]^n$ with $\mathbf{P} \ge \mathbf{Q}$, namely $P_j \ge Q_j$ for $j \in \{1, ..., n\}$,*

$$J_n\left(\mathbf{P},\mathbf{A},f\right)\ge J_n\left(\mathbf{Q},\mathbf{A},f\right)\ge 0, \tag{2.3}$$

i.e., the functional $J_n(\cdot, \mathbf{A}, f)$ is monotonic non-decreasing on $\left[\mathcal{B}_1^{++}(H)\right]^n$.

Proof. (i). If $\mathbf{P},\ \mathbf{Q}\in\left[\mathcal{B}_1^+(H)\right]^n$, then we have

$$\begin{aligned}
&J_n(\mathbf{P}+\mathbf{Q},\mathbf{A},f) \\
&=\sum_{j=1}^n \operatorname{tr}\left[(P_j+Q_j)\,f(A_j)\right] \\
&-\sum_{j=1}^n \operatorname{tr}(P_j+Q_j)\,f\left(\frac{\sum_{j=1}^n \operatorname{tr}((P_j+Q_j)\,A_j)}{\sum_{j=1}^n \operatorname{tr}(P_j+Q_j)}\right) \\
&=\sum_{j=1}^n \operatorname{tr}\left[P_j f(A_j)\right]+\sum_{j=1}^n \operatorname{tr}\left[Q_j f(A_j)\right] \\
&-\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right] f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)+\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right).
\end{aligned}\tag{2.4}$$

By the convexity of f we obtain

$$\begin{aligned}
&f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)+\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right) \\
&=f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_j)\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}+\sum_{j=1}^n \operatorname{tr}(Q_j)\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right) \\
&\le \frac{\sum_{j=1}^n \operatorname{tr}(P_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]} f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}\right) \\
&+\frac{\sum_{j=1}^n \operatorname{tr}(Q_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]} f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right] f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)+\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right) \\
&\ge -\frac{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]\sum_{j=1}^n \operatorname{tr}(P_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]} f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}\right) \\
&-\frac{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]\sum_{j=1}^n \operatorname{tr}(Q_j)}{\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]} f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right) \\
&=-\sum_{j=1}^n \operatorname{tr}(P_j)\, f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}\right)-\sum_{j=1}^n \operatorname{tr}(Q_j)\, f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right)
\end{aligned}$$

and by (2.4) we derive

$$\begin{aligned}
&J_n(\mathbf{P}+\mathbf{Q},\mathbf{A},f) \\
&\geq \sum_{j=1}^{n} \operatorname{tr}[P_j f(A_j)] + \sum_{j=1}^{n} \operatorname{tr}[Q_j f(A_j)] \\
&- \sum_{j=1}^{n} \operatorname{tr}(P_j) f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}{\sum_{j=1}^{n} \operatorname{tr}(P_j)}\right) - \sum_{j=1}^{n} \operatorname{tr}(Q_j)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j A_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) \\
&= \sum_{j=1}^{n} \operatorname{tr}[P_j f(A_j)] - \sum_{j=1}^{n} \operatorname{tr}(P_j) f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}{\sum_{j=1}^{n} \operatorname{tr}(P_j)}\right) \\
&+ \sum_{j=1}^{n} \operatorname{tr}[Q_j f(A_j)] - \sum_{j=1}^{n} \operatorname{tr}(Q_j)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(Q_j A_j)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j)}\right) \\
&= J_n(\mathbf{P},\mathbf{A},f) + J_n(\mathbf{Q},\mathbf{A},f) \geq 0
\end{aligned}$$

and the inequality (2.2) is proved.

