BASIC PROPERTIES OF RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp \left\langle A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}} x, x \right\rangle .$$

In this paper we show, among others, that

$$\left(\frac{\langle Ax, x\rangle}{\langle AB^{-1}Ax, x\rangle} \right)^{\langle Ax, x\rangle} \leq D_x(A|B) \leq \left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle} \right)^{\langle Ax, x\rangle}$$

for all A, $B > 0$ and $x \in H$ with $\|x\| = 1$. Several other properties of $D_x(\cdot|\cdot)$ are also provided.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [9].

For each unit vector $x \in H$, see also [12], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;

(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;

(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;

(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;

(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

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We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [9] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x \rangle - \Delta_x(A) \leq L(m, M) \left[\ln L(m, M) + \frac{M \ln m - m \ln M}{M - m} - 1 \right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [18]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1, \infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1, \infty)$.

In [10], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x \rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A \ln Ax, x \rangle) = \exp \langle \eta(A) x, x \rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp(-\langle tA \ln(tA) x, x \rangle) \\
&= \exp(-\langle tA (\ln t + \ln A) x, x \rangle) = \exp(-\langle (tA \ln t + tA \ln A) x, x \rangle) \\
&= \exp(-\langle Ax, x \rangle t \ln t) \exp(-t \langle A \ln Ax, x \rangle) \\
&= \exp \ln\left(t^{-\langle Ax, x \rangle t}\right) [\exp(-\langle A \ln Ax, x \rangle)]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax, x \rangle} [\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In the recent paper [3] we showed among others that, if $A,\ B>0$, then for all $x\in H, \|x\|=1$ and $t\in[0,1]$,

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A>0$ and $x\in H, \|x\|=1$.

Definition 1. *For positive invertible operators $A,\ B$ and $x\in H$ with $\|x\|=1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B):=\exp\langle S(A|B)x,x\rangle=\exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle.$$

We observe that for $A>0$,

$$D_x(A|1_H)=\exp\langle S(A|1_H)x,x\rangle=\exp(-\langle A\ln Ax,x\rangle)=\eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B>0$,

$$D_x(1_H|B):=\exp\langle S(1_H|B)x,x\rangle=\exp\langle\ln Bx,x\rangle=\Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant.*

Motivated by the above results, in this paper we show, among others, that

$$\left(\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}\right)^{\langle Ax,x\rangle}\leq D_x(A|B)\leq\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}$$

for all $A,\ B>0$ and $x\in H$ with $\|x\|=1$. Several other properties of $D_x(\cdot|\cdot)$ are also provided.

2. Relative Entropic Normalized Determinant

Kamei and Fujii [7], [8] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{2.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [16].

In general, we can define for positive operators $A,\ B$

$$S(A|B):=s\text{-}\lim_{\varepsilon\to0+}S(A+\varepsilon1_H|B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t)=-t\ln t$, the *operator entropy* has the following expression:

$$\eta(A)=-A\ln A=S(A|1_H)\geq0$$

for positive contraction A. This shows that the relative operator entropy (2.1) is a relative version of the operator entropy.

For $A=1_H$ in (2.1) we have

$$S(1_H|B)=\ln B$$

for positive contraction B.

Following [11, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} = B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \tag{2.2}$$

(ii) We have the inequalities

$$S(A|B) \leq A(\ln\|B\| - \ln A) \text{ and } S(A|B) \leq B - A; \tag{2.3}$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^*S(A|B)T \leq S(T^*AT|T^*BT).$$

(viii) The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \geq tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [1], [5], [13], [14], [15] and [17].

Observe that, if we replace in (2.2) B with A, then we get

$$\begin{aligned} S(B|A) &= A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\ &= A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} = -S(B|A) \tag{2.4}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

In [19], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t}, \tag{2.5}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu \in [0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν.

For $B = 1_H$ we have

$$A\natural_\nu 1_H = A^{1-\nu}$$

while for $A = 1_H$ we get

$$1_H\natural_\nu B = B^\nu$$

for any real number ν.

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [4]) by

$$T_t(A|B) := \frac{A\natural_t B - A}{t}.$$

We then have

$$T_t(A|1_H) := \frac{A\natural_t 1_H - A}{t} = \frac{A^{1-t} - A}{t},\ t > 0$$

and

$$T_t(1_H|B) := \frac{B^t - 1_H}{t},\ t > 0$$

for A, $B > 0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [7] for $0 < t \le 1$. However, it hods for any $t > 0$.

