SOME BOUNDS FOR THE RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle.$$

In this paper we show among others that, if $0 < A_i,\ B_i,\ i \in \{1, ..., n\}$ with $0 < mA_i \le B_i \le MA_i,\ i \in \{1, ..., n\}$, then for all $p_i \ge 0$ with $\sum_{i=1}^n p_i = 1$,

$$1 \le \frac{\left(\frac{\sum_{i=1}^n p_i\langle B_i x, x\rangle}{\sum_{i=1}^n p_i\langle A_i x, x\rangle}\right)^{\sum_{i=1}^n p_i\langle A_i x, x\rangle}}{\prod_{i=1}^n [D_x(A_i|B_i)]^{p_i}} \le \exp\left(\frac{(M-m)^2}{4mM}\sum_{i=1}^n p_i\langle A_i x, x\rangle\right)$$

and

$$\begin{aligned} 1 &\le \frac{\prod_{i=1}^n [D_x(A_i|B_i)]^{p_i}}{\left(\frac{\sum_{i=1}^n p_i\langle A_i x, x\rangle}{\sum_{i=1}^n p_i\left\langle A_i B_i^{-1} A_i x, x\right\rangle}\right)^{\sum_{i=1}^n p_i\langle A_i x, x\rangle}} \\ &\le \exp\left(\frac{(M-m)^2}{4mM}\sum_{i=1}^n p_i\langle A_i x, x\rangle\right) \end{aligned}$$

for all $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [9].

For each unit vector $x \in H$, see also [12], we have:

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(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [9] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [18]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [10], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\langle tA\ln(tA)x,x\rangle\right)\\
&=\exp\left(-\langle tA(\ln t+\ln A)x,x\rangle\right)=\exp\left(-\langle (tA\ln t+tA\ln A)x,x\rangle\right)\\
&=\exp\left(-\langle Ax,x\rangle t\ln t\right)\exp\left(-t\langle A\ln Ax,x\rangle\right)\\
&=\exp\ln\left(t^{-\langle Ax,x\rangle t}\right)\left[\exp\left(-\langle A\ln Ax,x\rangle\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In the recent paper [3] we showed among others that, if $A,\ B > 0$, then for all $x \in H, \|x\| = 1$ and $t \in [0,1]$,

$$\eta_x((1-t)A + tB) \geq (\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A > 0$ and $x \in H, \|x\| = 1$.

Definition 1. *For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B) := \exp\langle S(A|B)x,x\rangle = \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle.$$

We observe that for $A > 0$,

$$D_x(A|1_H) = \exp\langle S(A|1_H)x,x\rangle = \exp\left(-\langle A\ln Ax,x\rangle\right) = \eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B > 0$,

$$D_x(1_H|B) := \exp\langle S(1_H|B)x,x\rangle = \exp\langle \ln Bx,x\rangle = \Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant*.

In this paper we show among others that, if $0 < A_i,\ B_i,\ i \in \{1,...,n\}$ with $0 < mA_i \leq B_i \leq MA_i,\ i \in \{1,...,n\}$, then for all $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$,

$$1 \leq \frac{\left(\frac{\sum_{i=1}^n p_i\langle B_ix,x\rangle}{\sum_{i=1}^n p_i\langle A_ix,x\rangle}\right)^{\sum_{i=1}^n p_i\langle A_ix,x\rangle}}{\prod_{i=1}^n\left[D_x(A_i|B_i)\right]^{p_i}} \leq \exp\left(\frac{(M-m)^2}{4mM}\sum_{i=1}^n p_i\langle A_ix,x\rangle\right)$$

and

$$1 \leq \frac{\prod_{i=1}^n\left[D_x(A_i|B_i)\right]^{p_i}}{\left(\frac{\sum_{i=1}^n p_i\langle A_ix,x\rangle}{\sum_{i=1}^n p_i\left\langle A_iB_i^{-1}A_ix,x\right\rangle}\right)^{\sum_{i=1}^n p_i\langle A_ix,x\rangle}} \leq \exp\left(\frac{(M-m)^2}{4mM}\sum_{i=1}^n p_i\langle A_ix,x\rangle\right)$$

for all $x \in H$ with $\|x\| = 1$.