(ii) If $\mathbf{P} \geq \mathbf{Q}$, then $\mathbf{P} = \mathbf{P} - \mathbf{Q} + \mathbf{Q}$ and if we use the property (2.2), then we get

$$J_n(\mathbf{P},\mathbf{A},f) = J_n(\mathbf{P}-\mathbf{Q}+\mathbf{Q},\mathbf{A},f) \geq J_n(\mathbf{P}-\mathbf{Q},\mathbf{A},f) + J_n(\mathbf{Q},\mathbf{A},f),$$

which gives

$$J_n(\mathbf{P},\mathbf{A},f) - J_n(\mathbf{Q},\mathbf{A},f) \geq J_n(\mathbf{P}-\mathbf{Q},\mathbf{A},f) \geq 0$$

and the inequality (2.3) is proved. □

Corollary 1. *With the assumptions of Theorem 4 and if we assume that there exists the positive constants $m < M$ such that*

$$m\mathbf{Q} \leq \mathbf{P} \leq M\mathbf{Q}, \tag{2.5}$$

then

$$mJ_n(\mathbf{Q},\mathbf{A},f) \leq J_n(\mathbf{P},\mathbf{A},f) \leq MJ_n(\mathbf{Q},\mathbf{A},f). \tag{2.6}$$

Proof. Observe that for $\alpha > 0$ we have $J_n(\alpha\mathbf{Q},\mathbf{A}) = \alpha J_n(\mathbf{Q},\mathbf{A})$. Utilizing the monotonicity property (2.3) we have

$$J_n(m\mathbf{Q},\mathbf{A},f) \leq J_n(\mathbf{P},\mathbf{A},f) \leq J_n(M\mathbf{Q},\mathbf{A},f),$$

which imply the desired result (2.6). □

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $q = (q_1, ..., q_n)$, $q_j \geq 0$ with $j \in \{1, ..., n\}$ and $Q_n := \sum_{j=1}^{n} q_j > 0$. For p, $\mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1, ..., n\}$.

For $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$, we define the functional

$$J_n(\mathbf{q}; Q,\mathbf{A},f) := \sum_{j=1}^{n} q_j \operatorname{tr}[Qf(A_j)] - Q_n f\left(\frac{1}{Q_n}\sum_{j=1}^{n} q_j \operatorname{tr}(QA_j)\right),$$

where $Q_n := \sum_{j=1}^{n} q_j > 0$.

We observe that if we put $Q_j \to q_j Q$, $j \in \{1, ..., n\}$ then $J_n(\mathbf{q}, Q,\mathbf{A},f) = J_n(\mathbf{Q},\mathbf{A},f)$ and we can state the following result:

Theorem 5. *Assume that f is convex on the interval I, $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{A} := (A_1, ..., A_n)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \{1, ..., n\}$.*

(i) *For all p, $\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q}; Q, \mathbf{A}, f) \geq J_n(\mathbf{p}; Q; \mathbf{A}, f) + J_n(\mathbf{p}; Q, \mathbf{A}, f) \geq 0, \tag{2.7}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A}, f)$ is superadditive on $\mathcal{P}_n^+$;

(ii) *For p, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$*

$$J_n(\mathbf{p}; Q, \mathbf{A}, f) \geq J_n(\mathbf{q}; Q, \mathbf{A}, f) \geq 0, \tag{2.8}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A}, f)$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

Remark 1. *We observe that if all $q_j > 0$ then we have the inequality*

$$\begin{aligned} \min_{j\in\{1,...,n\}} \left\{\frac{p_j}{q_j}\right\} J_n(\mathbf{q}; Q, \mathbf{A}, f) &\leq J_n(\mathbf{p}; Q, \mathbf{A}, f) \\ &\leq \max_{j\in\{1,...,n\}} \left\{\frac{p_j}{q_j}\right\} J_n(\mathbf{p}; Q, \mathbf{A}, f). \end{aligned} \tag{2.9}$$

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j = \frac{1}{n}, j \in \{1, ..., n\}$, then we have the inequalities

$$\begin{aligned} n \min_{j\in\{1,...,n\}} \{p_j\} J_n(Q, \mathbf{A}, f) &\leq J_n(\mathbf{p}; Q, \mathbf{A}, f) \\ &\leq n \max_{j\in\{1,...,n\}} \{p_j\} J_n(Q, \mathbf{A}, f), \end{aligned} \tag{2.10}$$

where

$$J_n(Q, \mathbf{A}, f) := \frac{1}{n} \sum_{j=1}^{n} \operatorname{tr}[Qf(A_j)] - f\left(\frac{1}{n} \sum_{j=1}^{n} \operatorname{tr}(QA_j)\right). \tag{2.11}$$