Theorem 1. *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$T_t(A|B)(A\natural_t B)^{-1} A \le S(A|B) \le T_t(A|B). \tag{2.6}$$

In particular, we have for $t = 1$ that

$$\left(1_H - AB^{-1}\right) A \le S(A|B) \le B - A,\ [7] \tag{2.7}$$

and for $t = 2$ that

$$\frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) A \le S(A|B) \le \frac{1}{2}\left(BA^{-1}B - A\right). \tag{2.8}$$

The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}(A|B) := 2(A\sharp B - A)$$

and

$$T_{1/2}(A|B)\left(A\natural_{1/2}B\right)^{-1} A = 2\left(1_H - A(A\sharp B)^{-1}\right) A,$$

hence by (2.6) we get

$$2\left(1_H - A(A\sharp B)^{-1}\right) A \le S(A|B) \le 2(A\sharp B - A) \le B - A. \tag{2.9}$$

We have the following fundamental properties for the relative entropic normalized determinant:

Proposition 1. *Assume that A, $B > 0$ and $x \in H$ with $\|x\| = 1$.*

(1) *We have the upper bound*

$$D_x(A|B) \le \frac{\exp\langle Bx, x\rangle}{\exp\langle Ax, x\rangle};$$

(2) *For any C, D positive invertible operators we have that*

$$D_x(A + B|C + D) \ge D_x(A|C)\, D_x(B|D); \tag{2.10}$$

(3) *If* $B \leq C$ *then*
$$D_x(A|B) \leq D_x(A|C);$$
(4) *If* $B_n \downarrow B$ *then*
$$D_x(A|B_n) \downarrow D_x(A|B);$$
(5) *For* $\alpha > 0$ *we have*
$$D_x(\alpha A|\alpha B) = [D_x(A|B)]^{\alpha}.$$

The proof follows by the properties "(ii)-(iii)" above.

Corollary 1. *For* A, $B > 0$, α, $\beta > 0$ *and* $x \in H$ *with* $\|x\| = 1$, *we have*
$$\frac{\eta_x(A+B)}{\eta_x(A)\eta_x(B)} \geq \frac{\alpha^{\langle Ax,x\rangle}\beta^{\langle Bx,x\rangle}}{(\alpha+\beta)^{\langle (A+B)x,x\rangle}}. \tag{2.11}$$
In particular, for $\alpha = \beta = 1$, *we get*
$$\frac{\eta_x(A+B)}{\eta_x(A)\eta_x(B)} \geq \frac{1}{2^{\langle (A+B)x,x\rangle}}. \tag{2.12}$$

Proof. Observe that
$$\begin{aligned} D_x(A|\alpha 1_H) &= \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}\alpha 1_H A^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle \\ &= \exp\left\langle A^{\frac{1}{2}}\left(\ln \alpha 1_H - \ln A\right)A^{\frac{1}{2}}x, x\right\rangle \\ &= \exp\left(\langle Ax, x\rangle \ln\alpha - \langle A\ln Ax, x\rangle\right) = \alpha^{\langle Ax,x\rangle}\eta_x(A). \end{aligned}$$
Then by (2.10) for $C = \alpha 1_H$ and $D = \beta 1_H$ we have
$$D_x(A+B|(\alpha+\beta)1_H) \geq D_x(A|\alpha 1_H)D_x(B|\beta 1_H),$$
namely
$$(\alpha+\beta)^{\langle (A+B)x,x\rangle}\eta_x(A+B) \geq \alpha^{\langle Ax,x\rangle}\eta_x(A)\beta^{\langle Bx,x\rangle}\eta_x(B)$$
and the inequality (2.11) is obtained. □

Also, we have:

Corollary 2. *For* C, $D > 0$, γ, $\delta > 0$ *and* $x \in H$ *with* $\|x\| = 1$, *we have*
$$\frac{[\Delta_x(C+D)]^{\gamma+\delta}}{[\Delta_x(C)]^{\gamma}[\Delta_x(D)]^{\delta}} \geq \frac{(\gamma+\delta)^{\gamma+\delta}}{\gamma^{\gamma}\delta^{\delta}}. \tag{2.13}$$
In particular, for $\gamma = \delta = 1$, *we get*
$$\frac{[\Delta_x(C+D)]^2}{\Delta_x(C)\Delta_x(D)} \geq 4. \tag{2.14}$$

