A SUB-MULIPLICATIVE PROPERTY FOR THE ENTROPIC TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic trace P-determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp[-\operatorname{tr}(PA \ln A)].$$

In this paper we show among others that, if $A, B > 0$ are such that $AB + BA \geq 0$, then

$$\eta_P(A)\eta_P(B) \geq \eta_P(A + B).$$

If $0 < m \leq A \leq M$ and $0 < n \leq B \leq N$, then we have the reverse inequality

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A + B)} \leq \exp\left(\frac{M + N}{m + n}\right).$$

Moreover, if $2mn \geq \frac{1}{4}(M - m)(N - n)$, then

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A + B)} \geq \exp\left[\frac{2mn - \frac{1}{4}(M - m)(N - n)}{M + N}\right] \geq 1.$$

1. Introduction

In 1952, in the paper [3], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

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where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [7].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H) \mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|;\tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;
(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P\ge 0$ and $P\in\mathcal{B}_1(H)$, then for all $T\in\mathcal{B}(H)$, PT, $TP\in\mathcal{B}_1(H)$ and $\operatorname{tr}(PT)=\operatorname{tr}(TP)$. Also, since $P^{1/2}\in\mathcal{B}_2(H)$, $TP^{1/2}\in\mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2}=TP\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)=\operatorname{tr}(TP)$. Therefore, if $P\ge 0$ and $P\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(PT)=\operatorname{tr}(TP)=\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T\in\mathcal{B}(H)$.

If $T\ge 0$, then $P^{1/2}TP^{1/2}\ge 0$, which implies that $\operatorname{tr}(PT)\ge 0$ that shows that the functional $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n\to T$ for $n\to\infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty}\operatorname{tr}(PT_n)=\operatorname{tr}(PT)$, namely $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [1] and the references therein.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A)=\exp\operatorname{tr}((\ln A)P)=\exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right).\tag{1.12}$$

Assume that $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. We observe that we have the following elementary properties [2]:

(i) *continuity*: the map $A\to\Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t)=\Delta_P(A)^t$ for all $t>0$;
(iii) *homogeneity*: $\Delta_P(tA)=t\Delta_P(A)$ and $\Delta_P(tI)=t$ for all $t>0$;
(iv) *monotonicity*: $0<A\le B$ implies $\Delta_P(A)\le\Delta_P(B)$.

In [2], we presented some fundamental properties of this determinant. Among others we showed that

$$1\le\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\le\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]$$

and

$$1\le\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\le\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right],$$

for $A>0$ and $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

For the entropy function $\eta(t)=-t\ln t$, $t>0$, the operator entropy has the following expression:

$$\eta(A)=-A\ln A$$

for positive A.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A):=\exp\left[-\operatorname{tr}(PA\ln A)\right]=\exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\}=\exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(I) = 1 \text{ and } \eta_P(tI) = t^{-t} \tag{1.14}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that, if A, $B > 0$ are such that $AB + BA \geq 0$, then

$$\eta_P(A)\eta_P(B) \geq \eta_P(A+B).$$

If $0 < m \leq A \leq M$ and $0 < n \leq B \leq N$, then we have the reverse inequality

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \leq \exp\left(\frac{M+N}{m+n}\right).$$

Moreover, if $2mn \geq \frac{1}{4}\left(M-m\right)\left(N-n\right)$, then

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \geq \exp\left[\frac{2mn-\frac{1}{4}\left(M-m\right)\left(N-n\right)}{M+N}\right] \geq 1.$$

2. Main Results

We start with the following integral representation result:

Theorem 4. *Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. For any A, $B > 0$ we have*

$$\begin{aligned}
&\operatorname{tr}\left[P\left(A+B\right)\ln\left(A+B\right)\right]-\operatorname{tr}\left(PA\ln A\right)-\operatorname{tr}\left(PB\ln B\right) \\
&=\int_0^\infty \operatorname{tr}\left[P\left(A+B+\lambda\right)^{-1}K\left(A,B;\lambda\right)\left(A+B+\lambda\right)^{-1}\right]d\lambda,
\end{aligned} \tag{2.1}$$

where

$$K\left(A,B;\lambda\right) := AB+BA+B\left(A+\lambda\right)^{-1}AB+A\left(B+\lambda\right)^{-1}BA. \tag{2.2}$$

Proof. Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{\left(\lambda+t\right)\left(\lambda+1\right)} = \frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{\left(\lambda+t\right)\left(\lambda+1\right)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.3}$$

for all $t > 0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T = \int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}\,d\lambda \tag{2.4}$$

for all operators $T > 0$.