2. Main Results

We start the following main result

Theorem 1. *Assume that* $0 < A_i,\ B_i,\ i \in \{1, ..., n\}$, *then for all* $s > 0$ *and* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$

$$\exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - \frac{1}{s}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right) \tag{2.1}$$
$$\leq \frac{s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}{\prod_{i=1}^{n} [D_x (A_i|B_i)]^{p_i}}$$
$$\leq \exp\left(s\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right),$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$1 \leq \frac{\left(\frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}\right)^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}{\prod_{i=1}^{n} [D_x (A_i|B_i)]^{p_i}} \tag{2.2}$$
$$\leq \exp\left(\frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle \sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)^2}{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}\right),$$

for all $x \in H$, $\|x\| = 1$.

Proof. We use the following gradient inequality for differentiable convex function f on the open interval I

$$f'(s)(t - s) \leq f(t) - f(s) \leq f'(t)(t - s)$$

for all $t,\ s \in I$.

If we use the continuos functional calculus for the selfadjoint operator T with $\operatorname{Sp}(T) \subset I$, then

$$f'(s)(T - s1_H) \leq f(T) - f(s) 1_H \leq f'(T)(T - s1_H) \tag{2.3}$$

for all $s \in I$.

Let $I = (0, \infty)$. Since $0 < A_i^{-1/2} B_i A_i^{-1/2}$, $i \in \{1, ..., n\}$, by writing (2.3) for $T = A_i^{-1/2} B_i A_i^{-1/2}$ we get

$$f'(s)\left(A_i^{-1/2} B_i A_i^{-1/2} - s1_H\right) \tag{2.4}$$
$$\leq f\left(A_i^{-1/2} B_i A_i^{-1/2}\right) - f(s) 1_H$$
$$\leq f'\left(A_i^{-1/2} B_i A_i^{-1/2}\right)\left(A_i^{-1/2} B_i A_i^{-1/2} - s1_H\right)$$

for all $i \in \{1, ..., n\}$.

If we multiply both sides by $A_i^{1/2}>0$, we get

$$\begin{aligned}&f'(s)(B_i-sA_i)\\&\le A_i^{1/2}f\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{1/2}-f(s)A_i\\&\le A_i^{1/2}f'\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{-1/2}B_i-sA_i^{1/2}f'\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{1/2}\end{aligned}$$

for all $i\in\{1,...,n\}$.

Further if we multiply by $p_i\ge 0$ and sum over i from 1 to n, then we get

$$\begin{aligned}&f'(s)\left(\sum_{i=1}^n p_iB_i-s\sum_{i=1}^n p_iA_i\right)\\&\le\sum_{i=1}^n p_iA_i^{1/2}f\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{1/2}-f(s)\sum_{i=1}^n p_iA_i\\&\le\sum_{i=1}^n p_iA_i^{1/2}f'\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{-1/2}B_i\\&-s\sum_{i=1}^n p_iA_i^{1/2}f'\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{1/2}\end{aligned}$$

for all $s>0$.

Further, if we take the inner product over $x\in H$, $\|x\|=1$, then we get the scalar inequalities

$$\begin{aligned}&f'(s)\left(\sum_{i=1}^n p_i\langle B_ix,x\rangle-s\sum_{i=1}^n p_i\langle A_ix,x\rangle\right)\\&\le\sum_{i=1}^n p_i\left\langle A_i^{1/2}f\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{1/2}x,x\right\rangle-f(s)\sum_{i=1}^n p_i\langle A_ix,x\rangle\\&\le\sum_{i=1}^n p_i\left\langle A_i^{1/2}f'\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{-1/2}B_ix,x\right\rangle\\&-s\sum_{i=1}^n p_i\left\langle A_i^{1/2}f'\left(A_i^{-1/2}B_iA_i^{-1/2}\right)A_i^{1/2}x,x\right\rangle\end{aligned}\tag{2.5}$$

for all $s>0$ and $x\in H$, $\|x\|=1$.

By taking

$$s=\frac{\sum_{i=1}^n p_i\langle B_ix,x\rangle}{\sum_{i=1}^n p_i\langle A_ix,x\rangle}>0$$

in (2.5) we get the Jensen's type inequality

$$
\begin{aligned}
(2.6)\qquad 0 &\leq \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} f\left(A_i^{-1/2} B_i A_i^{-1/2}\right) A_i^{1/2} x, x\right\rangle \\
&- f\left(\frac{\sum_{i=1}^{n} p_i \left\langle B_i x, x\right\rangle}{\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle}\right) \sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle \\
&\leq \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} f'\left(A_i^{-1/2} B_i A_i^{-1/2}\right) A_i^{-1/2} B_i x, x\right\rangle \\
&- \frac{\sum_{i=1}^{n} p_i \left\langle B_i x, x\right\rangle}{\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle} \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} f'\left(A_i^{-1/2} B_i A_i^{-1/2}\right) A_i^{1/2} x, x\right\rangle
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