Proof. Observe that
$$\begin{aligned} D_x(\gamma 1_H|C) &= \exp\left\langle (\gamma 1_H)^{\frac{1}{2}}\left(\ln\left((\gamma 1_H)^{-\frac{1}{2}}C(\gamma 1_H)^{-\frac{1}{2}}\right)\right)(\gamma 1_H)^{\frac{1}{2}}x, x\right\rangle \\ &= \exp\langle \gamma(\ln C - \ln\gamma)x, x\rangle = \exp\left(\gamma\langle \ln Cx, x\rangle - \ln(\gamma^{\gamma})\right) \\ &= \frac{\exp(\gamma\langle \ln Cx, x\rangle)}{\exp\ln(\gamma^{\gamma})} = \left(\frac{\Delta_x(C)}{\gamma}\right)^{\gamma}. \end{aligned}$$
By (2.10) we have
$$D_x((\gamma+\delta)1_H|C+D) \geq D_x(\gamma 1_H|C)D_x(\delta 1_H|D),$$

namely

$$\left(\frac{\Delta_x(C+D)}{\gamma+\delta}\right)^{\gamma+\delta} \geq \left(\frac{\Delta_x(C)}{\gamma}\right)^{\gamma}\left(\frac{\Delta_x(D)}{\delta}\right)^{\delta}$$

□

Proposition 2. *Assume that $A,\ B>0$ and $x\in H$ with $\|x\|=1$.*

(a) *We have*

$$D_x\left(A|B\right) \leq \|B\|^{\langle Ax,x\rangle}\,\eta_x(A) \tag{2.15}$$

(aa) *For every operator T with $Tx\neq 0$, we have*

$$\left[D_{\frac{Tx}{\|Tx\|}}\left(A|B\right)\right]^{\|Tx\|^2} \leq D_x\left(T^*AT|T^*BT\right). \tag{2.16}$$

(aaa) *For every $C,\ D>0$*

$$D_x\left(tA+(1-t)\,B|tC+(1-t)\,D\right) \geq \left[D_x\left(A|C\right)\right]^t\left[D_x\left(B|D\right)\right]^{1-t} \tag{2.17}$$

for all $t\in[0,1]$.

Proof. a. By taking the inner product over $x\in H$ with $\|x\|=1$ in (ii) we get

$$\begin{aligned}
D_x\left(A|B\right) &= \exp\left\langle S\left(A|B\right)x,x\right\rangle \leq \exp\left\langle\left(\ln\|B\|\,A - A\ln A\right)x,x\right\rangle\\
&= \exp\left(\ln\|B\|\left\langle Ax,x\right\rangle - \left\langle A\ln Ax,x\right\rangle\right)\\
&= \exp\left(\ln\|B\|^{\langle Ax,x\rangle}\right)\exp\left(-\left\langle A\ln Ax,x\right\rangle\right)\\
&= \|B\|^{\langle Ax,x\rangle}\,\eta_x(A)
\end{aligned}$$

and the statement is proved.

aa. If we take the inner product over $x\in H$ with $\|x\|=1$ in (vii) then we get

$$\exp\left\langle T^*S\left(A|B\right)Tx,x\right\rangle \leq \exp\left\langle S\left(T^*AT|T^*BT\right)x,x\right\rangle = D_x\left(T^*AT|T^*BT\right).$$

Also, if $Tx\neq 0$,

$$\begin{aligned}
\exp\left\langle T^*S\left(A|B\right)Tx,x\right\rangle &= \exp\left\langle S\left(A|B\right)Tx,Tx\right\rangle\\
&= \exp\left\langle \|Tx\|^2 S\left(A|B\right)\frac{Tx}{\|Tx\|},\frac{Tx}{\|Tx\|}\right\rangle\\
&= \left(\exp\left\langle S\left(A|B\right)\frac{Tx}{\|Tx\|},\frac{Tx}{\|Tx\|}\right\rangle\right)^{\|Tx\|^2}\\
&= \left[D_{\frac{Tx}{\|Tx\|}}\left(A|B\right)\right]^{\|Tx\|^2},
\end{aligned}$$

which proves the statement.

aaa. If we take the inner product over $x\in H$ with $\|x\|=1$ in (viii), then we get for all $t\in[0,1]$ that

$$\begin{aligned}
&D_x\left(tA+(1-t)\,B|tC+(1-t)\,D\right)\\
&= \exp\left\langle S\left(tA+(1-t)\,B|tC+(1-t)\,D\right)x,x\right\rangle\\
&\geq \exp\left\langle\left[tS\left(A|C\right)+(1-t)\,S\left(B|D\right)\right]x,x\right\rangle\\
&= \exp\left[t\left\langle S\left(A|C\right)x,x\right\rangle + (1-t)\left\langle S\left(B|D\right)x,x\right\rangle\right]\\
&= \left(\exp\left\langle S\left(A|C\right)x,x\right\rangle\right)^t\left[\exp\left\langle S\left(B|D\right)x,x\right\rangle\right]^{1-t}\\
&= \left[D_x\left(A|C\right)\right]^t\left[D_x\left(B|D\right)\right]^{1-t}
\end{aligned}$$

and the statement is proved. □

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 3. *With the assumptions of Proposition 2,*

$$\int_0^1 D_x(tA + (1-t)B|tC + (1-t)D)dt \geq L(D_x(A|B), D_x(C|D)). \tag{2.18}$$

and

$$D_x\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq \int_0^1 [D_x((1-t)A + tB|(1-t)C + tD)]^{1/2} \times [D_x(tA + (1-t)B|tC + (1-t)D)]^{1/2} dt. \tag{2.19}$$