Observe that

$$\begin{aligned}\int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}\,d\lambda &= \int_0^\infty \frac{1}{\lambda+1}(T+\lambda-\lambda-1)(\lambda+T)^{-1}\,d\lambda \\ &= \int_0^\infty \left[(\lambda+1)^{-1} - (\lambda+T)^{-1}\right]d\lambda\end{aligned}$$

and then

$$\ln T = \int_0^\infty \left[(\lambda+1)^{-1} - (\lambda+T)^{-1}\right]d\lambda$$

giving the representation

$$T\ln T = \int_0^\infty \left[(\lambda+1)^{-1}T - T(\lambda+T)^{-1}\right]d\lambda \tag{2.5}$$

for all operators $T > 0$.

For A, $B > 0$ we have

$$\begin{aligned}&(A+B)\ln(A+B) - A\ln A - B\ln B \\ &= \int_0^\infty \left[(\lambda+1)^{-1}(A+B) - (A+B)(\lambda+(A+B))^{-1}\right]d\lambda \\ &- \int_0^\infty \left[(\lambda+1)^{-1}A - A(\lambda+A)^{-1}\right]d\lambda \\ &- \int_0^\infty \left[(\lambda+1)^{-1}B - B(\lambda+B)^{-1}\right]d\lambda \\ &= \int_0^\infty \left[A(\lambda+A)^{-1} + B(\lambda+B)^{-1} - (A+B)(\lambda+A+B)^{-1}\right]d\lambda.\end{aligned} \tag{2.6}$$

Now, observe that

$$\begin{aligned}&A(\lambda+A)^{-1} + B(\lambda+B)^{-1} - (A+B)(\lambda+A+B)^{-1} \\ &= (A+\lambda-\lambda)(\lambda+A)^{-1} + (B+\lambda-\lambda)(\lambda+B)^{-1} \\ &- (A+B+\lambda-\lambda)(\lambda+A+B)^{-1} \\ &= 1 - \lambda(\lambda+A)^{-1} + 1 - \lambda(\lambda+B)^{-1} - 1 + \lambda(\lambda+A+B)^{-1} \\ &= 1 + \lambda\left[(\lambda+A+B)^{-1} - (\lambda+A)^{-1} - (\lambda+B)^{-1}\right] \\ &= \lambda\left[(\lambda+A+B)^{-1} + \lambda^{-1} - (\lambda+A)^{-1} - (\lambda+B)^{-1}\right]\end{aligned}$$

Consider

$$L_\lambda := (A+B+\lambda)^{-1} + \lambda^{-1} - (A+\lambda)^{-1} - (B+\lambda)^{-1}. \tag{2.7}$$

Then by (2.6) we obtain the representation

$$(A+B)\ln(A+B)-A\ln A-B\ln B=\int_0^\infty \lambda L_\lambda d\lambda \tag{2.8}$$

for all $A,\ B>0$.

If we multiply both sides of (2.7) by $A+B+\lambda$, then we get

$$\begin{aligned}
W_\lambda &:= (A+B+\lambda)\,L_\lambda\,(A+B+\lambda)\\
&= (A+B+\lambda)+\lambda^{-1}(A+B+\lambda)^2\\
&- (A+B+\lambda)(A+\lambda)^{-1}(A+B+\lambda)\\
&- (A+B+\lambda)(B+\lambda)^{-1}(A+B+\lambda)\\
&= (A+B+\lambda)+\lambda^{-1}(A+B+\lambda)^2\\
&- (A+B+\lambda)-B(A+\lambda)^{-1}(A+B+\lambda)\\
&- A(B+\lambda)^{-1}(A+B+\lambda)-(A+B+\lambda)\\
&= \lambda^{-1}(A+B+\lambda)^2-B(A+\lambda)^{-1}B-B\\
&- A(B+\lambda)^{-1}A-A-(A+B+\lambda)
\end{aligned}$$