If we take the convex function $f(t) = -\ln t$, $t > 0$, in (2.5), then we get

$$
\begin{aligned}
&\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle - \frac{1}{s} \sum_{i=1}^{n} p_i \left\langle B_i x, x\right\rangle \\
&\leq \ln s \sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \ln\left(A_i^{-1/2} B_i A_i^{-1/2}\right) A_i^{1/2} x, x\right\rangle \\
&\leq s \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \left(A_i^{-1/2} B_i A_i^{-1/2}\right)^{-1} A_i^{1/2} x, x\right\rangle \\
&- \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \left(A_i^{-1/2} B_i A_i^{-1/2}\right)^{-1} A_i^{-1/2} B_i x, x\right\rangle,
\end{aligned}
$$

namely

$$
\begin{aligned}
&\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle - \frac{1}{s} \sum_{i=1}^{n} p_i \left\langle B_i x, x\right\rangle \\
&\leq \ln\left(s^{\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle}\right) - \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \ln\left(A_i^{-1/2} B_i A_i^{-1/2}\right) A_i^{1/2} x, x\right\rangle \\
&\leq s \sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle,
\end{aligned}
$$

for all $s > 0$ and $x \in H$, $\|x\| = 1$.

If we take the exponential, we then get

$$
\begin{aligned}
(2.7)\qquad &\exp\left(\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle - \frac{1}{s} \sum_{i=1}^{n} p_i \left\langle B_i x, x\right\rangle\right) \\
&\leq \frac{\exp\ln\left(s^{\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle}\right)}{\exp \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \ln\left(A_i^{-1/2} B_i A_i^{-1/2}\right) A_i^{1/2} x, x\right\rangle} \\
&\leq \exp\left(s \sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle\right),
\end{aligned}
$$

for all $s \in [m, M]$ and $x \in H$, $\|x\| = 1$.

Observe that

$$\begin{aligned}
&\exp \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \ln \left(A_i^{-1/2} B_i A_i^{-1/2} \right) A_i^{1/2} x, x \right\rangle \\
&= \prod_{i=1}^{n} \left(\exp \left\langle A_i^{1/2} \ln \left(A_i^{-1/2} B_i A_i^{-1/2} \right) A_i^{1/2} x, x \right\rangle \right)^{p_i} \\
&= \prod_{i=1}^{n} \left[D_x \left(A_i | B_i \right) \right]^{p_i}
\end{aligned}$$

and by (2.7) we derive (2.1). □

Remark 1. *The case of one pair of operators is as follows. If* $0 < A$, B, *then for all* $s > 0$

$$\begin{aligned}
\exp \left(\langle Ax, x \rangle - \frac{1}{s} \langle Bx, x \rangle \right) &\le \frac{s^{\langle Ax, x \rangle}}{D_x (A|B)} \\
&\le \exp \left(s \left\langle AB^{-1} Ax, x \right\rangle - \langle Ax, x \rangle \right), \qquad (2.8)
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$\begin{aligned}
1 &\le \frac{\left(\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} \right)^{\langle Ax, x \rangle}}{D_x (A|B)} \\
&\le \exp \left(\frac{\langle Bx, x \rangle \left\langle AB^{-1} Ax, x \right\rangle - \langle Ax, x \rangle^2}{\langle Ax, x \rangle} \right), \qquad (2.9)
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.*The case of two pairs of operators is as follows. If* $0 < A$, B, C, D, *then for all* $s > 0$,

$$\begin{aligned}
&\exp \left(\langle [(1-t) A + tC] x, x \rangle - \frac{1}{s} \langle [(1-t) B + tD] x, x \rangle \right) \qquad (2.10) \\
&\le \frac{s^{\langle [(1-t)A + tC] x, x \rangle}}{\left[D_x (A|B) \right]^{1-t} \left[D_x (C|D) \right]^t} \\
&\le \exp \left(s \left\langle \left[(1-t) AB^{-1} A + tCD^{-1} C \right] x, x \right\rangle - \langle [(1-t) A + tC] x, x \rangle \right),
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$\begin{aligned}
1 &\le \frac{\left(\frac{\langle [(1-t)B + tD] x, x \rangle}{\langle [(1-t)A + tC] x, x \rangle} \right)^{\langle [(1-t)A + tC] x, x \rangle}}{\left[D_x (A|B) \right]^{1-t} \left[D_x (C|D) \right]^t} \qquad (2.11) \\
&\le \exp \left(\frac{\langle [(1-t) B + tD] x, x \rangle}{\langle [(1-t) A + tC] x, x \rangle} \left\langle \left[(1-t) AB^{-1} A + tCD^{-1} C \right] x, x \right\rangle \right. \\
&\quad \left. - \langle [(1-t) A + tC] x, x \rangle \right),
\end{aligned}$$

for all $t \in [0, 1]$ *and* $x \in H$, $\|x\| = 1$.