For $n = 2$ and by choosing $p_1 = \alpha$, $p_2 = 1 - \alpha$ with $\alpha \in [0, 1]$, we get from (2.10) the inequality

$$\begin{aligned} &2\min\{\alpha, 1-\alpha\} \\ &\times \left[\frac{\operatorname{tr}[Qf(A)] + \operatorname{tr}[Qf(B)]}{2} - f\left(\operatorname{tr}\left[Q\left(\frac{A+B}{2}\right)\right]\right)\right] \\ &\leq (1-\alpha)\operatorname{tr}[Qf(A)] + \alpha \operatorname{tr}[Qf(B)] - f(\operatorname{tr}(Q[(1-\alpha)A + \alpha B])) \\ &\leq 2\max\{\alpha, 1-\alpha\} \\ &\times \left[\frac{\operatorname{tr}[Qf(A)] + \operatorname{tr}[Qf(B)]}{2} - f\left(\operatorname{tr}\left[Q\left(\frac{A+B}{2}\right)\right]\right)\right], \end{aligned} \tag{2.12}$$

where $f : I \to \mathbb{R}$ is a convex function and A and B are two bounded selfadjoint operators on the complex Hilbert space H with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq I$.

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $\mathcal{A}(H)$ the linear space of all sequences of selfadjoint operators defined on the complex Hilbert space, i.e.,

$$\mathcal{A}(H) = \left\{\mathbf{A} = (A_k)_{k\in\mathbb{N}} \mid A_k \text{ are selfadjoint operators on } H \text{ for all } k \in \mathbb{N}\right\}$$

and $\mathcal{S}_+\left(\mathcal{B}_1^{++}(H)\right)$ the family of positive sequences from $\mathcal{B}_1(H)$.

Let $f : I \to \mathbb{R}$ is a convex function on the interval I. We consider the functional

$$J_K(\mathbf{Q}, \mathbf{A}, f) := \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in K} \operatorname{tr}(Q_j) f\left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)}\right),$$

where $K \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{Q} \in \mathcal{S}_+\left(\mathcal{B}_1^{++}(H)\right)$, $\mathbf{A} \in \mathcal{A}(H)$ and $\operatorname{Sp}(A_j) \subseteq I$, $j \in \mathbb{N}$.

Theorem 6. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I and $\mathbf{Q} \in \mathcal{S}_+\left(\mathcal{B}_1^{++}(H)\right)$, $\mathbf{A} \in \mathcal{A}(H)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \mathbb{N}$.*

(i) *If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$, then we have the inequality*

$$J_{K\cup L}(\mathbf{Q}, \mathbf{A}, f) \geq J_K(\mathbf{Q}, \mathbf{A}, f) + J_L(\mathbf{Q}, \mathbf{A}, f) \geq 0, \tag{2.13}$$

i.e., $J_{\cdot}(\mathbf{Q}, \mathbf{A}, f)$ is super-additive as an index set functional.

(ii) *If $\emptyset \neq K \subset L$, then we have*

$$J_L(\mathbf{Q}, \mathbf{A}, f) \geq J_K(\mathbf{Q}, \mathbf{A}, f) \geq 0, \tag{2.14}$$

i.e., $J_{\cdot}(\mathbf{Q}, \mathbf{A}, f)$ is monotonic non-decreasing as an index set functional.

Proof. (i). If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$, then we have

$$\begin{aligned}
&J_{K\cup L}(\mathbf{Q}, \mathbf{A}, f) \\
&= \sum_{j\in K\cup L} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in K\cup L} \operatorname{tr}(Q_j) f\left(\frac{\sum_{j\in K\cup L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)}\right) \\
&= \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] + \sum_{j\in L} \operatorname{tr}[Q_j f(A_j)] \\
&- \sum_{j\in K\cup L} \operatorname{tr}(Q_j) \\
&\times f\left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j) \frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)} + \sum_{j\in L} \operatorname{tr}(Q_j) \frac{\sum_{j\in L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in L} \operatorname{tr}(Q_j)}}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)}\right) \\
&\geq \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] + \sum_{j\in L} \operatorname{tr}[Q_j f(A_j)] \\
&- \sum_{j\in K\cup L} \operatorname{tr}(Q_j) \left[\frac{\sum_{j\in K} \operatorname{tr}(Q_j)}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)} f\left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)}\right)\right. \\
&+ \left.\frac{\sum_{j\in L} \operatorname{tr}(Q_j)}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)} f\left(\frac{\sum_{j\in L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in L} \operatorname{tr}(Q_j)}\right)\right] \\
&= \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in K} \operatorname{tr}(Q_j) \left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)}\right) \\
&+ \sum_{j\in L} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in L} \operatorname{tr}(Q_j) f\left(\frac{\sum_{j\in L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in L} \operatorname{tr}(Q_j)}\right) \\
&= J_K(\mathbf{Q}, \mathbf{A}, f) + J_L(\mathbf{Q}, \mathbf{A}, f) \geq 0,
\end{aligned} \tag{2.15}$$