$$\begin{aligned}
&= \lambda^{-1}\left(A^2+AB+\lambda A+BA+B^2+\lambda B+\lambda A+\lambda B+\lambda^2\right)\\
&- B(A+\lambda)^{-1}B-2B-A(B+\lambda)^{-1}A-2A-\lambda\\
&= \lambda^{-1}\left(A^2+AB+BA+B^2\right)+2B+2A+\lambda\\
&- B(A+\lambda)^{-1}B-A(B+\lambda)^{-1}A-2A-2B-\lambda\\
&= \lambda^{-1}\left(A^2+AB+BA+B^2\right)-B(A+\lambda)^{-1}B-A(B+\lambda)^{-1}A\\
&= \lambda^{-1}\left[A^2+AB+BA+B^2-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A\right]\\
&= \lambda^{-1}\left[A^2+AB+BA+B^2-B\left(\lambda^{-1}A+1\right)^{-1}B-A\left(\lambda^{-1}B+1\right)^{-1}A\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&B^2-B\left(\lambda^{-1}A+1\right)^{-1}B\\
&= B\left(\lambda^{-1}A+1\right)^{-1}\left(\lambda^{-1}A+1\right)B-B\left(\lambda^{-1}A+1\right)^{-1}B\\
&= B\left(\lambda^{-1}A+1\right)^{-1}\left(\lambda^{-1}A+1-1\right)B\\
&= \lambda^{-1}B\left(\lambda^{-1}A+1\right)^{-1}AB=B(A+\lambda)^{-1}AB
\end{aligned}$$

and

$$\begin{aligned}
&A^2-A\left(\lambda^{-1}B+1\right)^{-1}A\\
&= A\left(\lambda^{-1}B+1\right)^{-1}\left(\lambda^{-1}B+1\right)A-A\left(\lambda^{-1}B+1\right)^{-1}A\\
&= A\left(\lambda^{-1}B+1\right)^{-1}\left(\lambda^{-1}B+1-1\right)A\\
&= \lambda^{-1}A\left(\lambda^{-1}B+1\right)^{-1}BA=A(B+\lambda)^{-1}BA.
\end{aligned}$$

Therefore

$$W_\lambda=\lambda^{-1}\left[AB+BA+B(A+\lambda)^{-1}AB+A(B+\lambda)^{-1}BA\right],$$

which gives that

$$L_\lambda := (A+B+\lambda)^{-1} W_\lambda (A+B+\lambda)^{-1}.$$

From (2.8) we then get the following representation that is of interest in itself:

$$\begin{aligned} &(A+B)\ln(A+B) - A\ln A - B\ln B \\ &= \int_0^\infty (A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1} d\lambda. \end{aligned} \tag{2.9}$$

Further, if we multiply both sides by $P^{1/2} \geq 0$, we get

$$\begin{aligned} &P^{1/2}(A+B)\ln(A+B)P^{1/2} - P^{1/2}A\ln AP^{1/2} - P^{1/2}B\ln BP^{1/2} \\ &= \int_0^\infty P^{1/2}(A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1} P^{1/2} d\lambda. \end{aligned}$$

If we take the trace and use its properties, we get the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 4 we have the representation*

$$\begin{aligned} &\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \\ &= \exp\int_0^\infty \operatorname{tr}\left[P(A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1}\right] d\lambda. \end{aligned} \tag{2.10}$$

Proof. If we take the exponential in (2.1), then we get

$$\begin{aligned} &\exp(\operatorname{tr}[P(A+B)\ln(A+B)] - \operatorname{tr}(PA\ln A) - \operatorname{tr}(PB\ln B)) \\ &= \exp\left(\int_0^\infty \operatorname{tr}\left[P(A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1}\right] d\lambda\right). \end{aligned}$$

Observe that

$$\begin{aligned} &\exp(\operatorname{tr}[P(A+B)\ln(A+B)] - \operatorname{tr}(PA\ln A) - \operatorname{tr}(PB\ln B)) \\ &= \frac{\exp[-\operatorname{tr}(PA\ln A)]\exp[-\operatorname{tr}(PB\ln B)]}{\exp[-\operatorname{tr}[P(A+B)\ln(A+B)]]} = \frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)}, \end{aligned}$$

and the identity (2.10) is thus proved. □

The symmetrized product of two operators $A, B \in B(H)$ is defined by $S(A,B) = AB + BA$. In general, the symmetrized product of two operators A, B is not positive (see for instance [9]). Also Gustafson [6] showed that if $0 \leq m \leq A \leq M$ and $0 \leq n \leq B \leq N$, then we have the lower bound

$$S(A,B) \geq 2mn - \frac{1}{4}(M-m)(N-n) =: k, \tag{2.11}$$

which can take positive or negative values depending on the parameters m, M, n, N.