Proof. If we take the integral over $t \in [0,1]$ in (2.17), then we get

$$\begin{aligned}\int_0^1 D_x(tA + (1-t)B|tC + (1-t)D)dt &\geq \int_0^1 [D_x(A|C)]^t [D_x(B|D)]^{1-t} dt \\ &= L(D_x(A|C), D_x(B|D))\end{aligned}$$

for all A, B, C, $D > 0$, which proves (2.18).

We get from (2.17) for $t = 1/2$ that

$$D_x\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq [D_x(A|C)]^{1/2} [D_x(B|D)]^{1/2}.$$

If we replace A by $(1-t)A + tB$, B by $tA + (1-t)B$, C by $(1-t)C + tD$ and D by $tC + (1-t)D$ we obtain

$$\begin{aligned}&D_x\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \\ &\geq [D_x((1-t)A + tB|(1-t)C + tD)]^{1/2} \\ &\times [D_x(tA + (1-t)B|tC + (1-t)D)]^{1/2}.\end{aligned}$$

By taking the integral, we derive the desired result (2.19). □

By the use of Theorem 1 we can also state:

Proposition 3. *Assume that A, $B > 0$ and $x \in H$ with $\|x\| = 1$. Then for any $t > 0$ we have*

$$\exp\left\langle T_t(A|B)(A\sharp_t B)^{-1} Ax, x\right\rangle \leq D_x(A|B) \leq \exp\langle T_t(A|B)x, x\rangle. \tag{2.20}$$

In particular, we have for $t = 1$ that

$$\frac{\exp\langle Ax, x\rangle}{\exp\langle AB^{-1}Ax, x\rangle} \leq D_x(A|B) \leq \frac{\exp\langle Bx, x\rangle}{\exp\langle Ax, x\rangle} \tag{2.21}$$

and for $t=2$ that

$$\left(\frac{\exp\langle Ax,x\rangle}{\left\langle\left(AB^{-1}\right)^{2}Ax,x\right\rangle}\right)^{\frac{1}{2}}\le D_{x}\left(A|B\right)\le\left(\frac{\exp\left\langle BA^{-1}Bx,x\right\rangle}{\exp\langle Ax,x\rangle}\right)^{\frac{1}{2}}. \tag{2.22}$$

We have the following bounds for the *normalized entropic determinant.*

Corollary 4. *Assume that $A>0$ and $x\in H$ with $\|x\|=1$. If $\alpha,\ t>0$, then*

$$\begin{aligned}&\alpha^{-\langle Ax,x\rangle}\exp\left\langle\frac{A-\alpha^{-t}A^{t+1}}{t}x,x\right\rangle\\&\le\eta_{x}(A)\\&\le\alpha^{-\langle Ax,x\rangle}\exp\left\langle\frac{\alpha^{t}A^{1-t}-A}{t}x,x\right\rangle.\end{aligned}\tag{2.23}$$

In particular, for $\alpha=1$, we get

$$\exp\left\langle\frac{A-A^{t+1}}{t}x,x\right\rangle\le\eta_{x}(A)\le\exp\left\langle\frac{A^{1-t}-A}{t}x,x\right\rangle, \tag{2.24}$$

for all $t>0$.

For $t=1$, we get

$$\begin{aligned}&\alpha^{-\langle Ax,x\rangle}\exp\left\langle\left(A-\alpha^{-1}A^{2}\right)x,x\right\rangle\\&\le\eta_{x}(A)\\&\le\alpha^{-\langle Ax,x\rangle}\exp\left\langle\left(\alpha 1_{H}-A\right)x,x\right\rangle,\end{aligned}\tag{2.25}$$

for all $\alpha>0$.

Also, for $\alpha=t=1$, we obtain

$$\exp\left\langle\left(A-A^{2}\right)x,x\right\rangle\le\eta_{x}(A)\le\exp\left\langle\left(1_{H}-A\right)x,x\right\rangle. \tag{2.26}$$

Proof. If we take $B=\alpha 1_{H}$ in (2.20), we get

$$\begin{aligned}\exp\left\langle T_{t}\left(A|\alpha 1_{H}\right)\left(A\sharp_{t}\left(\alpha 1_{H}\right)\right)^{-1}Ax,x\right\rangle&\le D_{x}\left(A|\alpha 1_{H}\right)\\&\le\exp\left\langle T_{t}\left(A|\alpha 1_{H}\right)x,x\right\rangle.\end{aligned}\tag{2.27}$$

Observe that

$$A\sharp_{t}\left(\alpha 1_{H}\right)=A^{1/2}\left(A^{-1/2}\left(\alpha 1_{H}\right)A^{-1/2}\right)^{t}A^{1/2}=\alpha^{t}A^{1-t}$$

and

$$T_{t}\left(A|\alpha 1_{H}\right)=\frac{A\sharp_{t}\left(\alpha 1_{H}\right)-A}{t}=\frac{\alpha^{t}A^{1-t}-A}{t}.$$

Also

$$\begin{aligned}T_{t}\left(A|\alpha 1_{H}\right)\left(A\sharp_{t}\left(\alpha 1_{H}\right)\right)^{-1}A&=\frac{\alpha^{t}A^{1-t}-A}{t}\left(\alpha^{t}A^{1-t}\right)^{-1}A\\&=\frac{A-A\left(\alpha^{t}A^{1-t}\right)^{-1}A}{t}\\&=\frac{A-\alpha^{-t}A^{t+1}}{t}.\end{aligned}$$

Then by (2.27) we get

$$\exp\left\langle \frac{A-\alpha^{-t}A^{t+1}}{t}x,x\right\rangle \leq \alpha^{\langle Ax,x\rangle}\eta_x(A) \leq \exp\left\langle \frac{\alpha^{t}A^{1-t}-A}{t}x,x\right\rangle$$

and the inequality (2.23) is obtained. □

We also have the following bounds for the *normalized determinant.*

Corollary 5. *Assume that $B>0$ and $x\in H$ with $\|x\|=1$. If $\beta,\ t>0$, then*

$$\beta\exp\left\langle \frac{1_H-\beta^{t}B^{-t}}{t}x,x\right\rangle \leq \Delta_x(B) \leq \beta\exp\left\langle \frac{\beta^{-t}B^{t}-1_H}{t}x,x\right\rangle. \tag{2.28}$$

In particular, for $\beta=1$, we get

$$\exp\left\langle \frac{1_H-B^{-t}}{t}x,x\right\rangle \leq \Delta_x(B) \leq \exp\left\langle \frac{B^{t}-1_H}{t}x,x\right\rangle, \tag{2.29}$$

for all $t>0$.

For $t=1$, we get

$$\beta\exp\left\langle \left(1_H-\beta B^{-1}\right)x,x\right\rangle \leq \Delta_x(B) \leq \beta\exp\left\langle \left(\beta^{-1}B-1_H\right)x,x\right\rangle, \tag{2.30}$$

for all $\beta>0$.

Also, for $\beta=t=1$, we obtain

$$\exp\left\langle \left(1_H-B^{-1}\right)x,x\right\rangle \leq \Delta_x(B) \leq \exp\left\langle \left(B-1_H\right)x,x\right\rangle. \tag{2.31}$$

Proof. We have from (2.20) for $A=\beta 1_H$ that

$$\begin{aligned}\exp\left\langle T_t\left(\beta 1_H|B\right)\left(\left(\beta 1_H\right)\sharp_t B\right)^{-1}\left(\beta 1_H\right)x,x\right\rangle &\leq D_x\left(\beta 1_H|B\right)\\ &\leq \exp\left\langle T_t\left(\beta 1_H|B\right)x,x\right\rangle.\end{aligned} \tag{2.32}$$

Observe that

$$\left(\beta 1_H\right)\sharp_t B = \left(\beta 1_H\right)^{1/2}\left(\left(\beta 1_H\right)^{-1/2}B\left(\beta 1_H\right)^{-1/2}\right)^{t}\left(\beta 1_H\right)^{1/2} = \beta^{1-t}B^{t},$$

and

$$T_t\left(\left(\beta 1_H\right)|B\right) := \frac{\left(\beta 1_H\right)\sharp_t B-\beta 1_H}{t} = \frac{\beta^{1-t}B^{t}-\beta 1_H}{t}.$$

Also,

$$\begin{aligned}T_t\left(\beta 1_H|B\right)\left(\left(\beta 1_H\right)\sharp_t B\right)^{-1}\left(\beta 1_H\right) &= \frac{\beta^{1-t}B^{t}-\beta 1_H}{t}\left(\beta^{1-t}B^{t}\right)^{-1}\beta\\ &= \frac{\beta-\beta\left(\beta^{1-t}B^{t}\right)^{-1}\beta}{t}\\ &= \frac{\beta-\beta^{t+1}B^{-t}}{t}.\end{aligned}$$

Then by (2.32) we get

$$\exp\left\langle \frac{\beta 1_H-\beta^{t+1}B^{-t}}{t}x,x\right\rangle \leq \left(\frac{\Delta_x(B)}{\beta}\right)^{\beta} \leq \exp\left\langle \frac{\beta^{1-t}B^{t}-\beta 1_H}{t}x,x\right\rangle.$$

By taking the power $1/\beta$ we get

$$\exp\left\langle \frac{\beta 1_H-\beta^{t+1}B^{-t}}{\beta t}x,x\right\rangle \leq \frac{\Delta_x(B)}{\beta} \leq \exp\left\langle \frac{\beta^{1-t}B^{t}-\beta 1_H}{\beta t}x,x\right\rangle,$$

which is equivalent to (2.28). □

3. Several Bounds

We have the following bounds for the relative entropic normalized determinant:

Theorem 2. *Assume that* A, $B>0$ *and* $x\in H$ *with* $\|x\|=1$. *Then for any* $s>0$ *we have*

$$s^{\langle Ax,x\rangle}\exp\left(\langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle\right) \le D_x(A|B) \le s^{\langle Ax,x\rangle}\exp\left(\frac{\langle Bx,x\rangle - s\langle Ax,x\rangle}{s}\right). \tag{3.1}$$

The best lower bound in the first inequality is

$$\left(\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}\right)^{\langle Ax,x\rangle} \le D_x(A|B), \tag{3.2}$$

while the best upper bound in the second inequality is

$$D_x(A|B) \le \left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}. \tag{3.3}$$

Proof. We use the gradient inequality for differentiable convex functions f on the open interval

$$f'(s)(t-s) \ge f(t)-f(s) \ge f'(t)(t-s)$$

for all t, $s\in I$.

If we write this inequality for the function ln on $(0,\infty)$, then we get

$$\frac{t}{s}-1 \ge \ln t - \ln s \ge 1-\frac{s}{t}$$

for all t, $s\in(0,\infty)$.

Using the functional calculus for positive operator $T>0$, we get

$$\frac{1}{s}T - 1_H \ge \ln T - \ln s 1_H \ge 1_H - sT^{-1}.$$

for all $s\in(0,\infty)$.

If we take $T=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}>0$, then we get

$$\frac{1}{s}A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_H \ge \ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \ln s 1_H \ge 1_H - sA^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$$

for all $s\in(0,\infty)$.

If we multiply both sides by $A^{\frac{1}{2}}>0$, then we get

$$\frac{1}{s}B - A \ge A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}} - (\ln s)A \ge A - sAB^{-1}A$$

for all $s\in(0,\infty)$.

Now, if we take the inner product for $x\in H$ with $\|x\|=1$, then we get

$$\frac{1}{s}\langle Bx,x\rangle - \langle Ax,x\rangle \ge \left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle - (\ln s)\langle Ax,x\rangle \ge \langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle$$

for all $s\in(0,\infty)$.

By taking the exponential, we derive

$$\exp\left(\frac{\langle Bx,x\rangle - s\langle Ax,x\rangle}{s}\right) \geq \frac{\exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle}{\exp\left[(\ln s)\langle Ax,x\rangle\right]}$$
$$\geq \exp\left(\langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle\right)$$

for all $s\in(0,\infty)$, which is equivalent to (3.1).

Now, consider the function

$$f(s) := s^{\langle Ax,x\rangle}\exp\left(\langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle\right),\ s\in(0,\infty).$$

We have

$$\begin{aligned} f'(s) &= \langle Ax,x\rangle s^{\langle Ax,x\rangle-1}\exp\left(\langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle\right)\\ &\quad - \left\langle AB^{-1}Ax,x\right\rangle s^{\langle Ax,x\rangle}\exp\left(\langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle\right)\\ &= s^{\langle Ax,x\rangle-1}\exp\left(\langle Ax,x\rangle - s\left\langle AB^{-1}Ax,x\right\rangle\right)\\ &\quad\times\left(\langle Ax,x\rangle - \left\langle AB^{-1}Ax,x\right\rangle s\right). \end{aligned}$$

We observe that the function f is increasing on $\left(0,\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}\right)$ and decreasing on $\left(\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle},\infty\right)$. Therefore

$$\sup_{s\in(0,\infty)} f(s) = f\left(\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}\right) = \left(\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}\right)^{\langle Ax,x\rangle},$$

which gives the best lower bound in (3.1).

Now, consider the function

$$g(s) := s^{\langle Ax,x\rangle}\exp\left(\frac{\langle Bx,x\rangle}{s} - \langle Ax,x\rangle\right),\ s\in(0,\infty).$$

We have

$$\begin{aligned} g'(s) &:= \langle Ax,x\rangle s^{\langle Ax,x\rangle-1}\exp\left(\frac{\langle Bx,x\rangle}{s} - \langle Ax,x\rangle\right)\\ &\quad + s^{\langle Ax,x\rangle}\exp\left(\frac{\langle Bx,x\rangle}{s} - \langle Ax,x\rangle\right)\left(-\frac{\langle Bx,x\rangle}{s^2}\right)\\ &= s^{\langle Ax,x\rangle-1}\exp\left(\frac{\langle Bx,x\rangle}{s} - \langle Ax,x\rangle\right)\left(\langle Ax,x\rangle - \frac{\langle Bx,x\rangle}{s}\right)\\ &= s^{\langle Ax,x\rangle-2}\exp\left(\frac{\langle Bx,x\rangle}{s} - \langle Ax,x\rangle\right)\left(\langle Ax,x\rangle s - \langle Bx,x\rangle\right). \end{aligned}$$

We observe that the function g is decreasing on $\left(0,\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)$ and increasing on $\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle},\infty\right)$. Therefore

$$\inf_{s\in(0,\infty)} g(s) = g\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right) = \left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle},$$

which gives the best upper bound in (3.1). $\square$

Corollary 6. *Assume that $A>0$ and $x\in H$ with $\|x\|=1$. Then for any $s>0$ we have*

$$\begin{aligned}(3.4)\qquad & s^{\langle Ax,x\rangle}\exp\left(\langle Ax,x\rangle - s\left\langle A^2x,x\right\rangle\right)\\ &\le \eta_x(A)\le s^{\langle Ax,x\rangle}\exp\left(\frac{1}{s}-\langle Ax,x\rangle\right).\end{aligned}$$

The best lower bound for $\eta_x(A)$ is obtained for $s=\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}$, namely

$$\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}\le \eta_x(A).$$

The best upper bound for $\eta_x(A)$ is obtained for $s=\langle Ax,x\rangle^{-1}$, namely

$$\eta_x(A)\le \langle Ax,x\rangle^{-\langle Ax,x\rangle}.$$

Proof. If we take $B=1_H$ in (3.1), then we get

$$s^{\langle Ax,x\rangle}\exp\left(\langle Ax,x\rangle - s\left\langle A^2x,x\right\rangle\right)\le \eta_x(A)\le s^{\langle Ax,x\rangle}\exp\left(\frac{1-s\langle Ax,x\rangle}{s}\right),$$

which is equivalent to (3.4). □

Corollary 7. *Assume that $B>0$ and $x\in H$ with $\|x\|=1$. Then for any $s>0$ we have*

$$(3.5)\qquad s\exp\left(1-s\left\langle B^{-1}x,x\right\rangle\right)\le \Delta_x(B)\le s\exp\left(\frac{\langle Bx,x\rangle - s}{s}\right).$$

The best lower bound for $\Delta_x(B)$ is obtained for $s=\left\langle B^{-1}x,x\right\rangle^{-1}$, namely

$$\left\langle B^{-1}x,x\right\rangle^{-1}\le \Delta_x(B).$$

The best upper bound for $\Delta_x(B)$ is obtained for $s=\langle Bx,x\rangle$, namely

$$\Delta_x(A)\le \langle Bx,x\rangle.$$

Theorem 3. *Assume that A, $B>0$ with the property that $0<mA\le B\le MA$ for some constants m, $M>0$ and $x\in H$ with $\|x\|=1$. Then*

$$(3.6)\qquad \left(\frac{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}}{S\left(\frac{M}{m}\right)}\right)^{\langle Ax,x\rangle}\le D_x\left(A|B\right)\le\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}$$

and

$$\begin{aligned}(3.7)\qquad 0 &\le \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}-\left[D_x\left(A|B\right)\right]^{\langle Ax,x\rangle^{-1}}\\ &\le L\left(m,M\right)\left[\ln L\left(m,M\right)+\frac{M\ln m - m\ln M}{M-m}-1\right].\end{aligned}$$

Proof. We observe that for $x \in H$ with $\|x\| = 1$

$$\begin{aligned}
D_x(A|B) &= \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle \\
&= \exp\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle \\
&= \exp\left[\left\|A^{\frac{1}{2}}x\right\|^2\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}, \frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}\right\rangle\right] \\
&= \left(\exp\left[\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}, \frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}\right\rangle\right]\right)^{\left\|A^{\frac{1}{2}}x\right\|^2} \\
&= \left(\exp\left[\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}, \frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}\right\rangle\right]\right)^{\langle Ax,x\rangle} \\
&= \left(\Delta_{A^{1/2}x/\left\|A^{1/2}x\right\|}(A^{-1/2}BA^{-1/2})\right)^{\langle Ax,x\rangle},
\end{aligned}$$

which gives that

$$[D_x(A|B)]^{\langle Ax,x\rangle^{-1}} = \Delta_{A^{1/2}x/\left\|A^{1/2}x\right\|}(A^{-1/2}BA^{-1/2}) \tag{3.8}$$

for $x \in H$ with $\|x\| = 1$.

Since $0 < mA \le B \le MB$ for the positive operators A, B is equivalent with $0 < m \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le M$, then by (1.4) for $A^{1/2}x/\left\|A^{1/2}x\right\|$ and for the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ we get

$$1 \le \frac{\left\langle A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{1/2}x/\left\|A^{1/2}x\right\|, A^{1/2}x/\left\|A^{1/2}x\right\|\right\rangle}{\Delta_{A^{1/2}x/\left\|A^{1/2}x\right\|}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})} \le S\left(\frac{M}{m}\right),$$

namely

$$1 \le \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle\,\Delta_{A^{1/2}x/\left\|A^{1/2}x\right\|}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})} \le S\left(\frac{M}{m}\right),$$

which gives by (3.8) that

$$1 \le \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle\,[D_x(A|B)]^{\langle Ax,x\rangle^{-1}}} \le S\left(\frac{M}{m}\right).$$

By taking the power $\langle Ax,x\rangle > 0$ we get

$$1 \le \frac{\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}}{D_x(A|B)} \le \left[S\left(\frac{M}{m}\right)\right]^{\langle Ax,x\rangle}.$$

From (1.2) we get

$$\begin{aligned}
0 &\le \left\langle A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{1/2}x/\left\|A^{1/2}x\right\|, A^{1/2}x/\left\|A^{1/2}x\right\|\right\rangle \\
&\quad - \Delta_{A^{1/2}x/\left\|A^{1/2}x\right\|}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \\
&\le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right],
\end{aligned}$$

namely

$$\begin{aligned} 0 &\leq \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle} - \left[D_x (A|B)\right]^{\langle Ax,x\rangle^{-1}} \\ &\leq L(m, M)\left[\ln L(m, M) + \frac{M \ln m - m \ln M}{M - m} - 1\right] \end{aligned}$$

for $x \in H$ with $\|x\| = 1$. □

Remark 1. *Assume that $B > 0$ with the property that $0 < m1_H \leq B \leq M1_H$ for some constants m, $M > 0$ and $x \in H$ with $\|x\| = 1$. Then by $A = 1_H$ in the above Theorem 3 we recapture the inequality (1.4) and (1.2).*

If we take $B = 1_H$ in Theorem 3, then for $0 < mA \leq 1_H \leq MA$ for some constants m, $M > 0$ and $x \in H$ with $\|x\| = 1$. Then

$$\left(\langle Ax, x\rangle S\left(\frac{M}{m}\right)\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax, x\rangle^{-\langle Ax,x\rangle} \tag{3.9}$$

and

$$\begin{aligned} 0 &\leq \langle Ax, x\rangle^{-1} - \left[\eta_x(A)\right]^{\langle Ax,x\rangle^{-1}} \\ &\leq L(m, M)\left[\ln L(m, M) + \frac{M \ln m - m \ln M}{M - m} - 1\right]. \end{aligned} \tag{3.10}$$

If $0 < n1_H \leq A \leq N1_H$, then by taking $m = N^{-1}$ and $M = n^{-1}$ we get $0 < mA \leq 1_H \leq MA$ and by (3.9) and (3.10) we obtain

$$\left[\langle Ax, x\rangle S\left(\frac{N}{n}\right)\right]^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax, x\rangle^{-\langle Ax,x\rangle} \tag{3.11}$$

and

$$\begin{aligned} 0 &\leq \langle Ax, x\rangle^{-1} - \left[\eta_x(A)\right]^{\langle Ax,x\rangle^{-1}} \\ &\leq \frac{L(n, N)}{nN}\left[\ln\left(\frac{L(n, N)}{nN}\right) + \frac{N \ln n - n \ln N}{N - n} - 1\right] \end{aligned} \tag{3.12}$$

for $x \in H$ with $\|x\| = 1$.

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