BOUNDS FOR RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA KANTOROVICH CONSTANT

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Abstract. For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle.$$

In this paper we show, among others, that, if the positive invertible operators A, B satisfy the condition $0 < mA \le B \le MA$ for some constants m, M, then

$$\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle - \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}x,x\right\rangle\right]} \\
&\le \frac{D_x(A|B)}{m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}} \\
&\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle + \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}x,x\right\rangle\right]} \\
&\le K\left(\frac{M}{m}\right)^{\langle Ax,x\rangle}
\end{aligned}$$

for all $x \in H$ with $\|x\| = 1$, where $K(\cdot)$ is the *Kantorovich constant*.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [11], [12], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [11].

For each unit vector $x \in H$, see also [14], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;

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(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha) A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [11] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [21]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [12], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp(-\langle tA\ln(tA)x, x\rangle) \\
&= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\
&= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\
&= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle} [\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In the recent paper [5] we showed among others that, if A, $B > 0$, then for all $x \in H, \|x\| = 1$ and $t \in [0,1]$,

$$\eta_x((1-t)A + tB) \geq (\eta_x(A))^{1-t} (\eta_x(B))^t .$$

Also we have the bounds

$$\left(\frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle}\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax, x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A > 0$ and $x \in H, \|x\| = 1$.

Definition 1. *For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B) := \exp\langle S(A|B)x, x\rangle = \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle.$$

We observe that for $A > 0$,

$$D_x(A|1_H) = \exp\langle S(A|1_H)x, x\rangle = \exp(-\langle A\ln Ax, x\rangle) = \eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B > 0$,

$$D_x(1_H|B) := \exp\langle S(1_H|B)x, x\rangle = \exp\langle \ln Bx, x\rangle = \Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant*.

2. Main Results

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{2.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu} \leq\right) K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{2.2}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (2.2) was obtained by Zuo et al. in [23] while the second by Liao et al. [17].

Theorem 1. *Assume that the positive invertible operators A, B satisfy the condition $0 < mA \le B \le MA$ for some constants m, M, then*

$$
\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle - \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}x,x\right\rangle\right]} \\
&\le \frac{D_x(A|B)}{m^{\frac{M\langle Ax,x\rangle-\langle Bx,x\rangle}{M-m}}M^{\frac{\langle Bx,x\rangle-m\langle Ax,x\rangle}{M-m}}} \\
&\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle + \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}x,x\right\rangle\right]} \\
&\le K\left(\frac{M}{m}\right)^{\langle Ax,x\rangle}
\end{aligned}
\tag{2.3}
$$

for all $x \in H$ with $\|x\| = 1$.

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0,1]$. Then

$$
\begin{aligned}
\min\{1-\nu,\nu\} &= \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right| \\
&= \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,
\end{aligned}
$$

$$
\begin{aligned}
\max\{1-\nu,\nu\} &= \frac{1}{2} + \left|\nu - \frac{1}{2}\right| = \frac{1}{2} + \left|\frac{t-m}{M-m} - \frac{1}{2}\right| \\
&= \frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,
\end{aligned}
$$

$$
(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t
$$

and

$$
m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.
$$

By using (2.2) we get

$$
\begin{aligned}
m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \\
&\le t \le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}
\end{aligned}
\tag{2.4}
$$

for $t \in [m, M]$.

By taking the log in (2.4) we get

$$
\begin{aligned}
& \frac{M-t}{M-m} \ln m+\frac{t-m}{M-m} \ln M \\
& \leq\left[\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right] \ln K\left(\frac{M}{m}\right) \\
& +\frac{M-t}{M-m} \ln m+\frac{t-m}{M-m} \ln M \\
& \leq \ln t \leq\left[\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right] \ln K\left(\frac{M}{m}\right) \\
& +\frac{M-t}{M-m} \ln m+\frac{t-m}{M-m} \ln M \\
& \leq \ln K\left(\frac{M}{m}\right)+\frac{M-t}{M-m} \ln m+\frac{t-m}{M-m} \ln M
\end{aligned} \tag{2.5}
$$

for $t \in [m, M]$.

