SOME BOUNDS FOR THE RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp \left\langle A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}} x, x \right\rangle .$$

Assume that $0 < mA \le B \le MA$ for some constants M and m. In this paper we show, among others, that

$$\begin{aligned} 1 \le & \left(\exp \left[\frac{\langle BA^{-1}Bx, x\rangle \langle Ax, x\rangle - \langle Bx, x\rangle^2}{\langle Ax, x\rangle} \right] \right)^{\frac{1}{2M^2}} \\ \le & \frac{\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle} \right)^{\langle Ax,x\rangle}}{D_x(A|B)} \\ \le & \left(\exp \left[\frac{\langle BA^{-1}Bx, x\rangle \langle Ax, x\rangle - \langle Bx, x\rangle^2}{\langle Ax, x\rangle} \right] \right)^{\frac{1}{2m^2}} \end{aligned}$$

for all $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [9].

For each unit vector $x \in H$, see also [12], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.
Key words and phrases. Positive operators, Normalized determinants, Inequalities.

(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^\alpha$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [9] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [18]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [10], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA\ln(tA)x, x\rangle) \\ &= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\ &= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\ &= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}[\eta_x(A)]^{-t} \tag{1.6}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_x(I)=1 \text{ and } \eta_x(tI)=t^{-t} \tag{1.7}$$

for $t>0$.

In the recent paper [3] we showed among others that, if $A,\ B>0$, then for all $x\in H, \|x\|=1$ and $t\in[0,1]$,

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A>0$ and $x\in H, \|x\|=1$.

Definition 1. *For positive invertible operators A, B and $x\in H$ with $\|x\|=1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B):=\exp\langle S(A|B)x,x\rangle=\exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle.$$

We observe that for $A>0$,

$$D_x(A|1_H)=\exp\langle S(A|1_H)x,x\rangle=\exp(-\langle A\ln Ax,x\rangle)=\eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B>0$,

$$D_x(1_H|B):=\exp\langle S(1_H|B)x,x\rangle=\exp\langle\ln Bx,x\rangle=\Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant.*

Assume that $0<mA\leq B\leq MA$ for some constants M and m. Motivated by the above results, in this paper we show, among others, that

$$\begin{aligned}1&\leq\left(\exp\left[\frac{\langle BA^{-1}Bx,x\rangle\langle Ax,x\rangle-\langle Bx,x\rangle^2}{\langle Ax,x\rangle}\right]\right)^{\frac{1}{2M^2}}\\&\leq\frac{\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}}{D_x(A|B)}\\&\leq\left(\exp\left[\frac{\langle BA^{-1}Bx,x\rangle\langle Ax,x\rangle-\langle Bx,x\rangle^2}{\langle Ax,x\rangle}\right]\right)^{\frac{1}{2m^2}}\end{aligned}$$

for all $x\in H$ with $\|x\|=1$.

2. Main Results

We start to the following logarithmic inequalities:

Lemma 1. *For any a, $b > 0$ we have*

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 = \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2. \tag{2.1}$$

Proof. It is easy to see that

$$\int_a^b \frac{b-t}{t^2}dt = \frac{b-a}{a} - \ln b + \ln a \tag{2.2}$$

for any a, $b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.3}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \ge \int_b^a \frac{t-b}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.4}$$

Therefore, by (2.3) and (2.4) we have for any a, $b > 0$ that

$$\int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (2.2) we then get the desired result (2.1). □

When some bounds for a, b are provided, then we have:

Corollary 1. *Assume that a, $b \in [m, M] \subset (0,\infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{m^2} \tag{2.5}$$

and, by swapping a with b,

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{2.6}$$

Theorem 1. *Assume that $0 < mA \le B \le MA$ for some constants M and m. Then for all $a \in [m, M]$ and $x \in H$, $\|x\| = 1$,*

$$1 \le \left(\exp\left[\left(\left\langle BA^{-1}Bx, x\right\rangle - 2a\langle Bx, x\rangle + a^2\langle Ax, x\rangle\right)\right]\right)^{\frac{1}{2M^2}} \le \frac{\left(\frac{a}{e}\right)^{\langle Ax,x\rangle}\left[\exp\langle Bx, x\rangle\right]^{\frac{1}{a}}}{D_x(A|B)} \le \left(\exp\left[\left(\left\langle BA^{-1}Bx, x\right\rangle - 2a\langle Bx, x\rangle + a^2\langle Ax, x\rangle\right)\right]\right)^{\frac{1}{2m^2}} \tag{2.7}$$

and

$$1 \leq \left(\exp\left[\left(\left\langle BA^{-1}Bx, x\right\rangle - 2a\left\langle Bx, x\right\rangle + a^2\left\langle Ax, x\right\rangle\right)\right]\right)^{\frac{1}{2M^2}} \leq \frac{D_x(A|B)}{(ae)^{\langle Ax,x\rangle}\left[\exp\left(-\left\langle AB^{-1}Ax, x\right\rangle\right)\right]^a} \leq \left(\exp\left[\left(\left\langle BA^{-1}Bx, x\right\rangle - 2a\left\langle Bx, x\right\rangle + a^2\left\langle Ax, x\right\rangle\right)\right]\right)^{\frac{1}{2m^2}}. \tag{2.8}$$

Proof. If we use the continuous functional calculus for selfadjoint operator T with spectrum $\operatorname{Sp} T$ in $[m, M]$ and the inequality (2.5) we have

$$\frac{1}{2}\frac{(T - a1_H)^2}{M^2} \leq \frac{T - a1_H}{a} - \ln T + (\ln a)\, 1_H \leq \frac{1}{2}\frac{(T - a1_H)^2}{m^2} \tag{2.9}$$

for all $a \in [m, M]$.

Since $0 < mA \leq B \leq MA$, hence by multiplying both sides by $A^{-1/2} > 0$ we get $0 < m \leq A^{-1/2}BA^{-1/2} \leq A$. By writing (2.9) for $T = A^{-1/2}BA^{-1/2}$ we get

$$\frac{1}{2}\frac{\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2}{M^2} \leq \frac{A^{-1/2}BA^{-1/2} - a1_H}{a} - \ln\left(A^{-1/2}BA^{-1/2}\right) + (\ln a)\, 1_H \leq \frac{1}{2}\frac{\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2}{m^2}$$

and by multiplying both sides by $A^{1/2} > 0$, we derive

$$\frac{1}{2M^2}A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 A^{1/2} \leq \frac{1}{a}B - A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} + (\ln a)\, A - A \leq \frac{1}{2m^2}A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 A^{1/2} \tag{2.10}$$

for all $a \in [m, M]$.

Observe that

$$A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 A^{1/2} = A^{1/2}\left(A^{-1/2}BA^{-1/2}A^{-1/2}BA^{-1/2} - 2aA^{-1/2}BA^{-1/2} + a^2 1_H\right)A^{1/2} = BA^{-1}B - 2B + a^2 A$$

and by (2.10) we get

$$\frac{1}{2M^2}\left(BA^{-1}B - 2B + a^2 A\right) \leq \frac{1}{a}B - A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} + (\ln a)\, A - A \leq \frac{1}{2m^2}\left(BA^{-1}B - 2B + a^2 A\right),$$

which gives that

$$\begin{aligned}(2.11)\qquad &\frac{1}{2M^2}\left\langle\left(BA^{-1}B-2aB+a^2A\right)x,x\right\rangle\\ &\leq\frac{1}{a}\langle Bx,x\rangle-\left\langle A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}x,x\right\rangle+\ln\left(\frac{a}{e}\right)^{\langle Ax,x\rangle}\\ &\leq\frac{1}{2m^2}\left\langle\left(BA^{-1}B-2aB+a^2A\right)x,x\right\rangle,\end{aligned}$$

for all $a\in[m,M]$ and $x\in H$, $\|x\|=1$.