which proves (2.13).

(ii). If $\emptyset \neq K \subset L$ with $L \diagdown K \neq \emptyset$, then we have by (2.13) that

$$\begin{aligned} J_L(\mathbf{Q}, \mathbf{A}, f) &= J_{K \cup (L \diagdown K)}(\mathbf{Q}, \mathbf{A}, f) \\ &\geq J_L(\mathbf{Q}, \mathbf{A}, f) + J_{L \diagdown K}(\mathbf{Q}, \mathbf{A}, f) \geq J_K(\mathbf{Q}, \mathbf{A}, f) \end{aligned}$$

and the inequality (2.14) is thus proved. □

Corollary 2. *Assume that f is convex on the interval I and $\mathbf{A} := (A_1, ..., A_n)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \{1, ..., n\}$. Then we have the inequality*

$$J_k(\mathbf{Q}, \mathbf{A}, f) \geq J_{k-1}(\mathbf{Q}, \mathbf{A}, f) \geq 0 \tag{2.16}$$

for any $k \in \{1, ..., n\}$ with $n \geq k \geq 2$.

We also have that

$$\begin{aligned} J_n(\mathbf{Q}, \mathbf{A}, f) \geq \max_{j,k \in \{1,...,n\}} &\Big\{ \operatorname{tr}[Q_j f(A_j)] + \operatorname{tr}[Q_k f(A_k)] \\ &- \operatorname{tr}(Q_j + Q_k) f\left(\frac{\operatorname{tr}(Q_j A_j) + \operatorname{tr}(Q_k A_k)}{\operatorname{tr}(Q_j + Q_k)}\right)\Big\} \\ &\geq 0. \end{aligned} \tag{2.17}$$

Now, consider the weighted functional

$$J_K(\mathbf{q}; Q, \mathbf{A}, f) := \sum_{j \in K} q_j \operatorname{tr}[Q f(A_j)] - Q_K f\left(\frac{1}{Q_K} \sum_{j \in K} q_j \operatorname{tr}(Q A_j)\right),$$

where $K \in \mathcal{P}_f(\mathbb{N})$, $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{q} \in \mathcal{P}_n^+$ with $Q_K := \sum_{j \in K} q_j > 0$.

Proposition 2. *If $K, L \in \mathcal{P}_f(\mathbb{N}) \diagdown \{\emptyset\}$ with $K \cap L = \emptyset$, then we have the inequality*

$$J_{K \cup L}(\mathbf{q}; Q, \mathbf{A}, f) \geq J_K(\mathbf{q}; Q, \mathbf{A}, f) + J_L(\mathbf{q}; Q, \mathbf{A}, f) \geq 0,$$

i.e., $J.(\mathbf{q}, Q, \mathbf{A}, f)$ is super-additive as an index set functional.

If $\emptyset \neq K \subset L$, then we have

$$J_L(\mathbf{q}; Q, \mathbf{A}, f) \geq J_K(\mathbf{q}; Q, \mathbf{A}, f) \geq 0,$$

i.e., $J.(\mathbf{Q}, \mathbf{A}, f)$ is monotonic non-decreasing as an index set functional.

We have the inequality

$$J_k(\mathbf{q}; Q, \mathbf{A}, f) \geq J_{k-1}(\mathbf{q}; Q, \mathbf{A}, f) \geq 0$$

for any $k \in \{1, ..., n\}$ with $n \geq k \geq 2$.

We also have the lower bound:

$$\begin{aligned} J_n(\mathbf{q}; Q, \mathbf{A}, f) \geq \max_{j,k \in \{1,...,n\}} &\Big\{ p_j \operatorname{tr}[Q f(A_j)] + p_k \operatorname{tr}[Q f(A_k)] \\ &- (p_j + p_k) f\left(\frac{p_j \operatorname{tr}(Q A_j) + p_k \operatorname{tr}(Q A_k)}{p_j + p_k}\right)\Big\} \\ &\geq 0. \end{aligned}$$

3. Entropic Determinant Inequalities

We define the entropic determinant functional

$$E_n(\mathbf{q};\mathbf{A},Q) .= \frac{\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{-\sum_{j=1}^n q_j \operatorname{tr}(QA_j)}}{\prod_{i=1}^n \left[\eta_Q(A_j)\right]^{q_i}},$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators $\mathbf{p} \in \mathcal{P}_n^+$ and $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$.

Theorem 7. *Let $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{A} := (A_1, ..., A_n)$ with $Sp(A_j) \subseteq I$, $j \in \{1, ..., n\}$.*

(i) *For all p, $\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$E_n(\mathbf{p}+\mathbf{q};\mathbf{A},Q) \geq E_n(\mathbf{p};\mathbf{A},Q)\, E_n(\mathbf{q};\mathbf{A},Q) \geq 1 \tag{3.1}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A})$ is super-multiplicative on $\mathcal{P}_n^+$;

(ii) *For p, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$*

$$E_n(\mathbf{p}; Q, \mathbf{A}) \geq E_n(\mathbf{q}; Q, \mathbf{A}) \geq 1, \tag{3.2}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A})$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

Proof. (i) Consider the convex function $f(t) = t \ln t$, $t > 0$. Observe that for $\mathbf{A} = (A_1, ..., A_n)$ an n-tuple of selfadjoint positive operators $\mathbf{p} \in \mathcal{P}_n^+$ and $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$, then

$$\begin{aligned}
&J_n(\mathbf{q}; Q, \mathbf{A}, \cdot \ln(\cdot)) \\
&= \sum_{j=1}^n q_j \operatorname{tr}(QA_j \ln A_j) \\
&- Q_n \left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right) \ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right) \\
&= \sum_{j=1}^n q_j \operatorname{tr}(QA_j \ln A_j) + \ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{-\sum_{j=1}^n q_j \operatorname{tr}(QA_j)}.
\end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned}
&\exp J_n(\mathbf{q}; Q, \mathbf{A}, \cdot \ln(\cdot)) \\
&= \exp\left[\sum_{j=1}^n q_j \operatorname{tr}(QA_j \ln A_j) + \ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{-\sum_{j=1}^n q_j \operatorname{tr}(QA_j)}\right] \\
&= \frac{\exp \ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{-\sum_{j=1}^n q_j \operatorname{tr}(QA_j)}}{\exp\left(\sum_{j=1}^n q_j \operatorname{tr}(-QA_j \ln A_j)\right)} \\
&= \frac{\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{-\sum_{j=1}^n q_j \operatorname{tr}(QA_j)}}{\prod_{i=1}^n \left[\eta_Q(A_j)\right]^{q_i}} \\
&= E_n(\mathbf{q}; \mathbf{A}, Q).
\end{aligned}$$

Therefore, by the properties of $J_n(\mathbf{q}; Q, \mathbf{A}, \cdot \ln(\cdot))$,

$$\begin{aligned}
E_n(\mathbf{p}+\mathbf{q}; \mathbf{A}, Q) &= \exp J_n(\mathbf{p}+\mathbf{q}, Q, \mathbf{A}, \cdot \ln(\cdot)) \\
&\geq \exp\left[J_n(\mathbf{p}, Q, \mathbf{A}, \cdot \ln(\cdot)) + J_n(\mathbf{q}, Q, \mathbf{A}, \cdot \ln(\cdot))\right] \\
&= \exp J_n(\mathbf{p}, Q, \mathbf{A}, \cdot \ln(\cdot)) \exp J_n(\mathbf{q}, Q, \mathbf{A}, \cdot \ln(\cdot)) \\
&= E_n(\mathbf{p}; \mathbf{A}, Q)\, E_n(\mathbf{q}; \mathbf{A}, Q).
\end{aligned}$$