If $0 < mI \leq T \leq MI$, then by using the continuous functional calculus for selfadjoint operators we get from (2.5) that

$$
\begin{aligned}
& \ln m \frac{M 1_H-T}{M-m}+\ln M \frac{T-m 1_H}{M-m} \\
& \leq\left[\frac{1}{2} 1_H-\frac{1}{M-m}\left|T-\frac{1}{2}(m+M) 1_H\right|\right] \ln K\left(\frac{M}{m}\right) \\
& +\ln m \frac{M 1_H-T}{M-m}+\ln M \frac{T-m 1_H}{M-m} \\
& \leq \ln T \leq\left[\frac{1}{2} 1_H+\frac{1}{M-m}\left|T-\frac{1}{2}(m+M) 1_H\right|\right] \ln K\left(\frac{M}{m}\right) \\
& +\ln m \frac{M 1_H-T}{M-m}+\ln M \frac{T-m 1_H}{M-m} \\
& \leq \ln K\left(\frac{M}{m}\right) 1_H+\ln m \frac{M 1_H-T}{M-m}+\ln M \frac{T-m 1_H}{M-m} .
\end{aligned} \tag{2.6}
$$

Since $0 < mA \leq B \leq MA$, hence by multiplying both sides by $A^{-1/2} > 0$ we get $0 < m \leq A^{-1/2}BA^{-1/2} \leq A$. By writing (2.6) for $T = A^{-1/2}BA^{-1/2}$ we get

$$
\begin{aligned}
& \ln m \frac{M 1_H-A^{-1/2} B A^{-1/2}}{M-m}+\ln M \frac{A^{-1/2} B A^{-1/2}-m 1_H}{M-m} \\
& \leq\left[\frac{1}{2} 1_H-\frac{1}{M-m}\left|A^{-1/2} B A^{-1/2}-\frac{1}{2}(m+M) 1_H\right|\right] \ln K\left(\frac{M}{m}\right) \\
& +\ln m \frac{M 1_H-A^{-1/2} B A^{-1/2}}{M-m}+\ln M \frac{A^{-1/2} B A^{-1/2}-m 1_H}{M-m}
\end{aligned}
$$

$$\le \ln\left(A^{-1/2}BA^{-1/2}\right)$$
$$\le \left[\frac{1}{2}1_H + \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|\right]\ln K\left(\frac{M}{m}\right)$$
$$+ \ln m \frac{M1_H - A^{-1/2}BA^{-1/2}}{M-m} + \ln M \frac{A^{-1/2}BA^{-1/2} - m1_H}{M-m}$$
$$\le \ln K\left(\frac{M}{m}\right)1_H + \ln m \frac{M1_H - A^{-1/2}BA^{-1/2}}{M-m} + \ln M \frac{A^{-1/2}BA^{-1/2} - m1_H}{M-m}.$$

Now, if we multiply this inequality both sides by $A^{1/2} > 0$, then we get

$$\ln m \frac{MA-B}{M-m} + \ln M \frac{B-mA}{M-m}$$
$$\le \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}A - \frac{1}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}\right]$$
$$+ \ln m \frac{MA-B}{M-m} + \ln M \frac{B-mA}{M-m}$$
$$\le A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}$$
$$\le \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}A + \frac{1}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}\right]$$
$$+ \ln m \frac{MA-B}{M-m} + \ln M \frac{B-mA}{M-m}$$
$$\le \ln K\left(\frac{M}{m}\right)A + \ln m \frac{MA-B}{M-m} + \ln M \frac{B-mA}{M-m}.$$

If we take the inner product over $x \in H$ with $\|x\| = 1$, then we get

$$\ln\left(m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}\right)$$
$$\le \ln K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle - \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}x,x\right\rangle\right]}$$
$$+ \ln\left(m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}\right)$$
$$\le \left\langle A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}x,x\right\rangle$$
$$\le \ln K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle + \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}x,x\right\rangle\right]}$$
$$+ \ln\left(m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}\right)$$
$$\le \ln K\left(\frac{M}{m}\right)^{\langle Ax,x\rangle} + \ln\left(m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}\right).$$

and by taking the exponential, we derive the desired result (2.3). □

Corollary 1. *Assume that the positive invertible operator B satisfies the condition $0 < m1_H \le B \le M1_H$ for some constants m, M, then*

$$
\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|B-\frac{1}{2}(m+M)1_H\right|x,x\right\rangle\right]} \\
&\le \frac{\eta_x(B)}{m^{\frac{M-\langle Bx,x\rangle}{M-m}}M^{\frac{\langle Bx,x\rangle-m}{M-m}}} \\
&\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|B-\frac{1}{2}(m+M)1_H\right|x,x\right\rangle\right]} \le K\left(\frac{M}{m}\right)
\end{aligned}
\tag{2.7}
$$

for all $x \in H$ with $\|x\| = 1$.

Also, we have:

Corollary 2. *Assume that the positive invertible operator A satisfies the condition $0 < n1_H \le A \le N1_H$ for some constants n, N, then*

$$
\begin{aligned}
1 &\le K\left(\frac{N}{n}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle-\frac{1}{N-n}\left\langle\left|nN-\frac{1}{2}(n+N)A\right|x,x\right\rangle\right]} \\
&\le \frac{\Delta_x(A)}{N^{\frac{N(n-\langle Ax,x\rangle)}{N-n}}n^{\frac{n(\langle Ax,x\rangle-N)}{N-n}}} \\
&\le K\left(\frac{N}{n}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle+\frac{1}{N-n}\left\langle\left|nN-\frac{1}{2}(n+N)A\right|x,x\right\rangle\right]} \\
&\le K\left(\frac{N}{n}\right)^{\langle Ax,x\rangle} \le K\left(\frac{N}{n}\right)^{N}
\end{aligned}
\tag{2.8}
$$

for all $x \in H$ with $\|x\| = 1$.

Proof. If we assume that $0 < mA \le 1_H \le MA$ for some constants m, M, then for $B = 1_H$ in (2.3) we get

$$
\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle-\frac{1}{M-m}\left\langle\left|1-\frac{1}{2}(m+M)A\right|x,x\right\rangle\right]} \\
&\le \frac{\Delta_x(A)}{m^{\frac{M\langle Ax,x\rangle-1}{M-m}}M^{\frac{1-m\langle Ax,x\rangle}{M-m}}} \\
&\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle+\frac{1}{M-m}\left\langle\left|1-\frac{1}{2}(m+M)A\right|x,x\right\rangle\right]} \\
&\le K\left(\frac{M}{m}\right)^{\langle Ax,x\rangle}
\end{aligned}
\tag{2.9}
$$

for all $x \in H$ with $\|x\| = 1$.

If $0 < n1_H \leq A \leq N1_H$, then $\frac{1}{N}A \leq 1_H \leq \frac{1}{n}A$ and by taking $M = \frac{1}{n}$ and $m = \frac{1}{N}$ in (2.9), we get

$$1 \leq K\left(\frac{\frac{1}{n}}{\frac{1}{N}}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle - \frac{1}{\frac{1}{n}-\frac{1}{N}}\left\langle\left|1-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{N}\right)A\right|x,x\right\rangle\right]}$$
$$\leq \frac{\Delta_x(A)}{\left(\frac{1}{N}\right)^{\frac{\frac{1}{n}\langle Ax,x\rangle - 1}{\frac{1}{n}-\frac{1}{N}}}\left(\frac{1}{n}\right)^{\frac{1-\frac{1}{N}\langle Ax,x\rangle}{\frac{1}{n}-\frac{1}{N}}}}$$
$$\leq K\left(\frac{\frac{1}{n}}{\frac{1}{N}}\right)^{\left[\frac{1}{2}\langle Ax,x\rangle + \frac{1}{\frac{1}{n}-\frac{1}{N}}\left\langle\left|1-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{N}\right)A\right|x,x\right\rangle\right]} \leq K\left(\frac{\frac{1}{n}}{\frac{1}{N}}\right)^{\langle Ax,x\rangle},$$

which is equivalent to (2.8). □

3. Related Results

We also have:

Theorem 2. *Assume that the positive invertible operators A, B satisfy the condition $0 < mA \leq B \leq MA$ for some constants m, M, then*

$$(3.1) \qquad 1 \leq \frac{D_x(A|B)}{m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}}$$
$$\leq \exp\left[\frac{\left\langle (MA-B)A^{-1}(B-mA)x,x\right\rangle}{Mm}\right] \leq \exp\left[\frac{(M-m)^2}{4Mm}\langle Ax,x\rangle\right]$$

for all $x \in H$ with $\|x\| = 1$.

Proof. In [1] we obtained the following reverses of Young's inequality:

$$1 \leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right],$$

where a, $b > 0$, $\nu \in [0,1]$.

This is equivalent, by taking the logarithm, with

$$0 \leq \ln((1-\nu)a + \nu b) - (1-\nu)\ln a - \nu \ln b \leq \nu(1-\nu)\frac{(b-a)^2}{ba}$$

where a, $b > 0$, $\nu \in [0,1]$.

If we take $a = m$, $b = M$, $t \in [m, M]$ and $\nu = \frac{t-m}{M-m} \in [0,1]$, then we get

$$0 \leq \ln t - \frac{M-t}{M-m}\ln m - \frac{t-m}{M-m}\ln M \leq \frac{(M-t)(t-m)}{(M-m)^2}\frac{(M-m)^2}{Mm}$$
$$= \frac{(M-t)(t-m)}{Mm}.$$

Using the continuous functional calculus for selfadjoint operator T with $0 < m1_H \leq T \leq M1_H$ we have that

$$(3.2) \qquad 0 \leq \ln T - \ln m\frac{M1_H - T}{M-m} - \ln M\frac{T - m1_H}{M-m} \leq \frac{(M1_H - T)(T - m1_H)}{Mm}$$
$$\leq \frac{(M-m)^2}{4Mm}1_H.$$

By writing (3.2) for $T = A^{-1/2}BA^{-1/2}$, we get

$$\begin{aligned}
0 &\leq \ln\left(A^{-1/2}BA^{-1/2}\right) \\
&\quad - \ln m \frac{M1_H - A^{-1/2}BA^{-1/2}}{M-m} - \ln M \frac{A^{-1/2}BA^{-1/2} - m1_H}{M-m} \\
&\leq \frac{\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)}{Mm} \\
&\leq \frac{(M-m)^2}{4Mm} 1_H.
\end{aligned}$$

Now, if we multiply this inequality both sides by $A^{1/2} > 0$, then we get

$$\begin{aligned}
0 &\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} - \ln m \frac{MA - B}{M-m} - \ln M \frac{B - mA}{M-m} \\
&\leq \frac{A^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)A^{1/2}}{Mm} \\
&\leq \frac{(M-m)^2}{4Mm} A,
\end{aligned}$$

namely

$$\begin{aligned}
0 &\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} - \ln m \frac{MA - B}{M-m} - \ln M \frac{B - mA}{M-m} \\
&\leq \frac{(MA - B)A^{-1}(B - mA)}{Mm} \leq \frac{(M-m)^2}{4Mm} A,
\end{aligned}$$

If we take the inner product over $x \in H$ with $\|x\| = 1$, then we get

$$\begin{aligned}
0 &\leq \left\langle A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}x, x\right\rangle \\
&\quad - \ln\left(m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}\right) \\
&\leq \frac{\left\langle (MA - B)A^{-1}(B - mA)x, x\right\rangle}{Mm} \leq \frac{(M-m)^2}{4Mm}\langle Ax, x\rangle.
\end{aligned}$$

If we take the exponential, then we get the desired result (3.1). □

Corollary 3. *Assume that the positive invertible operator B satisfies the condition $0 < m1_H \leq B \leq M1_H$ for some constants m, M, then*

$$\begin{aligned}
1 &\leq \frac{\eta_x(B)}{m^{\frac{M - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m}{M-m}}} \\
&\leq \exp\left[\frac{\langle (M1_H - B)(B - m1_H)x, x\rangle}{Mm}\right] \leq \exp\left[\frac{(M-m)^2}{4Mm}\right]
\end{aligned} \tag{3.3}$$

for all $x \in H$ with $\|x\| = 1$.

Also, we have:

Corollary 4. *Assume that the positive invertible operator A satisfies the condition $0 < n1_H \leq A \leq N1_H$ for some constants n, N, then*

$$1 \leq \frac{\Delta_x(A)}{N^{\frac{N(n-\langle Ax,x\rangle)}{N-n}} n^{\frac{n(\langle Ax,x\rangle - N)}{N-n}}} \tag{3.4}$$
$$\leq \exp\left[\left\langle (A - n1_H) A^{-1} (N1_H - A) x, x\right\rangle\right] \leq \exp\left[\frac{(N-n)^2}{4Nn} \langle Ax, x\rangle\right]$$

for all $x \in H$ with $\|x\| = 1$.

In [2] we obtained the following refinement and reverse of Young's inequality:

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \tag{3.5}$$
$$\leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}}$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right],$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Theorem 3. *Assume that the positive invertible operators A, B satisfy the condition $0 < mA \leq B \leq MA$ for some constants m, M, then*

$$1 \leq \exp\left[\frac{\left\langle (MA - B) A^{-1} (B - mA) x, x\right\rangle}{2M^2}\right] \tag{3.6}$$
$$\leq \frac{D_x(A|B)}{m^{\frac{M\langle Ax,x\rangle - \langle Bx,x\rangle}{M-m}} M^{\frac{\langle Bx,x\rangle - m\langle Ax,x\rangle}{M-m}}}$$
$$\leq \exp\left[\frac{\left\langle (MA - B) A^{-1} (B - mA) x, x\right\rangle}{2m^2}\right] \leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2 \langle Ax, x\rangle\right]$$

for all $x \in H$ with $\|x\| = 1$.

Proof. From (3.5) we have

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{m}{M}\right)^2\right]$$
$$\leq \frac{(1-\nu)m + \nu M}{m^{1-\nu}M^{\nu}} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m} - 1\right)^2\right],$$

for $\nu \in [0,1]$.

By taking the logarithm, we obtain

$$\frac{1}{2}\nu(1-\nu)\left(1 - \frac{m}{M}\right)^2 \tag{3.7}$$
$$\leq \ln((1-\nu)m + \nu M) - (1-\nu)\ln m - \nu \ln M$$
$$\leq \frac{1}{2}\nu(1-\nu)\left(\frac{M}{m} - 1\right)^2,$$

for $\nu \in [0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$\begin{aligned}\frac{(M-t)(t-m)}{2M^2}&\le \ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M\\&\le\frac{(M-t)(t-m)}{2m^2}\end{aligned}$$

for all $t\in[m,M]$.

As above, we have

$$\begin{aligned}&\frac{\left(M-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m\right)}{2M^2}\\&\le\ln\left(A^{-1/2}BA^{-1/2}\right)\\&-\ln m\frac{M1_H-A^{-1/2}BA^{-1/2}}{M-m}-\ln M\frac{A^{-1/2}BA^{-1/2}-m1_H}{M-m}\\&\le\frac{\left(M-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m\right)}{2m^2}\end{aligned}$$

and by utilising a similar argument to the one in Theorem 2, we derive the desired result (3.6). □

Corollary 5. *Assume that the positive invertible operator B satisfies the condition $0<m1_H\le B\le M1_H$ for some constants m, M, then*

$$\begin{aligned}1&\le\exp\left[\frac{\langle(M1_H-B)(B-m1_H)x,x\rangle}{2M^2}\right]\\&\le\frac{\eta_x(B)}{m^{\frac{M-\langle Bx,x\rangle}{M-m}}M^{\frac{\langle Bx,x\rangle-m}{M-m}}}\\&\le\exp\left[\frac{\langle(M1_H-B)(B-m1_H)x,x\rangle}{2m^2}\right]\le\exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right]\end{aligned}\tag{3.8}$$

for all $x\in H$ with $\|x\|=1$.

Finally, we can also state that:

Corollary 6. *Assume that the positive invertible operator A satisfies the condition $0<n1_H\le A\le N1_H$ for some constants n, N, then*

$$\begin{aligned}1&\le\exp\left[\frac{n\left\langle(A-n1_H)A^{-1}(N1_H-A)x,x\right\rangle}{2N}\right]\\&\le\frac{\Delta_x(A)}{N^{\frac{N(n-\langle Ax,x\rangle)}{N-n}}n^{\frac{n(\langle Ax,x\rangle-N)}{N-n}}}\\&\le\exp\left[\frac{N\left\langle(A-n1_H)A^{-1}(N1_H-A)x,x\right\rangle}{2n}\right]\le\exp\left[\frac{(N-n)^2}{4Nn}\langle Ax,x\rangle\right]\\&\le\exp\left[\frac{(N-n)^2}{4n}\right]\end{aligned}\tag{3.9}$$

for all $x\in H$ with $\|x\|=1$.

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