Corollary 2. *Let A, $B > 0$ and assume that $S(A,B) \geq k$ for some real constant k, then*

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \geq \exp\left(k\operatorname{tr}\left[P(A+B)^{-1}\right]\right). \tag{2.12}$$

If $k \geq 0$, then

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \geq \exp\left(k\operatorname{tr}\left[P(A+B)^{-1}\right]\right) \geq 1. \tag{2.13}$$

Proof. Since for all A, $B > 0$,

$$(B+\lambda)^{-1} B > 0, \;\; (A+\lambda)^{-1} A > 0$$

for $\lambda \geq 0$, then

$$A\,(B+\lambda)^{-1} BA, \;\; B\,(A+\lambda)^{-1} AB \geq 0$$

that gives

$$A\,(B+\lambda)^{-1} BA + B\,(A+\lambda)^{-1} AB \geq 0,$$

which implies that

$$(A+B+\lambda)^{-1} \left[A\,(B+\lambda)^{-1} BA + B\,(A+\lambda)^{-1} AB\right] (A+B+\lambda)^{-1} \geq 0$$

for$\lambda \geq 0$.

By the representation (2.1) we then get

$$\begin{aligned} &(A+B)\ln(A+B) - A\ln A - B\ln B \\ &\geq \int_0^\infty (A+B+\lambda)^{-1} S(A,B)\,(A+B+\lambda)^{-1}\, d\lambda \\ &\geq k \int_0^\infty (A+B+\lambda)^{-2}\, d\lambda = k\,(A+B)^{-1} \end{aligned}$$

since

$$t^{-1} = \int_0^\infty (\lambda+t)^{-2}\, d\lambda \text{ for } t > 0.$$

If we multiply both sides by $P^{1/2} \geq 0$, we get

$$\operatorname{tr}\left[P\,(A+B)\ln(A+B)\right] - \operatorname{tr}(PA\ln A) - \operatorname{tr}(PB\ln B) \geq k\operatorname{tr}\left[P\,(A+B)^{-1}\right].$$

By taking the exponential and using the equality (2.10) we obtain (2.12). □

Remark 1. *If* $0 \leq m \leq A \leq M$ *and* $0 \leq n \leq B \leq N$, *then*

$$\begin{aligned} &\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \\ &\geq \exp\left(\left[2mn - \frac{1}{4}(M-m)(N-n)\right]\operatorname{tr}\left[P\,(A+B)^{-1}\right]\right). \end{aligned} \tag{2.14}$$

If $2mn \geq \frac{1}{4}(M-m)(N-n)$, *then*

$$\eta_P(A)\eta_P(B) \geq \eta_P(A+B).$$

Corollary 3. *Assume that* A, $B > 0$ *with* $A + B \leq L$ *for some positive constant* L, *then*

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \leq L\operatorname{tr}\left[P\,(A+B)^{-1}\right]. \tag{2.15}$$

Moreover, if $0 < \ell \leq A + B$ *for some constant* $\ell > 0$, *then*

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \leq \frac{L}{\ell}. \tag{2.16}$$

Proof. Assume that A, $B \geq 0$. Observe that for $\lambda > 0$

$$(A+\lambda)^{-1} A = (A+\lambda)^{-1}(A+\lambda-\lambda) = 1-\lambda(A+\lambda)^{-1},$$

which shows that

$$0 \leq (A+\lambda)^{-1} A \leq 1.$$

If we multiply this inequality both sides by B, then we get

$$0 \leq B(A+\lambda)^{-1} AB \leq B^2.$$

Similarly,

$$0 \leq A(B+\lambda)^{-1} BA \leq A^2.$$

Therefore

$$0 \leq B(A+\lambda)^{-1} AB + A(B+\lambda)^{-1} BA \leq A^2 + B^2$$

and

$$\begin{aligned} L(A,B;\lambda) &= AB + BA + B(A+\lambda)^{-1} AB + A(B+\lambda)^{-1} BA \\ &\leq AB + BA + A^2 + B^2 = (A+B)^2 \leq L, \end{aligned}$$

which implies that

$$\begin{aligned} &(A+B+\lambda)^{-1} L(A,B;\lambda)(A+B+\lambda)^{-1} \\ &\leq L(A+B+\lambda)^{-1}(A+B+\lambda)^{-1} \end{aligned}$$

for $\lambda > 0$.