(ii) The monotonicity of $E_n(\cdot; \mathbf{A}, Q)$ follows by the monotonicity of $J_n(\cdot, Q, \mathbf{A}, \cdot \ln(\cdot))$. □

Corollary 3. *Let $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{A} := (A_1, ..., A_n)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \{1, ..., n\}$. Then*

$$\begin{aligned}
[E_n(\mathbf{q}; Q, \mathbf{A})]^{\min_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}} &\leq E_n(\mathbf{p}; \mathbf{A}, Q) \\
&\leq [E_n(\mathbf{q}; Q, \mathbf{A})]^{\max_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
E_n(Q, \mathbf{A})^{n \min_{j\in\{1,...,n\}}\{p_j\}} &\leq E_n(\mathbf{p}; \mathbf{A}, Q) \\
&\leq [E_n(Q, \mathbf{A})]^{n \max_{j\in\{1,...,n\}}\{p_j\}},
\end{aligned} \tag{3.4}$$

where

$$E_n(\mathbf{A}, Q) := \frac{\left(\frac{1}{n}\sum_{j=1}^n \operatorname{tr}(QA_j)\right)^{-\frac{1}{n}\sum_{j=1}^n \operatorname{tr}(QA_j)}}{\prod_{i=1}^n \left[\eta_Q(A_j)\right]^{1/n}}.$$

For $n=2$ and by choosing $p_1=\alpha$, $p_2=1-\alpha$ with $\alpha\in[0,1]$, we get from (3.7) the inequality for two positive operators A, B

$$1\leq\left(\frac{\left[\operatorname{tr}\left(Q\frac{A+B}{2}\right)\right]^{-\operatorname{tr}\left(Q\frac{A+B}{2}\right)}}{\left[\eta_Q(A)\right]^{1/2}\left[\eta_Q(B)\right]^{1/2}}\right)^{2\min\{\alpha,1-\alpha\}} \tag{3.5}$$
$$\leq\frac{\left[\operatorname{tr}(Q((1-\alpha)A+\alpha B))\right]^{-\operatorname{tr}(Q((1-\alpha)A+\alpha B))}}{\left[\eta_Q(A)\right]^{1-\alpha}\left[\eta_Q(B)\right]^{\alpha}}$$
$$\leq\left(\frac{\left[\operatorname{tr}\left(Q\frac{A+B}{2}\right)\right]^{-\operatorname{tr}\left(Q\frac{A+B}{2}\right)}}{\left[\eta_Q(A)\right]^{1/2}\left[\eta_Q(B)\right]^{1/2}}\right)^{2\max\{\alpha,1-\alpha\}}.$$

We also consider

$$E_K(\mathbf{q};Q,\mathbf{A}):=\frac{\left[\operatorname{tr}\left(\frac{1}{Q_K}\sum_{j\in K}q_jQA_j\right)\right]^{-\operatorname{tr}\left(\sum_{j\in K}q_jQA_j\right)}}{\prod_{j\in K}\left[\eta_Q(A_j)\right]^{q_i}},$$

where $K\in\mathcal{P}_f(\mathbb{N})$.

Proposition 3. *If K, $L\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ with $K\cap L=\emptyset$, then we have the inequality*

$$E_{K\cup L}(\mathbf{p}+\mathbf{q};\mathbf{A},Q)\geq E_K(\mathbf{p};\mathbf{A},Q)\,E_L(\mathbf{q};\mathbf{A},Q)\geq 1 \tag{3.6}$$

i.e., $E_\cdot(\mathbf{q};Q,\mathbf{A})$ is super-multiplicative as an index set functional.
If $\emptyset\neq K\subset L$, then we have

$$E_L(\mathbf{q};\mathbf{A},Q)\geq E_K(\mathbf{p};\mathbf{A},Q)\geq 1, \tag{3.7}$$

i.e., $E_\cdot(\mathbf{q};Q,\mathbf{A})$ is monotonic non-decreasing as an index set functional.
We have the inequality

$$E_k(\mathbf{q};\mathbf{A},Q)\geq E_{k-1}(\mathbf{p};\mathbf{A},Q)\geq 1 \tag{3.8}$$

for any $k\in\{1,...,n\}$ with $n\geq k\geq 2$.
We also have that

$$E_n(\mathbf{q};Q,\mathbf{A})\geq\max_{j,k\in\{1,\dots,n\}}\frac{\left[\operatorname{tr}\left(Q\frac{q_jA_j+q_kA_k}{q_j+q_k}\right)\right]^{-\operatorname{tr}\left(Q\frac{q_jA_j+q_kA_k}{q_j+q_k}\right)}}{\left[\eta_Q(A_j)\right]^{q_j}\left[\eta_Q(A_k)\right]^{q_k}}\geq 1. \tag{3.9}$$

4. Other Properties

We define $\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ the class of non-negative operators Q from $\mathcal{B}_1(H)$ with $\operatorname{tr}(Q)=1$. We observe that, if Q_1, $Q_2\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ then for all $t\in[0,1]$, $(1-t)Q_1+tQ_2\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ showing that $\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ is a convex subset of $\mathcal{B}_1(H)$. Also, if $Q_n\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ and $Q_n\to Q$ in the operator norm topology, then also $Q\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$.

Proposition 4. *The mapping $\eta_\cdot(A)$ is convex on $\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ for all positive invertible operator A.*

Proof. Let $Q_1, Q_2 \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ then for all $t \in [0,1]$,

$$\begin{aligned}\eta_{(1-t)Q_1+tQ_2}(A) &= \exp\left[-\operatorname{tr}\left(\left((1-t)Q_1+tQ_2\right)A\ln A\right)\right] \\ &= \exp\left\{-\left[(1-t)\operatorname{tr}\left(Q_1 A\ln A\right)+t\operatorname{tr}\left(Q_2 A\ln A\right)\right]\right\} \\ &= \left(\exp\left[-\operatorname{tr}\left(Q_1 A\ln A\right)\right]\right)^{(1-t)}\left(\exp\left[-\operatorname{tr}\left(Q_2 \ln A\right)\right]\right)^t \\ &\le (1-t)\exp\left[-\operatorname{tr}\left(Q_1 A\ln A\right)\right]+t\exp\left[-\operatorname{tr}\left(Q_2 A\ln A\right)\right] \\ &= (1-t)\,\eta_{Q_1}(A)+t\eta_{Q_2}(A),\end{aligned}$$

which proves the convexity of $\eta_{\cdot}(A)$. □

Using Jensen's inequality we have

$$\eta_{\sum_{k=1}^n p_k Q_k}(A) \le \sum_{k=1}^n p_k \eta_{Q_k}(A) \tag{4.1}$$

for all $Q_k \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$, $p_k \ge 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$.

By Hermite-Hadamard integral inequalities we also have

$$\eta_{\frac{P+Q}{2}}(A) \le \int_0^1 \eta_{(1-t)P+tQ}(A)\,dt \le \frac{1}{2}\left[\eta_P(A)+\eta_Q(A)\right] \tag{4.2}$$

for all $P, Q \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$.

Since

$$\begin{aligned}&\int_0^1 \eta_{(1-t)P+tQ}(A)\,dt \\ &= \int_0^1 \exp\left[(1-t)\operatorname{tr}(-PA\ln A)+t\operatorname{tr}(-QA\ln A)\right]dt \\ &= \begin{cases}\frac{\eta_Q(A)-\eta_P(A)}{\operatorname{tr}((P-Q)A\ln A)} \text{ if } \operatorname{tr}((Q-P)A\ln A)\neq 0 \\ \\ \exp\left[\operatorname{tr}(-PA\ln A)\right] \text{ if } \operatorname{tr}((Q-P)A\ln A)=0,\end{cases}\end{aligned}$$

hence

$$\eta_{\frac{P+Q}{2}}(A) \le \frac{\eta_Q(A)-\eta_P(A)}{\operatorname{tr}((P-Q)A\ln A)} \le \frac{1}{2}\left[\eta_P(A)+\eta_Q(A)\right] \tag{4.3}$$

provided $\operatorname{tr}((Q-P)A\ln A)\neq 0$.

Theorem 8. *For all $P, Q \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ and positive invertible operator A such that $\operatorname{tr}((Q-P)A\ln A)\neq 0$ we have*

$$\begin{aligned}0 &\le \frac{\eta_Q(A)-\eta_P(A)}{\operatorname{tr}((P-Q)A\ln A)}-\eta_{\frac{P+Q}{2}}(A) \\ &\le \frac{1}{8}\operatorname{tr}((P-Q)A\ln A)\left[\eta_Q(A)-\eta_P(A)\right]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}0 &\le \frac{1}{2}\left[\eta_P(A)+\eta_Q(A)\right]-\frac{\eta_Q(A)-\eta_P(A)}{\operatorname{tr}((P-Q)A\ln A)} \\ &\le \frac{1}{8}\operatorname{tr}((P-Q)A\ln A)\left[\eta_Q(A)-\eta_P(A)\right].\end{aligned} \tag{4.5}$$

Proof. For $P,\ Q\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$, we consider the function $\varphi_{P,Q}:[0,1]\to(0,\infty)$,

$$\varphi_{P,Q}(t):=\eta_{(1-t)P+tQ}(A),\ t\in[0,1].$$

Observe that

$$\varphi_{P,Q}(t)=\exp\left[(1-t)\operatorname{tr}(-PA\ln A)+t\operatorname{tr}(-QA\ln A)\right],$$

for all $t\in[0,1]$. Obviously, the function $\varphi_{P,Q}$ is also a convex function on $[0,1]$.

The function $\varphi_{P,Q}$ is differentiable on $(0,1)$ and

$$\varphi'_{P,Q}(t)=\operatorname{tr}(-(Q-P)A\ln A)\exp\left[(1-t)\operatorname{tr}(-PA\ln A)+t\operatorname{tr}(-QA\ln A)\right].$$

The lateral derivatives $\varphi'_{+P,Q}(0)$ and $\varphi'_{-P,Q}(1)$ also exist,

$$\varphi'_{+P,Q}(0)=\operatorname{tr}(-(Q-P)A\ln A)\,\eta_P(A)$$

and

$$\varphi'_{-P,Q}(1)=\operatorname{tr}(-(Q-P)A\ln A)\,\eta_Q(A).$$

In [1] we obtained the following reverse of first Hermite-Hadamard inequality for the convex function $f:[a,b]\to\mathbb{R}$,

$$0\le\frac{1}{b-a}\int_a^b f(s)\,ds-f\left(\frac{a+b}{2}\right)\le\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right],$$

with $\frac{1}{8}$ the best possible constant.

Therefore

$$0\le\int_0^1\varphi_{P,Q}(t)\,dt-\varphi_{P,Q}\left(\frac{1}{2}\right)\le\frac{1}{8}\left[\varphi'_{-P,Q}(1)-\varphi'_{+P,Q}(0)\right],$$

namely

$$\begin{aligned}0&\le\int_0^1\eta_{(1-t)P+tQ}(A)\,dt-\eta_{\frac{P+Q}{2}}(A)\\&\le\frac{1}{8}\operatorname{tr}(-(Q-P)A\ln A)\left[\eta_Q(A)-\eta_P(A)\right],\end{aligned}$$

which gives (4.4).

In [2] we also obtained the reverse of the second Hermite-Hadamard inequality for the convex function $f:[a,b]\to\mathbb{R}$,

$$0\le\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds\le\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right],$$

with $\frac{1}{8}$ the best possible constant. Applying this inequality, we derive the inequality (4.5). □

References

[1] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *Journal of Inequalities in Pure and Applied Mathematics*, **Volume 3**, Issue 2, Article 31, 2002.

[2] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *Journal of Inequalities in Pure and Applied Mathematics*, **Volume 3**, Issue 3, Article 35, 2002

[3] S. S. Dragomir, Trace inequalities for operators in Hilbert spaces: a survey of recent results, *Aust. J. Math. Anal. Appl.* Vol. **19** (2022), No. 1, Art. 1, 202 pp. `[Online https://ajmaa.org/searchroot/files/pdf/v19n1/v19i1p1.pdf]`.

[4] S. S. Dragomir, Some properties of trace class P-determinant of positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art. 15, 14 pp. [Online https://rgmia.org/papers/v25/v25a16.pdf].

[5] B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. of Math.* (2) **55** (1952), 520–530.

[6] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[7] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.

[8] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645

[9] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[10] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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