If we take the exponential in (2.11), then we get

$$\begin{aligned}(2.12)\quad 1&\leq\exp\left[\frac{1}{2M^2}\left(\left\langle BA^{-1}Bx,x\right\rangle-2a\langle Bx,x\rangle+a^2\langle Ax,x\rangle\right)\right]\\ &\leq\exp\left[\frac{1}{a}\langle Bx,x\rangle-\left\langle A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}x,x\right\rangle+\ln\left(\frac{a}{e}\right)^{\langle Ax,x\rangle}\right]\\ &\leq\exp\left[\frac{1}{2m^2}\left(\left\langle BA^{-1}Bx,x\right\rangle-2a\langle Bx,x\rangle+a^2\langle Ax,x\rangle\right)\right],\end{aligned}$$

for all $a\in[m,M]$ and $x\in H$, $\|x\|=1$.

This is equivalent to (2.7).

From (2.6) we get

$$\frac{1}{2}\frac{\left(T-a1_H\right)^2}{M^2}\leq\ln T-\ln(ae)1_H+aT^{-1}\leq\frac{1}{2}\frac{(T-a)^2}{m^2}$$

for selfadjoint operator T with spectrum in $[m,M]$ and for all $a\in[m,M]$.

If we take $T=A^{-1/2}BA^{-1/2}$, then we get

$$\begin{aligned}\frac{1}{2M^2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2&\leq\ln A^{-1/2}BA^{-1/2}-\ln(ae)1_H+aA^{1/2}B^{-1}A^{1/2}\\ &\leq\frac{1}{2m^2}\left(A^{-1/2}BA^{-1/2}-a\right)^2\end{aligned}$$

and by multiplying both sides by $A^{1/2}>0$, we derive

$$\begin{aligned}&\frac{1}{2M^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2A^{1/2}\\ &\leq A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2}-\ln(ae)A+aAB^{-1}A\\ &\leq\frac{1}{2m^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-a\right)^2A^{1/2}\end{aligned}$$

for all $a\in[m,M]$.

If we take the inner product over $x\in H$, $\|x\|=1$, then we get

$$\begin{aligned}&\frac{1}{2M^2}\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2A^{1/2}x,x\right\rangle\\ &\leq\left\langle A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2}x,x\right\rangle-\ln(ae)\langle Ax,x\rangle+a\left\langle AB^{-1}Ax,x\right\rangle\\ &\leq\frac{1}{2m^2}\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2A^{1/2}x,x\right\rangle,\end{aligned}$$

which produces the inequality (2.8). □

Corollary 2. *With the assumptions of Theorem 1, we have*

$$
\begin{aligned}
1 &\leq \left(\exp\left[\frac{\left\langle BA^{-1}Bx,x\right\rangle \langle Ax,x\rangle - \langle Bx,x\rangle^2}{\langle Ax,x\rangle}\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}}{D_x\left(A|B\right)} \\
&\leq \left(\exp\left[\frac{\left\langle BA^{-1}Bx,x\right\rangle \langle Ax,x\rangle - \langle Bx,x\rangle^2}{\langle Ax,x\rangle}\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
1 &\leq \left(\exp\left[\frac{\left\langle BA^{-1}Bx,x\right\rangle \langle Ax,x\rangle - \langle Bx,x\rangle^2}{\langle Ax,x\rangle}\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{D_x\left(A|B\right)}{\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}\left[\exp\left(\frac{\langle Ax,x\rangle^2 - \langle Bx,x\rangle\langle AB^{-1}Ax,x\rangle}{\langle Bx,x\rangle}\right)\right]^{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}}} \\
&\leq \left(\exp\left[\frac{\left\langle BA^{-1}Bx,x\right\rangle \langle Ax,x\rangle - \langle Bx,x\rangle^2}{\langle Ax,x\rangle}\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.14}
$$

for all $x \in H$, $\|x\| = 1$.

The proof follows by Theorem 1 for $a = \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}$, which, due to the condition $mA \leq B \leq MA$, belongs to the interval $[m, M]$ for all $x \in H$, $\|x\| = 1$.

Remark 1. *Assume that $0 < m1_H \leq B \leq M1_H$ for some constants M and m. Then by Corollary 2 for $A = 1_H$ we have*

$$
\begin{aligned}
1 &\leq \left(\exp\left[\left\langle B^2x,x\right\rangle - \langle Bx,x\rangle^2\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\langle Bx,x\rangle}{\eta_x(B)} \\
&\leq \left(\exp\left[\left\langle B^2x,x\right\rangle - \langle Bx,x\rangle^2\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
1 &\leq \left(\exp\left[\left\langle B^2x,x\right\rangle - \langle Bx,x\rangle^2\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\eta_x(B)}{\langle Bx,x\rangle\left[\exp\left(\langle Bx,x\rangle^{-1} - \langle B^{-1}x,x\rangle\right)\right]^{\langle Bx,x\rangle}} \\
&\leq \left(\exp\left[\left\langle B^2x,x\right\rangle - \langle Bx,x\rangle^2\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.16}
$$

for all $x \in H$, $\|x\| = 1$.

Assume that $0 < mA \leq 1 \leq MA$ *for some constants* M *and* m*. Then by Corollary 2 for* $B = 1_H$ *we have*

$$1 \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{1}{2M^2}} \leq \frac{\langle Ax, x \rangle^{-\langle Ax, x \rangle}}{\Delta_x(A)} \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{1}{2m^2}} \tag{2.17}$$

and

$$1 \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{1}{2M^2}} \leq \frac{\Delta_x(A)}{\langle Ax, x \rangle^{-\langle Ax, x \rangle} \left[\exp \left[-\left(\langle A^2x, x \rangle - \langle Ax, x \rangle^2 \right) \right] \right]^{\frac{1}{\langle Ax, x \rangle}}} \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{1}{2m^2}} \tag{2.18}$$

for all $x \in H$, $\|x\| = 1$.

If $0 < n1_H \leq A \leq N1_H$*, then* $0 < \frac{1}{N} A \leq 1_H \leq \frac{1}{n} A$ *and by taking* $m = \frac{1}{N}$ *and* $M = \frac{1}{n}$ *in (2.17) and (2.18), then we get*

$$1 \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{n^2}{2}} \leq \frac{\langle Ax, x \rangle^{-\langle Ax, x \rangle}}{\Delta_x(A)} \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{N^2}{2}} \tag{2.19}$$

and

$$1 \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{n^2}{2}} \leq \frac{\Delta_x(A)}{\langle Ax, x \rangle^{-\langle Ax, x \rangle} \left[\exp \left[-\left(\langle A^2x, x \rangle - \langle Ax, x \rangle^2 \right) \right] \right]^{\frac{1}{\langle Ax, x \rangle}}} \leq \left(\exp \left[\frac{\langle A^{-1}x, x \rangle \langle Ax, x \rangle - 1}{\langle Ax, x \rangle} \right] \right)^{\frac{N^2}{2}} \tag{2.20}$$

for all $x \in H$, $\|x\| = 1$.

We have the following reverses of Schwarz's inequality

$$0 \le \left\langle B^2 x, x\right\rangle - \langle Bx, x\rangle^2 \le \frac{1}{4}(M-m)^2, \text{ [11, p. 28]}$$

and

$$0 \le \left\langle B^2 x, x\right\rangle - \langle Bx, x\rangle^2 \le \frac{(M-m)^2}{4mM}\langle Bx, x\rangle^2, \text{ [11, p. 27]},$$

for all $x \in H$, $\|x\| = 1$, where $0 < m1_H \le B \le M1_H$ for some constants M and m.

By (2.15) we get

$$\frac{\langle Bx, x\rangle}{\eta_x(B)} \le \left(\exp\left[\left\langle B^2 x, x\right\rangle - \langle Bx, x\rangle^2\right]\right)^{\frac{1}{2m^2}} \le \begin{cases} \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right] \\ \exp\left[\frac{1}{8mM}\left(\frac{M}{m}-1\right)^2\langle Bx, x\rangle^2\right] \end{cases} \tag{2.21}$$

while by (2.16) we derive

$$\begin{aligned} &\frac{\eta_x(B)}{\langle Bx, x\rangle\left[\exp\left(\langle Bx, x\rangle^{-1} - \langle B^{-1}x, x\rangle\right)\right]^{\langle Bx, x\rangle}} \\ &\le \left(\exp\left[\left\langle B^2 x, x\right\rangle - \langle Bx, x\rangle^2\right]\right)^{\frac{1}{2m^2}} \\ &\le \begin{cases} \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right] \\ \exp\left[\frac{1}{8mM}\left(\frac{M}{m}-1\right)^2\langle Bx, x\rangle^2\right] \end{cases} \end{aligned} \tag{2.22}$$

for all $x \in H$, $\|x\| = 1$, where $0 < m1_H \le B \le M1_H$ for some constants M and m.

We know the following reverse inequalities hold as well

$$\left\langle A^{-1}x, x\right\rangle \langle Ax, x\rangle \le \frac{(N+v)^2}{4nN}, \text{ [11, p. 24]}$$

and

$$\left\langle A^{-1}x, x\right\rangle - \langle Ax, x\rangle^{-1} \le \frac{\left(\sqrt{N}-\sqrt{n}\right)^2}{nN}, \text{ [11, p. 28]},$$

for all $x \in H$, $\|x\| = 1$, where $0 < n1_H \le A \le N1_H$ for some constants N and n.

From (2.19) we then get

$$\begin{aligned} \frac{\langle Ax, x\rangle^{-\langle Ax, x\rangle}}{\Delta_x(A)} &\le \left(\exp\left[\frac{\left\langle A^{-1}x, x\right\rangle \langle Ax, x\rangle - 1}{\langle Ax, x\rangle}\right]\right)^{\frac{N^2}{2}} \\ &\le \exp\left[\frac{N(N-n)^2}{8n}\langle Ax, x\rangle^{-1}\right], \end{aligned} \tag{2.23}$$

while from (2.20) we obtain

$$\begin{aligned}
&\frac{\Delta_x(A)}{\langle Ax,x\rangle^{-\langle Ax,x\rangle}\left[\exp\left[-\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right]\right]^{\frac{1}{\langle Ax,x\rangle}}}\\
&\leq\left(\exp\left[\frac{\langle A^{-1}x,x\rangle\langle Ax,x\rangle-1}{\langle Ax,x\rangle}\right]\right)^{\frac{N^2}{2}}\\
&\leq\exp\left[\frac{N\left(\sqrt{N}-\sqrt{n}\right)^2}{2n}\right]
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, where $0<n1_H\leq A\leq N1_H$ for some constants N and n.

3. Some Related Results

If we take in (2.1) $a=1$ and $b=u\in(0,\infty)$, then we get

$$\begin{aligned}
\frac{1}{2}\left(1-\frac{\min\{1,u\}}{\max\{1,u\}}\right)^2&=\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,u\}}\\
&\leq u-1-\ln u\\
&\leq\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,u\}}=\frac{1}{2}\left(\frac{\max\{1,u\}}{\min\{1,u\}}-1\right)^2
\end{aligned}\tag{3.1}$$

and if we take $a=u$ and $b=1$, then we also get

$$\begin{aligned}
\frac{1}{2}\left(1-\frac{\min\{1,u\}}{\max\{1,u\}}\right)^2&=\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,u\}}\\
&\leq\ln u-\frac{u-1}{u}\\
&\leq\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,u\}}=\frac{1}{2}\left(\frac{\max\{1,u\}}{\min\{1,u\}}-1\right)^2.
\end{aligned}\tag{3.2}$$

If $u\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\}\leq\min\{1,u\}\leq\min\{1,K\}$$

and

$$\max\{1,k\}\leq\max\{1,u\}\leq\max\{1,K\}.$$

By (3.1) and (3.2) we get the *local bounds*

$$\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,K\}}\leq u-1-\ln u\leq\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,k\}}\tag{3.3}$$

and

$$\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,K\}}\leq\ln u-\frac{u-1}{u}\leq\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,k\}}\tag{3.4}$$

for any $u\in[k,K]\subset(0,\infty)$.

Theorem 2. *Assume that* $0 < mA \leq B \leq MA$ *for some constants* M *and* m. *Then*

$$1 \leq \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\left\langle BA^{-1}Bx,x\right\rangle - 2\langle Bx,x\rangle + \langle Ax,x\rangle\right)\right] \tag{3.5}$$
$$\leq \frac{\exp\left(\langle Bx,x\rangle - \langle Ax,x\rangle\right)}{D_x(A|B)}$$
$$\leq \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\left\langle BA^{-1}Bx,x\right\rangle - 2\langle Bx,x\rangle + \langle Ax,x\rangle\right)\right],$$

and

$$1 \leq \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\left\langle BA^{-1}Bx,x\right\rangle - 2\langle Bx,x\rangle + \langle Ax,x\rangle\right)\right] \tag{3.6}$$
$$\leq \frac{D_x(A|B)}{\exp\left(\langle Ax,x\rangle - \langle AB^{-1}Ax,x\rangle\right)}$$
$$\leq \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\left\langle BA^{-1}Bx,x\right\rangle - 2\langle Bx,x\rangle + \langle Ax,x\rangle\right)\right]$$

for all $x \in H$, $\|x\| = 1$.

Proof. From (3.3) we have for the selfadjoint operator T with $0 < m1_H \leq T \leq M1_H$ that

$$\frac{1}{2\max^2\{1,M\}}(T-1_H)^2 \leq T - 1_H - \ln T \leq \frac{1}{2\min^2\{1,m\}}(T-1_H)^2.$$

If we write this inequality for $T = A^{-1/2}BA^{-1/2}$, then we get

$$\frac{1}{2\max^2\{1,M\}}\left(A^{-1/2}BA^{-1/2} - 1_H\right)^2$$
$$\leq A^{-1/2}BA^{-1/2} - 1_H - \ln\left(A^{-1/2}BA^{-1/2}\right)$$
$$\leq \frac{1}{2\min^2\{1,m\}}\left(A^{-1/2}BA^{-1/2} - 1_H\right)^2.$$

If we multiply this inequality both sides by $A^{1/2} > 0$, then we get

$$\frac{1}{2\max^2\{1,M\}}A^{1/2}\left(A^{-1/2}BA^{-1/2} - 1_H\right)^2 A^{1/2} \tag{3.7}$$
$$\leq B - A - A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}$$
$$\leq \frac{1}{2\min^2\{1,m\}}A^{1/2}\left(A^{-1/2}BA^{-1/2} - 1_H\right)^2 A^{1/2}.$$

Observe that

$$A^{1/2}\left(A^{-1/2}BA^{-1/2} - 1_H\right)^2 A^{1/2}$$
$$= A^{1/2}\left(A^{-1/2}BA^{-1}BA^{-1/2} - 2A^{-1/2}BA^{-1/2} + 1_H\right)A^{1/2}$$
$$= BA^{-1}B - 2B + A$$

and by (3.7) we get

$$\begin{aligned}0 &\leq \frac{1}{2\max^2\{1,M\}}\left(BA^{-1}B-2B+A\right)\\&\leq B-A-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\\&\leq \frac{1}{2\min^2\{1,m\}}\left(BA^{-1}B-2B+A\right),\end{aligned}$$

which gives that

$$\begin{aligned}0 &\leq \frac{1}{2\max^2\{1,M\}}\left(\left\langle BA^{-1}Bx,x\right\rangle-2\langle Bx,x\rangle+\langle Ax,x\rangle\right)\\&\leq \langle Bx,x\rangle-\langle Ax,x\rangle-\left\langle A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}x,x\right\rangle\\&\leq \frac{1}{2\min^2\{1,m\}}\left(\left\langle BA^{-1}Bx,x\right\rangle-2\langle Bx,x\rangle+\langle Ax,x\rangle\right)\end{aligned}\tag{3.8}$$

for all $x\in H$, $\|x\|=1$.

If we take the exponential in (3.8), then we get the desired result (3.5).

By (3.4) we obtain

$$\begin{aligned}\frac{1}{2\max^2\{1,M\}}\left(T-1_H\right)^2 &\leq \ln T-1_H+T^{-1}\\&\leq \frac{1}{2\min^2\{1,m\}}\left(T-1_H\right)^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{2\max^2\{1,M\}}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2\\&\leq \ln\left(A^{-1/2}BA^{-1/2}\right)-1_H+A^{1/2}B^{-1}A^{1/2}\\&\leq \frac{1}{2\min^2\{1,m\}}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2.\end{aligned}$$

If we multiply this inequality both sides by $A^{1/2}>0$, then we get

$$\begin{aligned}&\frac{1}{2\max^2\{1,M\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2}\\&\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}-A+AB^{-1}A\\&\leq \frac{1}{2\min^2\{1,m\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2}.\end{aligned}$$

By taking the inner product over $x\in H$, $\|x\|=1$, we deduce (3.6). □

Remark 2. *Assume that* $0 < m1_H \leq B \leq M1_H$ *for some constants* M *and* m. *Then by Theorem 2 we get*

$$1 \leq \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\left\langle B^2x,x\right\rangle - 2\left\langle Bx,x\right\rangle + 1\right)\right] \leq \frac{\exp\left(\left\langle Bx,x\right\rangle - 1\right)}{\eta_x(B)} \leq \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\left\langle B^2x,x\right\rangle - 2\left\langle Bx,x\right\rangle + 1\right)\right] \tag{3.9}$$

and

$$1 \leq \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\left\langle B^2x,x\right\rangle - 2\left\langle Bx,x\right\rangle + 1\right)\right] \leq \frac{\eta_x(B)}{\exp\left(1 - \left\langle B^{-1}x,x\right\rangle\right)} \leq \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\left\langle B^2x,x\right\rangle - 2\left\langle Bx,x\right\rangle + 1\right)\right] \tag{3.10}$$

for all $x \in H$, $\|x\| = 1$.

If $0 < n1_H \leq A \leq N1_H$ *for some constants* N *and* n, *then by Theorem 2 we also obtain*

$$1 \leq \exp\left[\frac{1}{2}\min\left\{1,n^2\right\}\left(\left\langle A^{-1}x,x\right\rangle + \left\langle Ax,x\right\rangle - 2\right)\right] \leq \frac{\exp\left(1 - \left\langle Ax,x\right\rangle\right)}{\Delta_x(A)} \leq \exp\left[\frac{1}{2}\max\left\{1,N^2\right\}\left(\left\langle A^{-1}x,x\right\rangle + \left\langle Ax,x\right\rangle - 2\right)\right] \tag{3.11}$$

and

$$1 \leq \exp\left[\frac{1}{2}\min\left\{1,n^2\right\}\left(\left\langle A^{-1}x,x\right\rangle + \left\langle Ax,x\right\rangle - 2\right)\right] \leq \frac{\Delta_x(A)}{\exp\left(\left\langle Ax,x\right\rangle - \left\langle A^2x,x\right\rangle\right)} \leq \exp\left[\frac{1}{2}\max\left\{1,N^2\right\}\left(\left\langle A^{-1}x,x\right\rangle + \left\langle Ax,x\right\rangle - 2\right)\right] \tag{3.12}$$

for all $x \in H$, $\|x\| = 1$.

Observe also that for $u \in [k,K] \subset (0,\infty)$ we have

$$1 - \frac{\min\{1,u\}}{\max\{1,u\}} \geq 1 - \frac{\min\{1,K\}}{\max\{1,k\}} \geq 0$$

and

$$0 \leq \frac{\max\{1,u\}}{\min\{1,u\}} - 1 \leq \frac{\max\{1,K\}}{\min\{1,k\}} - 1.$$

Now, by (3.1) and (3.2) we get the *global bounds*

$$\frac{1}{2}\left(1 - \frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \leq u - 1 - \ln u \leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}} - 1\right)^2 \tag{3.13}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \le \ln u - \frac{u-1}{u} \le \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{3.14}$$

for all $u \in [k,K] \subset (0,\infty)$.

Theorem 3. *Assume that $0 < mA \le B \le MA$ for some constants M and m. Then*

$$\begin{aligned} 1 &\le \exp\left[\frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2 \langle Ax,x\rangle\right] \\ &\le \frac{\exp\left(\langle Bx,x\rangle - \langle Ax,x\rangle\right)}{D_x\left(A|B\right)} \\ &\le \exp\left[\frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2 \langle Ax,x\rangle\right] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} 1 &\le \exp\left[\frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2 \langle Ax,x\rangle\right] \\ &\le \frac{D_x\left(A|B\right)}{\exp\left(\langle Ax,x\rangle - \langle AB^{-1}Ax,x\rangle\right)} \\ &\le \exp\left[\frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2 \langle Ax,x\rangle\right] \end{aligned} \tag{3.16}$$

for all $x \in H$, $\|x\| = 1$.

Proof. From (3.13) we have for the selfadjoint operator T with $0 < m1_H \le T \le M1_H$ that

$$\frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2 \le T - 1_H - \ln T \le \frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2.$$

If we write this inequality for $T = A^{-1/2}BA^{-1/2}$, then we get

$$\begin{aligned} \frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2 &\le A^{-1/2}BA^{-1/2} - 1_H - \ln\left(A^{-1/2}BA^{-1/2}\right) \\ &\le \frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2. \end{aligned}$$

If we multiply this inequality both sides by $A^{1/2} > 0$, then we get

$$\begin{aligned} &\frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2 A \\ &\le B - A - A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\ &\le \frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2 A. \end{aligned}$$

By employing now a similar argument to the one in the proof of Theorem 2 we derive (3.15).

The inequality (3.16) follows in a similar way from (3.14). $\square$

Remark 3. *Assume that* $0 < m1_H \leq B \leq M1_H$ *for some constants* M *and* m. *Then by Theorem 3 we get*

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right] \leq \frac{\exp(\langle Bx, x\rangle - 1)}{\eta_x(B)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right], \tag{3.17}$$

and

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right] \leq \frac{\eta_x(B)}{\exp(1 - \langle B^{-1}x, x\rangle)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right] \tag{3.18}$$

for all $x \in H$, $\|x\| = 1$.

If $0 < n1_H \leq A \leq N1_H$ *for some constants* N *and* n, *then by Theorem 2 we also obtain*

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, N\}}{\max\{1, n\}}\right)^2 \langle Ax, x\rangle\right] \leq \frac{\exp(1 - \langle Ax, x\rangle)}{\Delta_x(A)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, N\}}{\min\{1, n\}} - 1\right)^2 \langle Ax, x\rangle\right] \tag{3.19}$$

and

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, N\}}{\max\{1, n\}}\right)^2 \langle Ax, x\rangle\right] \leq \frac{\Delta_x(A)}{\exp(\langle Ax, x\rangle - \langle A^2x, x\rangle)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, N\}}{\min\{1, n\}} - 1\right)^2 \langle Ax, x\rangle\right] \tag{3.20}$$

for all $x \in H$, $\|x\| = 1$.

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