By taking the integral and using the identity (2.1) we derive

$$\begin{aligned} &(A+B)\ln(A+B) - A\ln A - B\ln B \\ &\leq L\int_0^\infty (A+B+\lambda)^{-1}(A+B+\lambda)^{-1} dt = L(A+B)^{-1}, \end{aligned}$$

which proves the desired inequality (2.15). □

Remark 2. *We observe that, if* $0 < m \leq A \leq M$ *and* $0 < n \leq B \leq N$, *then* $0 < m+n \leq A+B \leq M+N$ *and by (2.16) we obtain the simple upper bound*

$$\frac{\eta_P(A)\eta_P(B)}{\eta_P(A+B)} \leq \exp\left(\frac{M+N}{m+n}\right). \tag{2.17}$$

3. Related Results

The following integral inequalities also hold:

Theorem 5. *Assume that* $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$. *Let* A, $B \geq 0$ *with* $AB + BA \geq 0$, *then*

$$\begin{aligned} \eta_P(A+B) &\leq \int_0^1 \eta_P((1-t)A+tB)\eta_P((1-t)B+tA)dt \\ &\leq \int_0^1 \eta_P^2((1-t)A+tB)dt \end{aligned} \tag{3.1}$$

and, if $A+B \leq L$, *then also*

$$\begin{aligned} &\int_0^1 \eta_P((1-t)A+tB)\eta_P((1-t)B+tA)dt \\ &\leq \eta_P(A+B)\exp\left[L\operatorname{tr}\left(P(A+B)^{-1}\right)\right]. \end{aligned} \tag{3.2}$$

Proof. We have

$$
\begin{aligned}
&((1-t)A+tB)((1-t)B+tA) \\
&= (1-t)^2 AB + t(1-t)B^2 + t(1-t)A^2 + t^2 BA
\end{aligned}
$$

and

$$
\begin{aligned}
&((1-t)B+tA)((1-t)A+tB) \\
&= (1-t)^2 BA + t(1-t)A^2 + (1-t)tB^2 + t^2 AB.
\end{aligned}
$$

Therefore, since $AB+BA \geq 0$, then

$$
\begin{aligned}
&((1-t)A+tB)((1-t)B+tA) \\
&+ ((1-t)B+tA)((1-t)A+tB) \\
&= (1-t)^2 AB + t(1-t)B^2 + t(1-t)A^2 + t^2 BA \\
&+ (1-t)^2 BA + t(1-t)A^2 + (1-t)tB^2 + t^2 AB \\
&= 2t(1-t)A^2 + 2t(1-t)B^2 + \left[(1-t)^2 + t^2\right](AB+BA) \\
&\geq 0
\end{aligned}
$$

for all $t \in [0,1]$.

By utilizing (2.13) for $(1-t)A+tB$ and $(1-t)B+tA$, $t \in [0,1]$, we get

$$\eta_P((1-t)A+tB)\eta_P((1-t)B+tA) \geq \eta_P(A+B).$$

If we integrate over $t \in [0,1]$, then we get

$$
\begin{aligned}
\eta_P(A+B) &\leq \int_0^1 \eta_P((1-t)A+tB)\eta_P((1-t)B+tA)dt \\
&\leq \left(\int_0^1 \eta_P^2((1-t)A+tB)dt\right)^{1/2} \left(\int_0^1 \eta_P^2((1-t)B+tA)dt\right)^{1/2} \\
&= \int_0^1 \eta_P^2((1-t)A+tB)dt,
\end{aligned}
$$

which proves (3.1).

From (2.15) we get

$$\eta_P((1-t)A+tB)\eta_P((1-t)B+tA) \leq \eta_P(A+B)\exp\left[L \operatorname{tr}\left(P(A+B)^{-1}\right)\right]$$

for all $t \in [0,1]$, which by integration gives (3.2). □

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa