Multivariate Fuzzy-Random and stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions induced Neural Network Approximations

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Abstract

In this article we study the degree of approximation of multivariate pointwise and uniform convergences in the q-mean to the Fuzzy-Random unit operator of multivariate Fuzzy-Random Quasi-Interpolation arctangent, algebraic, Gudermannian and generalized symmetric activation functions based neural network operators. These multivariate Fuzzy-Random operators arise in a natural way among multivariate Fuzzy-Random neural networks. The rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function. The plain stochastic extreme analog of this theory is also met in detail for the stochastic analogs of the operators: the stochastic full quasi-interpolation operators, the stochastic Kantorovich type operators and the stochastic quadrature type operators.

AMS 2020 Mathematics Subject Classification: 26A15, 26E50, 41A17, 41A25, 41A99, 47S40, 60H25, 60H30.

Key words and phrases: Fuzzy-Random analysis, Fuzzy-Random neural networks and operators, Fuzzy-Random modulus of continuity, Fuzzy-Random functions, Stochastic processes, Jackson type fuzzy and probabilistic inequalities.

RGMIA Res. Rep. Coll. **25** (2022), Art. 57, 33 p. Received 04/05/22

1 Fuzzy-Random Functions and Stochastic processes Background

See also [18], Ch. 22, pp. 497-501.

We start with

Definition 1 *(see [35]) Let* $\mu : \mathbb{R} \to [0,1]$ *with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ *(μ is called a convex fuzzy subset).*

(iii) μ *is upper semicontinuous on* $\mathbb{R}$*, i.e.,* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ *is compact in* $\mathbb{R}$ *(where* $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$*, for any* $x_0 \in \mathbb{R}$*, where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ *and* $\mu \in \mathbb{R}_{\mathcal{F}}$ *define* $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ *and* $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\ r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [35]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u, v) := \sup_{r \in [0,1]} \max\left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [35], with the properties

$$\begin{gathered} D(u \oplus w, v \oplus w) = D(u, v), \quad \forall\ u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) = |k| D(u, v), \quad \forall\ u, v \in \mathbb{R}_{\mathcal{F}}, \forall\ k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \quad \forall\ u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{gathered} \tag{1}$$

Let (M, d) metric space and $f, g : M \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^* (f, g) := \sup_{x \in M} D (f (x), g (x)) .$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\, r \in [0, 1]$.

$\overset{*}{\sum}$ denotes the fuzzy summation, $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ the neutral element with respect to $\oplus$. For more see also [36], [37].

We need

Definition 2 *(see also [30], Definition 13.16, p. 654) Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have*

$$g^{-1} (U) = \{s \in X; g (s) \in U\} \in \mathcal{B}). \tag{2}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}} (X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}} (X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n (s) \overset{\text{"q-mean"}}{\underset{n \to +\infty}{\to}} g (s)$ if

$$\lim_{n \to +\infty} \int_X D (g_n (s), g (s))^q \, P (ds) = 0. \tag{3}$$

Remark 3 *(see [30], p. 654) If $f, g \in \mathcal{L}_{\mathcal{F}} (X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F (s) = D (f (s), g (s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G (u, v) = D (u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H (s) = (f (s), g (s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.*

Definition 4 *(see [30], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}} (X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f (t) (s) = f (t, s)$, $t \in T$, $s \in X$.*

Remark 5 *(see [30], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f (t, s) = f (t)$, $\forall\, t \in T$, $s \in X$.*

Remark 6 *(see [30], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.*

Remark 7 *(see [30], p. 655) Let $f, g : T \to \mathcal{L}_{\mathcal{F}} (X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,*

$$\begin{aligned} (f \oplus g) (t, s) &= f (t, s) \oplus g (t, s) , \\ (k \odot f) (t, s) &= k \odot f (t, s) , \;\; t \in T, s \in X, \; k \in \mathbb{R}. \end{aligned}$$

Definition 8 *(see also Definition 13.18, pp. 655-656, [30]) For a fuzzy-random function* $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, *we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} =$$

$$\sup\left\{\left(\int_X D^q\left(f(x,s), f(y,s)\right) P(ds)\right)^{\frac{1}{q}} : x, y \in W,\ \ \|x-y\|_\infty \leq \delta\right\},$$

$0 < \delta$, $1 \leq q < \infty$.

Definition 9 *([16]) Here* $1 \leq q < +\infty$. *Let* $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, *be a fuzzy random function. We call* f *a (q-mean) uniformly continuous fuzzy random function over* W, *iff* $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ *:whenever* $\|x-y\|_\infty \leq \delta$, $x, y \in W$, *implies that*

$$\int_X \left(D\left(f(x,s), f(y,s)\right)\right)^q P(ds) \leq \varepsilon.$$

We denote it as $f \in C_{FR}^{U_q}(W)$.

Proposition 10 *([16]) Let* $f \in C_{FR}^{U_q}(W)$, *where* $W \subseteq \mathbb{R}^N$ *is convex.*

Then $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} < \infty$, *any* $\delta > 0$.

Proposition 11 *([16]) Let* $f, g : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, *be fuzzy random functions. It holds*

(i) $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$ *is nonnegative and nondecreasing in* $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f,0)_{L^q} = 0$, *iff* $f \in C_{FR}^{U_q}(W)$.

We mention

Definition 12 *(see also [6]) Let* $f(t,s)$ *be a random function (stochastic process) from* $W \times (X, \mathcal{B}, P)$, $W \subseteq \mathbb{R}^N$, *into* $\mathbb{R}$, *where* $(X, \mathcal{B}, P)$ *is a probability space. We define the q-mean multivariate first modulus of continuity of* f *by*

$$\Omega_1(f,\delta)_{L^q} :=$$

$$\sup\left\{\left(\int_X |f(x,s) - f(y,s)|^q P(ds)\right)^{\frac{1}{q}} : x, y \in W,\ \ \|x-y\|_\infty \leq \delta\right\}, \tag{4}$$

$\delta > 0$, $1 \leq q < \infty$.

The concept of f being (q-mean) uniformly continuous random function is defined the same way as in Definition 9, just replace D by $|\cdot|$, etc. We denote it as $f \in C_{\mathbb{R}}^{U_q}(W)$.

Similar properties as in Propositions 10, 11 are valid for $\Omega_1(f,\delta)_{L^q}$.

Also we have

Proposition 13 *([3]) Let* $Y(t,\omega)$ *be a real valued stochastic process such that* Y *is continuous in* $t \in [a,b]$*. Then* Y *is jointly measurable in* (t,ω)*.*

According to [28], p. 94 we have the following

Definition 14 *Let* $(Y,\mathcal{T})$ *be a topological space, with its* σ*-algebra of Borel sets* $\mathcal{B} := \mathcal{B}(Y,\mathcal{T})$ *generated by* $\mathcal{T}$*. If* $(X,\mathcal{S})$ *is a measurable space, a function* $f : X \to Y$ *is called measurable iff* $f^{-1}(B) \in \mathcal{S}$ *for all* $B \in \mathcal{B}$*.*

By Theorem 4.1.6 of [28], p. 89 f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We mention

Theorem 15 *(see [28], p. 95) Let* $(X,\mathcal{S})$ *be a measurable space and* (Y,d) *be a metric space. Let* f_n *be measurable functions from* X *into* Y *such that for all* $x \in X$*,* $f_n(x) \to f(x)$ *in* Y*. Then* f *is measurable. I.e.,* $\lim_{n\to\infty} f_n = f$ *is measurable.*

We need also

Proposition 16 *([16]) Let* f, g *be fuzzy random variables from* $\mathcal{S}$ *into* $\mathbb{R}_{\mathcal{F}}$*. Then*

(i) Let $c \in \mathbb{R}$*, then* $c \odot f$ *is a fuzzy random variable.*

(ii) $f \oplus g$ *is a fuzzy random variable.*

Proposition 17 *Let* $Y\left(\overrightarrow{t},\omega\right)$ *be a real valued multivariate random function (stochastic process) such that* Y *is continuous in* $\overrightarrow{t} \in \prod_{i=1}^{N}[a_i,b_i]$*. Then* Y *is jointly measurable in* $\left(\overrightarrow{t},\omega\right)$ *and* $\int_{\prod_{i=1}^{N}[a_i,b_i]} Y\left(\overrightarrow{t},\omega\right) d\overrightarrow{t}$ *is a real valued random variable.*

Proof. Similar to Proposition 18.14, p. 353 of [7]. ∎

2 About neural networks background

2.1 About the arctangent activation function

We consider the

$$\arctan x = \int_0^x \frac{dz}{1+z^2}, \quad x \in \mathbb{R}. \tag{5}$$

We will be using

$$h\left(x\right):=\frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right)=\frac{2}{\pi}\int_{0}^{\frac{\pi x}{2}}\frac{dz}{1+z^{2}},\ \ x\in\mathbb{R}, \tag{6}$$

which is a sigmoid type function and it is strictly increasing. We have that

$$h\left(0\right)=0,\ \ h\left(-x\right)=-h\left(x\right),\ \ h\left(+\infty\right)=1,\ \ h\left(-\infty\right)=-1,$$

and

$$h'\left(x\right)=\frac{4}{4+\pi^{2}x^{2}}>0,\ \ \text{all } x\in\mathbb{R}. \tag{7}$$

We consider the activation function

$$\psi_{1}\left(x\right):=\frac{1}{4}\left(h\left(x+1\right)-h\left(x-1\right)\right),\ \ x\in\mathbb{R}, \tag{8}$$

and we notice that

$$\psi_{1}\left(-x\right)=\psi_{1}\left(x\right), \tag{9}$$

it is an even function.

Since $x+1>x-1$, then $h\left(x+1\right)>h\left(x-1\right)$, and $\psi_{1}\left(x\right)>0$, all $x\in\mathbb{R}$.

We see that

$$\psi_{1}\left(0\right)=\frac{1}{\pi}\arctan\frac{\pi}{2}\cong 18.31. \tag{10}$$

Let $x>0$, we have that

$$\psi_{1}'\left(x\right)=\frac{1}{4}\left(h'\left(x+1\right)-h'\left(x-1\right)\right)=$$

$$\frac{-4\pi^{2}x}{\left(4+\pi^{2}\left(x+1\right)^{2}\right)\left(4+\pi^{2}\left(x-1\right)^{2}\right)}<0. \tag{11}$$

That is

$$\psi_{1}'\left(x\right)<0,\ \text{for } x>0. \tag{12}$$

That is ψ_{1} is strictly decreasing on $[0,\infty)$ and clearly is strictly increasing on $(-\infty,0]$, and $\psi_{1}'\left(0\right)=0$.

Observe that

$$\begin{aligned}&\lim_{x\to+\infty}\psi_{1}\left(x\right)=\tfrac{1}{4}\left(h\left(+\infty\right)-h\left(+\infty\right)\right)=0,\\&\text{and}\\&\lim_{x\to-\infty}\psi_{1}\left(x\right)=\tfrac{1}{4}\left(h\left(-\infty\right)-h\left(-\infty\right)\right)=0.\end{aligned} \tag{13}$$

That is the x-axis is the horizontal asymptote on ψ_{1}.

All in all, ψ_{1} is a bell symmetric function with maximum $\psi_{1}\left(0\right)\cong 18.31$.

We need

Theorem 18 *([19], p. 286) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_1 (x-i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{14}$$

Theorem 19 *([19], p. 287) It holds*

$$\int_{-\infty}^{\infty} \psi_1 (x) \, dx = 1. \tag{15}$$

So that $\psi_1 (x)$ is a density function on $\mathbb{R}$.

We mention

Theorem 20 *([19], p. 288) Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_1 (nx-k) < \frac{2}{\pi^2 \left(n^{1-\alpha} - 2 \right)} =: c_1 (\alpha, n). \tag{16}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We need

Theorem 21 *([19], p. 289) Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k)} < \frac{1}{\psi_1 (1)} \cong \mathbf{0.0868} =: \boldsymbol{\alpha}_1, \ \ \forall \ \mathbf{x} \in [a,b]. \tag{17}$$

Note 22 *([19], pp. 290-291)*

i) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k) \neq 1, \tag{18}$$

for at least some $x \in [a,b]$.

ii) For large enough $n \in \mathbb{N}$ *we always obtain* $\lceil na \rceil \leq \lfloor nb \rfloor$. *Also* $a \leq \frac{k}{n} \leq b$, *iff* $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 18, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k) \leq 1. \tag{19}$$

We introduce (see [24])

$$Z_1(x_1,...,x_N) := Z_1(x) := \prod_{i=1}^{N} \psi_1(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{20}$$

Denote by $a = (a_1,...,a_N)$ and $b = (b_1,...,b_N)$.

It has the properties:

(i) $Z_1(x) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_1(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z_1(x_1-k_1,...,x_N-k_N) = 1, \tag{21}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_1(nx-k) = 1, \tag{22}$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_1(x)\,dx = 1, \tag{23}$$

that is Z_1 is a multivariate density function.

(v) It is clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_1(nx-k) < \frac{2}{\pi^2 (n^{1-\beta} - 2)} = c_1(\beta, n), \tag{24}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

(vi) By Theorem 21 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_1(nx-k)} < \frac{1}{(\psi_1(1))^N} \cong (0.0868)^N =: \gamma_1(N), \tag{25}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right), \ n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_1(nx-k) \neq 1, \tag{26}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Above it is $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty = (-\infty, ... -\infty)$ upon the multivariate context.

2.2 About the algebraic activation function

Here see also [20].

We consider the generator algebraic function

$$\varphi(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}}, \quad m \in \mathbb{N},\, x \in \mathbb{R}, \tag{27}$$

which is a sigmoidal type of function and is a strictly increasing function.

We see that $\varphi(-x) = -\varphi(x)$ with $\varphi(0) = 0$. We get that

$$\varphi'(x) = \frac{1}{\left(1 + x^{2m}\right)^{\frac{2m+1}{2m}}} > 0, \ \ \forall\, x \in \mathbb{R}, \tag{28}$$

proving φ as strictly increasing over $\mathbb{R}, \varphi'(x) = \varphi'(-x)$. We easily find that $\lim_{x \to +\infty} \varphi(x) = 1$, $\varphi(+\infty) = 1$, and $\lim_{x \to -\infty} \varphi(x) = -1$, $\varphi(-\infty) = -1$.

We consider the activation function

$$\psi_2(x) = \frac{1}{4}\left[\varphi(x+1) - \varphi(x-1)\right]. \tag{29}$$

Clearly it is $\psi_2(x) = \psi_2(-x)$, $\forall\, x \in \mathbb{R}$, so that ψ_2 is an even function and symmetric with respect to the y-axis. Clealry $\psi_2(x) > 0$, $\forall\, x \in \mathbb{R}$.

Also it is

$$\psi_2(0) = \frac{1}{2\sqrt[2m]{2}}. \tag{30}$$

By [20], we have that $\psi_2'(x) < 0$ for $x > 0$. That is ψ_2 is strictly decreasing over $(0, +\infty)$.

Clearly, ψ_2 is strictly increasing over $(-\infty, 0)$ and $\psi_2'(0) = 0$.

Furthermore we obtain that

$$\lim_{x \to +\infty} \psi_2(x) = \frac{1}{4}\left[\varphi(+\infty) - \varphi(+\infty)\right] = 0, \tag{31}$$

and

$$\lim_{x \to -\infty} \psi_2(x) = \frac{1}{4}\left[\varphi(-\infty) - \varphi(-\infty)\right] = 0. \tag{32}$$

That is the x-axis is the horizontal asymptote of ψ_2.

Conclusion, ψ_2 is a bell shape symmetric function with maximum

$$\psi_2(0) = \frac{1}{2\sqrt[2m]{2}}, \quad m \in \mathbb{N}. \tag{33}$$

We need

Theorem 23 *([20]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_2 (x-i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{34}$$

Theorem 24 *([20]) It holds*

$$\int_{-\infty}^{\infty} \psi_2 (x) \, dx = 1. \tag{35}$$

Theorem 25 *([20]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_2 (nx-k) < \frac{1}{4m \left(n^{1-\alpha} - 2\right)^{2m}} =: c_2 (\alpha, n), \quad m \in \mathbb{N}. \tag{36}$$

We need

Theorem 26 *([20]) Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k)} < 2 \left(\sqrt[2m]{1+4^m} \right) =: \alpha_2, \tag{37}$$

$\forall \ x \in [a,b]$, $m \in \mathbb{N}$.

Note 27 *1) By [20] we have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k) \neq 1, \tag{38}$$

for at least some $x \in [a,b]$.

2) Let $[a,b] \subset \mathbb{R}$. For large $n \in \mathbb{N}$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general it holds that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k) \leq 1. \tag{39}$$

We introduce (see also [25])

$$Z_2 (x_1, ..., x_N) := Z_2 (x) := \prod_{i=1}^{N} \psi_2 (x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{40}$$

It has the properties:

(i) $Z_2(x) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_2(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \ldots \sum_{k_N=-\infty}^{\infty} Z_2(x_1-k_1, \ldots, x_N-k_N) = 1, \tag{41}$$

where $k := (k_1, \ldots, k_n) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_2(nx-k) = 1, \tag{42}$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_2(x)\,dx = 1, \tag{43}$$

that is Z_2 is a multivariate density function.

(v) It is clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_2(nx-k) < \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}} = c_2(\beta,n), \tag{44}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

(vi) By Theorem 26 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_2(nx-k)} < \frac{1}{\left(\psi_2(1)\right)^N} \cong \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N := \gamma_2(N), \tag{45}$$

$\forall \ x \in \left(\prod_{i=1}^N [a_i, b_i]\right), \ n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_2(nx-k) \neq 1, \tag{46}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

2.3 About the Gudermannian activation function

See also [21], [34].

Here we consider $gd(x)$ the Gudermannian function [34], which is a sigmoid function, as a generator function:

$$\sigma(x) = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{\cosh t} =: gd(x),\ x \in \mathbb{R}. \tag{47}$$

Let the normalized generator sigmoid function

$$f(x) := \frac{4}{\pi}\sigma(x) = \frac{4}{\pi}\int_0^x \frac{dt}{\cosh t} = \frac{8}{\pi}\int_0^x \frac{1}{e^t + e^{-t}}dt, \quad x \in \mathbb{R}. \tag{48}$$

Here

$$f'(x) = \frac{4}{\pi \cosh x} > 0, \quad \forall\ x \in \mathbb{R},$$

hence f is strictly increasing on $\mathbb{R}$.

Notice that $\tanh(-x) = -\tanh x$ and $\arctan(-x) = -\arctan x$, $x \in \mathbb{R}$.

So, here the neural network activation function will be:

$$\psi_3(x) = \frac{1}{4}\left[f(x+1) - f(x-1)\right],\ x \in \mathbb{R}. \tag{49}$$

By [21], we get that

$$\psi_3(x) = \psi_3(-x),\quad \forall\ x \in \mathbb{R}, \tag{50}$$

i.e. it is even and symmetric with respect to the y-axis. Here we have $f(+\infty) = 1$, $f(-\infty) = -1$ and $f(0) = 0$. Clearly it is

$$f(-x) = -f(x),\ \forall\ x \in \mathbb{R}, \tag{51}$$

an odd function, symmetric with respect to the origin. Since $x+1 > x-1$, and $f(x+1) > f(x-1)$, we obtain $\psi_3(x) > 0$, $\forall\ x \in \mathbb{R}$.

By [21], we have that

$$\psi_3(0) = \frac{2}{\pi}gd(1) \cong 0.551. \tag{52}$$

By [21] ψ_3 is strictly decreasing on $(0, +\infty)$, and strictly increasing on $(-\infty, 0)$, and $\psi_3'(0) = 0$.

Also we have that

$$\lim_{x\to+\infty}\psi_3(x) = \lim_{x\to-\infty}\psi_3(x) = 0, \tag{53}$$

that is the x-axis is the horizontal asymptote for ψ_3.

Conclusion, ψ_3 is a bell shaped symmetric function with maximum $\psi_3(0) \cong 0.551$.

We need

Theorem 28 *([21]) It holds that*

$$\sum_{i=-\infty}^{\infty} \psi_3 (x-i) = 1, \;\; \forall \; x \in \mathbb{R}. \tag{54}$$

Theorem 29 *([21]) We have that*

$$\int_{-\infty}^{\infty} \psi_3 (x) \, dx = 1. \tag{55}$$

So $\psi_3 (x)$ is a density function.

Theorem 30 *([21]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_3 (nx-k) < \frac{4}{\pi e^{(n^{1-\alpha}-2)}} = \frac{4e^2}{\pi e^{n^{1-\alpha}}} =: c_3 (\alpha, n) . \tag{56}$$

Theorem 31 *([21]) Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3 (nx-k)} < \frac{\pi}{gd (2)} \cong 2.412 =: \alpha_3, \tag{57}$$

$\forall \; x \in [a,b]$.

We make

Remark 32 *([21])*
(i) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3 (nx-k) \neq 1, \tag{58}$$

for at least some $x \in [a,b]$.
(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.
In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3 (nx-k) \leq 1. \tag{59}$$

We introduce (see also [23])

$$Z_3 (x_1, ..., x_N) := Z_3 (x) := \prod_{i=1}^{N} \psi_3 (x_i) , \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \; N \in \mathbb{N}. \tag{60}$$

It has the properties:

(i) $Z_3(x) > 0, \ \forall \, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_3(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z_3(x_1-k_1, \dots, x_N-k_N) = 1, \tag{61}$$

where $k := (k_1, \dots, k_n) \in \mathbb{Z}^N$, $\forall \, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_3(nx-k) = 1, \tag{62}$$

$\forall \, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_3(x)\, dx = 1, \tag{63}$$

that is Z_3 is a multivariate density function.

(v) It is also clear that

$$\sum_{\substack{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}}^{\infty} Z_3(nx-k) < \frac{4e^2}{\pi e^{n^{1-\beta}}} = c_3(\beta, n), \tag{64}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

(vi) By Theorem 31 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_3(nx-k)} < \left(\frac{\pi}{gd(2)} \right)^N \cong (2.412)^N =: \gamma_3(N), \tag{65}$$

$\forall \, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $\ n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_3(nx-k) \neq 1, \tag{66}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

2.4 About the generalized symmetrical activation function

Here we consider the generalized symmetrical sigmoid function ([22], [29])

$$f_1(x) = \frac{x}{\left(1 + |x|^{\mu}\right)^{\frac{1}{\mu}}}, \quad \mu > 0, \, x \in \mathbb{R}. \tag{67}$$

This has applications in immunology and protection from disease together with probability theory. It is also called a symmetrical protection curve.

The parameter μ is a shape parameter controling how fast the curve approaches the asymptotes for a given slope at the inflection point. When $\mu = 1$ f_1 is the absolute sigmoid function, and when $\mu = 2$, f_1 is the square root sigmoid function. When $\mu = 1.5$ the function approximates the arctangent function, when $\mu = 2.9$ it approximates the logistic function, and when $\mu = 3.4$ it approximates the error function. Parameter μ is estimated in the likelihood maximization ([29]). For more see [29].

Next we study the particular generator sigmoid function

$$f_2(x) = \frac{x}{\left(1 + |x|^{\lambda}\right)^{\frac{1}{\lambda}}}, \quad \lambda \text{ is an odd number, } x \in \mathbb{R}. \tag{68}$$

We have that $f_2(0) = 0$, and

$$f_2(-x) = -f_2(x), \tag{69}$$

so f_2 is symmetric with respect to zero.

When $x \geq 0$, we get that ([22])

$$f_2'(x) = \frac{1}{\left(1 + x^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} > 0, \tag{70}$$

that is f_2 is strictly increasing on $[0, +\infty)$ and f_2 is strictly increasing on $(-\infty, 0]$. Hence f_2 is strictly increasing on $\mathbb{R}$.

We also have $f_2(+\infty) = f_2(-\infty) = 1$.

Let us consider the activation function ([22]):

$$\psi_4(x) = \frac{1}{4}\left[f_2(x+1) - f_2(x-1)\right] =$$

$$\frac{1}{4}\left[\frac{(x+1)}{\left(1 + |x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(x-1)}{\left(1 + |x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right]. \tag{71}$$

Clearly it holds ([22])

$$\psi_4(x) = \psi_4(-x), \quad \forall \, x \in \mathbb{R}. \tag{72}$$

and

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}}, \tag{73}$$

and $\psi_4(x) > 0$, $\forall\ x \in \mathbb{R}$.

Following [22], we have that ψ_4 is strictly decreasing over $[0, +\infty)$, and ψ_4 is strictly increasing on $(-\infty, 0]$, by ψ_4-symmetry with respect to y-axis, and $\psi_4'(0) = 0$.

Clearly it is

$$\lim_{x \to +\infty} \psi_4(x) = \lim_{x \to -\infty} \psi_4(x) = 0, \tag{74}$$

therefore the x-axis is the horizontal asymptote of $\psi_4(x)$.

The value

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}},\ \lambda \text{ is an odd number}, \tag{75}$$

is the maximum of ψ_4, which is a bell shaped function.

We need

Theorem 33 *([22]) It holds*

$$\sum_{i=-\infty}^{\infty} \psi_4(x - i) = 1, \ \ \forall\ x \in \mathbb{R}. \tag{76}$$

Theorem 34 *([22]) We have that*

$$\int_{-\infty}^{\infty} \psi_4(x)\, dx = 1. \tag{77}$$

So that $\psi_4(x)$ is a density function on $\mathbb{R}$.

We need

Theorem 35 *([22]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\left\{ \begin{array}{c} j = -\infty \\ : |nx - j| \geq n^{1-\alpha} \end{array} \right.}^{\infty} \psi_4(nx - j) < \frac{1}{2\lambda \left(n^{1-\alpha} - 2\right)^{\lambda}} =: c_4(\alpha, n), \tag{78}$$

where $\lambda \in \mathbb{N}$ is an odd number.

We also need

Theorem 36 *([22]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_4(|nx - k|)} < 2\sqrt[\lambda]{1 + 2^{\lambda}} =: \alpha_4, \tag{79}$$

where λ is an odd number, $\forall\ x \in [a, b]$.

We make

Remark 37 *([22]) (1) We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4\left(nx-k\right)\neq 1,\quad \textit{for at least some } x\in\left[a,b\right]. \tag{80}$$

(2) Let $[a,b]\subset\mathbb{R}$. *For large enough* n *we always obtain* $\lceil na\rceil\leq\lfloor nb\rfloor$. *Also* $a\leq\frac{k}{n}\leq b$, *iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$.

In general it holds that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4\left(nx-k\right)\leq 1. \tag{81}$$

We introduce (see also [26])

$$Z_4\left(x_1,...,x_N\right):=Z_4\left(x\right):=\prod_{i=1}^{N}\psi_4\left(x_i\right),\quad x=\left(x_1,...,x_N\right)\in\mathbb{R}^N,\ N\in\mathbb{N}. \tag{82}$$

It has the properties:

(i) $Z_4\left(x\right)>0,\ \forall\ x\in\mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty}Z_4\left(x-k\right):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}Z_4\left(x_1-k_1,...,x_N-k_N\right)=1, \tag{83}$$

where $k:=\left(k_1,...,k_n\right)\in\mathbb{Z}^N$, $\forall\ x\in\mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty}Z_4\left(nx-k\right)=1, \tag{84}$$

$\forall\ x\in\mathbb{R}^N$; $n\in\mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N}Z_4\left(x\right)dx=1, \tag{85}$$

that is Z_4 is a multivariate density function.

(v) It is clear that

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z_4\left(nx-k\right)<\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}=c_4\left(\beta,n\right), \tag{86}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, λ is odd.

(vi) By Theorem 36 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_4 (nx - k)} < \left(2 \sqrt[\lambda]{1 + 2^{\lambda}} \right)^N =: \gamma_4 (N) , \tag{87}$$

$\forall \ x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $n \in \mathbb{N}$, λ is odd.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_4 (nx - k) \neq 1, \tag{88}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Set

$$\lceil na \rceil := (\lceil na_1 \rceil , ..., \lceil na_N \rceil) ,$$

$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor , ..., \lfloor nb_N \rfloor) ,$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$, $k := (k_1, ..., k_N)$.

Let $f \in C \left(\prod_{i=1}^N [a_i, b_i] \right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We define the multivariate averaged positive linear quasi-interpolation neural network operators ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i] \right)$); $j = 1, 2, 3, 4$:

$$_jA_n (f, x_1, ..., x_N) := \ _jA_n (f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f \left(\frac{k}{n} \right) Z_j (nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_j (nx - k)} = \tag{89}$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f \left(\frac{k_1}{n}, ..., \frac{k_N}{n} \right) \left(\prod_{i=1}^N \psi_j (nx_i - k_i) \right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_j (nx_i - k_i) \right)} .$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $f \in C_B \left(\mathbb{R}^N \right)$ we define ($j = 1, 2, 3, 4$)

$$_jB_n (f, x) := \ _jB_n (f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f \left(\frac{k}{n} \right) Z_j (nx - k) := \tag{90}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f \left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n} \right) \left(\prod_{i=1}^N \psi_j (nx_i - k_i) \right) ,$$

$n \in \mathbb{N}$, $\forall \ x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate full quasi-interpolation neural network operators.

Also for $f \in C_B\left(\mathbb{R}^N\right)$ we define the multivariate Kantorovich type neural network operators $(j = 1, 2, 3, 4)$

$$ {}_jC_n\left(f, x\right) := {}_jC_n\left(f, x_1, ..., x_N\right) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(t\right) dt \right) Z_j\left(nx - k\right) := \tag{91}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f\left(t_1, ..., t_N\right) dt_1 ... dt_N \right)$$

$$\cdot \left(\prod_{i=1}^{N} \psi_j\left(nx_i - k_i\right) \right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operators of quadrature type ${}_jD_n\left(f, x\right)$, $n \in \mathbb{N}$, as follows. Let $\theta = \left(\theta_1, ..., \theta_N\right) \in \mathbb{N}^N$, $\overline{r} = \left(r_1, ..., r_N\right) \in \mathbb{Z}_+^N$, $w_{\overline{r}} = w_{r_1, r_2, ... r_N} \geq 0$, such that $\sum\limits_{\overline{r}=0}^{\theta} w_{\overline{r}} = \sum\limits_{r_1=0}^{\theta_1} \sum\limits_{r_2=0}^{\theta_2} ... \sum\limits_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}\left(f\right) := \delta_{n, k_1, k_2, ..., k_N}\left(f\right) := \sum_{\overline{r}=0}^{\theta} w_{\overline{r}} f\left(\frac{k}{n} + \frac{\overline{r}}{n\theta} \right) :=$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N} \right), \tag{92}$$

where $\frac{\overline{r}}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N} \right)$; $j = 1, 2, 3, 4$.

We put

$$ {}_jD_n\left(f, x\right) := {}_jD_n\left(f, x_1, ..., x_N\right) := \sum_{k=-\infty}^{\infty} \delta_{nk}\left(f\right) Z_j\left(nx - k\right) := \tag{93}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, ..., k_N}\left(f\right) \left(\prod_{i=1}^{N} \psi_j\left(nx_i - k_i\right) \right),$$

$\forall\ x \in \mathbb{R}^N$.

For the next we need, for $f \in C\left(\prod_{i=1}^{N} \left[a_i, b_i\right]\right)$ the first multivariate modulus of continuity

$$\omega_1\left(f, h\right) := \sup_{\substack{x, y \in \prod_{i=1}^{N} \left[a_i, b_i\right] \\ \left\|x - y\right\|_{\infty} \leq h}} \left|f\left(x\right) - f\left(y\right)\right|, \quad h > 0. \tag{94}$$

It holds that

$$\lim_{h \to 0} \omega_1 (f, h) = 0. \tag{95}$$

Similarly it is defined for $f \in C_B \left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$) the $\omega_1 (f, h)$, and it has the property (95), given that $f \in C_U \left(\mathbb{R}^N\right)$ (uniformly continuous functions on $\mathbb{R}^N$).

We mention

Theorem 38 *(see [23], [24], [25], [26]) Let* $f \in C \left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$; $j = 1, 2, 3, 4$. *Then*

1)

$$\left| {}_jA_n (f, x) - f (x) \right| \leq \gamma_j (N) \left[\omega_1 \left(f, \frac{1}{n^\beta} \right) + 2c_j (\beta, n) \|f\|_\infty \right] =: \lambda_{j1}, \tag{96}$$

and

2)

$$\left\| {}_jA_n (f) - f \right\|_\infty \leq \lambda_{j1}. \tag{97}$$

We notice that $\lim_{n \to \infty} {}_jA_n (f) = f$, *pointwise and uniformly.*

In this article we extend Theorem 38 to the fuzzy-random level.

We mention

Theorem 39 *(see [23], [24], [25], [26]) Let* $f \in C_B \left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$; $j = 1, 2, 3, 4$. *Then*

1)

$$\left| {}_jB_n (f, x) - f (x) \right| \leq \omega_1 \left(f, \frac{1}{n^\beta} \right) + 2c_j (\beta, n) \|f\|_\infty =: \lambda_{j2}, \tag{98}$$

2)

$$\left\| {}_jB_n (f) - f \right\|_\infty \leq \lambda_{j2}. \tag{99}$$

Given that $f \in \left(C_U \left(\mathbb{R}^N\right) \cap C_B \left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n \to \infty} {}_jB_n (f) = f$, *uniformly.*

We also need

Theorem 40 *(see [23], [24], [25], [26]) Let* $f \in C_B \left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$; $j = 1, 2, 3, 4$. *Then*

1)

$$\left| {}_jC_n (f, x) - f (x) \right| \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + 2c_j (\beta, n) \|f\|_\infty =: \lambda_{j3}, \tag{100}$$

2)

$$\left\| {}_jC_n (f) - f \right\|_\infty \leq \lambda_{j3}. \tag{101}$$

Given that $f \in \left(C_U \left(\mathbb{R}^N\right) \cap C_B \left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n \to \infty} {}_jC_n (f) = f$, *uniformly.*

We also need

Theorem 41 *(see [23], [24], [25], [26]) Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta}>2$; $j=1,2,3,4$. *Then*

1)

$$\left|{}_jD_n\left(f,x\right)-f\left(x\right)\right| \leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+2c_j\left(\beta,n\right)\|f\|_{\infty}=\lambda_{j3}, \tag{102}$$

2)

$$\left\|{}_jD_n\left(f\right)-f\right\|_{\infty} \leq \lambda_{j3}. \tag{103}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\rightarrow\infty} {}_jD_n\left(f\right)=f$, *uniformly.*

In this article we extend Theorems 39, 40, 41 to the random level.

We are also motivated by [1] - [16] and continuing [17]. For general knowledge on neural networks we recommend [31], [32], [33].

3 Main Results

I) q-mean Approximation by Fuzzy-Random arctangent, algebraic, Gudermannian and generalized symmetric activation functions based Quasi-Interpolation Neural Network Operators

All terms and assumptions here as in Sections 1, 2.

Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $1 \leq q < +\infty$, $n, N \in \mathbb{N}$, $0<\beta<1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $\left(X, \mathcal{B}, P\right)$ probability space, $s \in X$; $j=1,2,3,4$.

We define the following multivariate fuzzy random arctangent, algebraic, Gudermannian and generalized symmetric activation functions based quasi-interpolation linear neural network operators

$$\left({}_jA_n^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right):=\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n},s\right) \odot \frac{Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)}, \tag{104}$$

(see also (89).

We present

Theorem 42 *Let* $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $0<\beta<1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta}>2$, $1 \leq q < +\infty$. *Assume that* $\int_X \left(D^*\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^q P\left(ds\right) < \infty$; $j=1,2,3,4$. *Then*

1)

$$\left(\int_X D^q\left(\left({}_jA_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}} \leq \tag{105}$$

$$\gamma_j(N)\left\{\Omega_1\left(f,\frac{1}{n^\beta}\right)_{L^q}+2c_j(\beta,n)\left(\int_X\left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}\right\}=:\lambda_{j1}^{(\mathcal{FR})},$$

2)

$$\left\|\left(\int_X D^q\left(\left({}_jA_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\right\|_{\infty,\left(\prod_{i=1}^N[a_i,b_i]\right)}\leq\lambda_{j1}^{(\mathcal{FR})}, \tag{106}$$

where $\gamma_j(N)$ as in (25), (45), (65), (87) and $c_j(\beta,n)$ as in (24), (44), (64), (86).

Proof. We notice that

$$D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)\leq D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),\widetilde{o}\right)+D\left(f(\overrightarrow{x},s),\widetilde{o}\right) \tag{107}$$

$$\leq 2D^*\left(f(\cdot,s),\widetilde{o}\right).$$

Hence

$$D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)\leq 2^qD^{*q}\left(f(\cdot,s),\widetilde{o}\right), \tag{108}$$

and

$$\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\leq 2\left(\int_X\left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^qP(ds)\right)^{\frac{1}{q}}. \tag{109}$$

We observe that

$$D\left(\left({}_jA_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)= \tag{110}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z_j(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k)},f(\overrightarrow{x},s)\odot 1\right)=$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z_j(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k)},f(\overrightarrow{x},s)\odot\frac{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k)}\right)= \tag{111}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)},\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\overrightarrow{x},s\right)\odot\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)$$

$$\leq\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right). \tag{112}$$

So that

$$D\left(\left({}_jA_n^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x},s\right)\right)\leq$$

$$\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)= \tag{113}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)+$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right).$$

Hence it holds

$$\left(\int_X D^q\left(\left({}_jA_n^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq \tag{114}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}+$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum\limits_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(nx-k\right)}\right)\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq$$

$$\left(\frac{1}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z_j (nx - k)} \right) \cdot \left\{ \Omega_1^{(\mathcal{F})} \left(f, \frac{1}{n^\beta} \right)_{L^q} + \right. \tag{115}$$

$$\left. 2 \left(\int_X (D^* (f(\cdot, s), \widetilde{o}))^q P(ds) \right)^{\frac{1}{q}} \left(\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z_j (nx - k) \right) \right\}$$

(by (24), (25); (44), (45); (64), (65); (86), (87))

$$\leq \gamma_j (N) \left\{ \Omega_1^{(\mathcal{F})} \left(f, \frac{1}{n^\beta} \right)_{L^q} + 2 c_j (\beta, n) \left(\int_X (D^* (f(\cdot, s), \widetilde{o}))^q P(ds) \right)^{\frac{1}{q}} \right\}. \tag{116}$$

We have proved claim. ■

Conclusion 43 *By Theorem 42 we obtain the pointwise and uniform convergences with rates in the q-mean and D-metric of the operator* ${}_jA_n^{\mathcal{FR}}$ *to the unit operator for* $f \in C_{\mathcal{FR}}^{U_q} \left(\prod_{i=1}^N [a_i, b_i] \right)$, $j = 1, 2, 3, 4$.

II) 1-mean Approximation by Stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions based full Quasi-Interpolation Neural Network Operators

Let $g \in C_{\mathcal{R}}^{U_1} (\mathbb{R}^N)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define

$${}_jB_n^{(\mathcal{R})} (g) (\overrightarrow{x}, s) := \sum_{\overrightarrow{k}=-\infty}^{\infty} g \left(\frac{\overrightarrow{k}}{n}, s \right) Z_j \left(n \overrightarrow{x} - \overrightarrow{k} \right), \; j = 1, 2, 3, 4, \tag{117}$$

(see also (90)).

We give

Theorem 44 *Let* $g \in C_{\mathcal{R}}^{U_1} (\mathbb{R}^N)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta} > 2$, $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$; $j = 1, 2, 3, 4$. *Then*

1)

$$\int_X \left| \left({}_jB_n^{(\mathcal{R})} (g) \right) (\overrightarrow{x}, s) - g (\overrightarrow{x}, s) \right| P(ds) \leq \tag{118}$$

$$\left\{ \Omega_1 \left(g, \frac{1}{n^\beta} \right)_{L^1} + 2 c_j (\beta, n) \|g\|_{\infty, \mathbb{R}^N, X} \right\} =: \mu_{j1}^{(\mathcal{R})},$$

2)

$$\left\| \int_X \left| \left({}_jB_n^{(\mathcal{R})}(g) \right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \right\|_{\infty, \mathbb{R}^N} \leq \mu_{j1}^{(\mathcal{R})}. \tag{119}$$

Proof. Since $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, then

$$\left| g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right| \leq 2\|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{120}$$

Hence

$$\int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right| P(ds) \leq 2\|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{121}$$

We observe that

$$\begin{gathered} \left({}_jB_n^{(\mathcal{R})}(g) \right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) = \\ \sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s \right) Z_j(nx - k) - g(\overrightarrow{x}, s) \sum_{\overrightarrow{k}=-\infty}^{\infty} Z_j(nx - k) = \\ \left(\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right) Z_j(nx - k). \end{gathered} \tag{122}$$

However it holds

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right| Z_j(nx - k) \leq 2\|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{123}$$

Hence

$$\begin{gathered} \left| \left({}_jB_n^{(\mathcal{R})}(g) \right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| \leq \\ \sum_{\overrightarrow{k}=-\infty}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right| Z_j(nx - k) = \\ \sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right| Z_j(nx - k) + \\ \sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s \right) - g(\overrightarrow{x}, s) \right| Z_j(nx - k). \end{gathered} \tag{124}$$

Furthermore it holds

$$\left(\int_X \left| \left({}_jB_n^{(\mathcal{R})}(g) \right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \right) \leq$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty}\left(\int_X\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\right)Z_j\left(nx-k\right)+ \tag{125}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}\left(\int_X\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\right)Z_j\left(nx-k\right)\leq$$

$$\Omega_1\left(g,\frac{1}{n^{\beta}}\right)_{L^1}+2\left\|g\right\|_{\infty,\mathbb{R}^N,X}\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}Z_j\left(nx-k\right)\leq$$

$$\Omega_1\left(g,\frac{1}{n^{\beta}}\right)_{L^1}+2c_j\left(\beta,n\right)\left\|g\right\|_{\infty,\mathbb{R}^N,X},$$

proving the claim. ■

Conclusion 45 *By Theorem 44 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* ${}_jB_n^{(\mathcal{R})}$ *to the unit operator for* $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $j=1,2,3,4$.

III) 1-mean Approximation by Stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions based multivariate Kantorovich type neural network operator

Let $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, with $\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty$, $(X,\mathcal{B},P)$ probability space, $s\in X$.

We define ($j=1,2,3,4$):

$${}_jC_n^{(\mathcal{R})}\left(g\right)\left(\overrightarrow{x},s\right):=\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}g\left(\overrightarrow{t},s\right)d\overrightarrow{t}\right)Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right), \tag{126}$$

(see also (91)).

We present

Theorem 46 *Let* $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, *with* $n^{1-\beta}>2$; $j=1,2,3,4$, $\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty$. *Then*

1)

$$\int_X\left|\left({}_jC_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\leq$$

$$\left[\Omega_1\left(g,\frac{1}{n}+\frac{1}{n^{\beta}}\right)_{L^1}+2c_j\left(\beta,n\right)\left\|g\right\|_{\infty,\mathbb{R}^N,X}\right]=:\gamma_{j1}^{(\mathcal{R})}, \tag{127}$$

2)

$$\left\|\int_X\left|\left({}_jC_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\right\|_{\infty,\mathbb{R}^N}\leq\gamma_{j1}^{(\mathcal{R})}. \tag{128}$$

Proof. Since $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, then

$$\left|n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t} - g\left(\overrightarrow{x}, s\right)\right| = \left|n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left(g\left(\overrightarrow{t}, s\right) - g\left(\overrightarrow{x}, s\right)\right) d\overrightarrow{t}\right| \leq$$

$$n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left|g\left(\overrightarrow{t}, s\right) - g\left(\overrightarrow{x}, s\right)\right| d\overrightarrow{t} \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{129}$$

Hence

$$\int_X \left|n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t} - g\left(\overrightarrow{x}, s\right)\right| P(ds) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{130}$$

We observe that

$$\left({}_jC_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t}\right) Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) - g\left(\overrightarrow{x}, s\right) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t}\right) Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) - g\left(\overrightarrow{x}, s\right) \sum_{\overrightarrow{k}=-\infty}^{\infty} Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) = \tag{131}$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left[\left(n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t}\right) - g\left(\overrightarrow{x}, s\right)\right] Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left(g\left(\overrightarrow{t}, s\right) - g\left(\overrightarrow{x}, s\right)\right) d\overrightarrow{t}\right] Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right).$$

However it holds

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left|g\left(\overrightarrow{t}, s\right) - g\left(\overrightarrow{x}, s\right)\right| d\overrightarrow{t}\right] Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{132}$$

Hence

$$\left|\left({}_jC_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right)\right| \leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left|g\left(\overrightarrow{t}, s\right) - g\left(\overrightarrow{x}, s\right)\right| d\overrightarrow{t}\right] Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) = \tag{133}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left[n^{N} \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left| g\left(\overrightarrow{t}, s \right) - g\left(\overrightarrow{x}, s \right) \right| d\overrightarrow{t} \right] Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right) + \tag{134}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left[n^{N} \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left| g\left(\overrightarrow{t}, s \right) - g\left(\overrightarrow{x}, s \right) \right| d\overrightarrow{t} \right] Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right) =$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left[n^{N} \int_{0}^{\frac{1}{n}} \left| g\left(\overrightarrow{t} + \frac{\overrightarrow{k}}{n}, s \right) - g\left(\overrightarrow{x}, s \right) \right| d\overrightarrow{t} \right] Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right) +$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left[n^{N} \int_{0}^{\frac{1}{n}} \left| g\left(\overrightarrow{t} + \frac{\overrightarrow{k}}{n}, s \right) - g\left(\overrightarrow{x}, s \right) \right| d\overrightarrow{t} \right] Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right). \tag{135}$$

Furthermore it holds

$$\left(\int_{X} \left| \left({}_{j}C_{n}^{(\mathcal{R})}\left(g \right) \right) \left(\overrightarrow{x}, s \right) - g\left(\overrightarrow{x}, s \right) \right| P\left(ds \right) \right) \underset{\text{(by Fubini's theorem)}}{\leq}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left[n^{N} \int_{0}^{\frac{1}{n}} \left(\int_{X} \left| g\left(\overrightarrow{t} + \frac{\overrightarrow{k}}{n}, s \right) - g\left(\overrightarrow{x}, s \right) \right| P\left(ds \right) \right) d\overrightarrow{t} \right] Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right) +$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left[n^{N} \int_{0}^{\frac{1}{n}} \left(\int_{X} \left| g\left(\overrightarrow{t} + \frac{\overrightarrow{k}}{n}, s \right) - g\left(\overrightarrow{x}, s \right) \right| P\left(ds \right) \right) d\overrightarrow{t} \right] Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right) \leq \tag{136}$$

$$\Omega_{1}\left(g, \frac{1}{n} + \frac{1}{n^{\beta}} \right)_{L^{1}} + 2\left\| g \right\|_{\infty, \mathbb{R}^{N}, X} \sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n}-\overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z_{j}\left(n\overrightarrow{x} - \overrightarrow{k} \right) \leq$$

$$\Omega_{1}\left(g, \frac{1}{n} + \frac{1}{n^{\beta}} \right)_{L^{1}} + 2c_{j}\left(\beta, n \right) \left\| g \right\|_{\infty, \mathbb{R}^{N}, X}, \tag{137}$$

proving the claim. ■

Conclusion 47 *By Theorem 46 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* ${}_{j}C_{n}^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_{1}}\left(\mathbb{R}^{N} \right)$, $j = 1, 2, 3, 4$.

IV) 1**-mean Approximation by Stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions based multivariate quadrature type neural network operator**

Let $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$, $j=1,2,3,4$.

We define

$$
{}_jD_n^{(\mathcal{R})}(g)(\overrightarrow{x}, s):=\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(\delta_{n\overrightarrow{k}}(g)\right)(s)\, Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right), \tag{138}
$$

where

$$
\left(\delta_{n\overrightarrow{k}}(g)\right)(s):=\sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\, g\left(\frac{\overrightarrow{k}}{n}+\frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right), \tag{139}
$$

(see also (92), (93)).

We finally give

Theorem 48 *Let* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta}>2$; $j=1,2,3,4$, $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$. *Then*

1)

$$
\int_X\left|\left({}_jD_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s)-g(\overrightarrow{x}, s)\right| P(ds) \leq
$$

$$
\left\{\Omega_1\left(g, \frac{1}{n}+\frac{1}{n^\beta}\right)_{L^1}+2 c_j(\beta, n)\|g\|_{\infty,\mathbb{R}^N,X}\right\}=: \gamma_{j1}^{(\mathcal{R})}, \tag{140}
$$

2)

$$
\left\|\int_X\left|\left({}_jD_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s)-g(\overrightarrow{x}, s)\right| P(ds)\right\|_{\infty,\mathbb{R}^N} \leq \gamma_{j1}^{(\mathcal{R})}. \tag{141}
$$

Proof. Notice that

$$
\left|\left(\delta_{n\overrightarrow{k}}(g)\right)(s)-g(\overrightarrow{x}, s)\right|=
$$

$$
\left|\sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\left(g\left(\frac{\overrightarrow{k}}{n}+\frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right)-g(\overrightarrow{x}, s)\right)\right| \leq
$$

$$
\sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\left|g\left(\frac{\overrightarrow{k}}{n}+\frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right)-g(\overrightarrow{x}, s)\right| \leq 2\|g\|_{\infty,\mathbb{R}^N,X}<\infty. \tag{142}
$$

Hence

$$
\int_X\left|\left(\delta_{n\overrightarrow{k}}(g)\right)(s)-g(\overrightarrow{x}, s)\right| P(ds) \leq 2\|g\|_{\infty,\mathbb{R}^N,X}<\infty. \tag{143}
$$

We observe that

$$
\left({}_jD_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s)-g(\overrightarrow{x}, s)=
$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(\delta_{n\overrightarrow{k}}\left(g\right)\right)\left(s\right)Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)-g\left(\overrightarrow{x},s\right)=$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(\left(\delta_{n\overrightarrow{k}}\left(g\right)\right)\left(s\right)-g\left(\overrightarrow{x},s\right)\right)Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right). \tag{144}$$

Thus

$$\left|{}_jD_n^{(\mathcal{R})}\left(g\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|\leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left|\left(\delta_{n\overrightarrow{k}}\left(g\right)\right)\left(s\right)-g\left(\overrightarrow{x},s\right)\right|Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)\leq 2\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty. \tag{145}$$

Hence it holds

$$\left|\left({}_jD_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|\leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left|\left(\delta_{n\overrightarrow{k}}\left(g\right)\right)\left(s\right)-g\left(\overrightarrow{x},s\right)\right|Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)=$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty}\left|\left(\delta_{n\overrightarrow{k}}\left(g\right)\right)\left(s\right)-g\left(\overrightarrow{x},s\right)\right|Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)+$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}\left|\left(\delta_{n\overrightarrow{k}}\left(g\right)\right)\left(s\right)-g\left(\overrightarrow{x},s\right)\right|Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right). \tag{146}$$

Furthermore we derive

$$\left(\int_X\left|\left({}_jD_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\right)\leq$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty}\ \sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}}w_{\overrightarrow{r}}\left(\int_X\left|g\left(\frac{\overrightarrow{k}}{n}+\frac{\overrightarrow{r}}{n\overrightarrow{\theta}},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\right)Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)$$

$$+\left(\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)\right)2\left\|g\right\|_{\infty,\mathbb{R}^N,X}\leq \tag{147}$$

$$\Omega_1\left(g,\frac{1}{n}+\frac{1}{n^{\beta}}\right)_{L^1}+2c_j\left(\beta,n\right)\left\|g\right\|_{\infty,\mathbb{R}^N,X}, \tag{148}$$

proving the claim. ■

Conclusion 49 *From Theorem 48 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* ${}_jD_n^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $j = 1, 2, 3, 4$.

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