Corollary 1. *Assume that* $0 < A_i$, $i \in \{1, ..., n\}$, *then for all* $s > 0$ *and* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$

$$\exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - \frac{1}{s}\right) \leq \frac{s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}{\prod_{i=1}^{n} [\eta_x(A_i)]^{p_i}} \leq \exp\left(s\sum_{i=1}^{n} p_i \left\langle A_i^2 x, x\right\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right), \tag{2.12}$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$1 \leq \frac{\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)^{-\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}{\prod_{i=1}^{n} [\Delta_x(A_i)]^{p_i}} \leq \exp\left(\frac{\sum_{i=1}^{n} p_i \left\langle A_i^2 x, x\right\rangle - \left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)^2}{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}\right), \tag{2.13}$$

for all $x \in H$, $\|x\| = 1$.

For one operator $A > 0$, we have

$$\exp\left(\langle Ax, x\rangle - \frac{1}{s}\right) \leq \frac{s^{\langle Ax, x\rangle}}{\eta_x(A)} \leq \exp\left(s\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle\right), \tag{2.14}$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$1 \leq \frac{\langle Ax, x\rangle^{-\langle Ax, x\rangle}}{\Delta_x(A)} \leq \exp\left(\frac{\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2}{\langle Ax, x\rangle}\right), \tag{2.15}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 2. *Assume that* $0 < B_i$, $i \in \{1, ..., n\}$, *then for all* $s > 0$ *and* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$

$$\exp\left(1 - \frac{1}{s}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right) \leq \frac{s}{\prod_{i=1}^{n} [\Delta_x(B_i)]^{p_i}} \leq \exp\left(s\sum_{i=1}^{n} p_i \left\langle B_i^{-1} x, x\right\rangle - 1\right), \tag{2.16}$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$1 \leq \frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}{\prod_{i=1}^{n} [\Delta_x(B_i)]^{p_i}} \leq \exp\left(\sum_{i=1}^{n} p_i \langle B_i x, x\rangle \sum_{i=1}^{n} p_i \left\langle B_i^{-1} x, x\right\rangle - 1\right), \tag{2.17}$$

for all $x \in H$, $\|x\| = 1$.

For one operator $B > 0$ we have

$$\exp\left(1 - \frac{1}{s}\langle Bx, x\rangle\right) \leq \frac{s}{\Delta_x(B_i)} \leq \exp\left(s\left\langle B^{-1}x, x\right\rangle - 1\right), \tag{2.18}$$

for all $x \in H$, $\|x\| = 1$.

In particular,

$$1 \leq \frac{\langle Bx, x\rangle}{\Delta_x(B)} \leq \exp\left(\langle Bx, x\rangle \left\langle B^{-1}x, x\right\rangle - 1\right), \tag{2.19}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 3. *Assume that* $0 < A_i,\ B_i,\ i \in \{1, ..., n\}$, *then for all* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$

$$1 \leq \frac{\prod_{i=1}^{n} [D_x(A_i|B_i)]^{p_i}}{\left(\frac{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}{\sum_{i=1}^{n} p_i\left\langle A_i B_i^{-1} A_i x, x\right\rangle}\right)^{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}} \leq \exp\left(\frac{\sum_{i=1}^{n} p_i\left\langle A_i B_i^{-1} A_i x, x\right\rangle \sum_{i=1}^{n} p_i\langle B_i x, x\rangle - \left(\sum_{i=1}^{n} p_i\langle A_i x, x\rangle\right)^2}{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}\right) \tag{2.20}$$

for all $x \in H$, $\|x\| = 1$.

In particular, we have

$$1 \leq \frac{\prod_{i=1}^{n} [\eta_x(A_i)]^{p_i}}{\left(\frac{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}{\sum_{i=1}^{n} p_i\left\langle A_i^2 x, x\right\rangle}\right)^{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}} \leq \exp\left(\frac{\sum_{i=1}^{n} p_i\left\langle A_i^2 x, x\right\rangle - \left(\sum_{i=1}^{n} p_i\langle A_i x, x\rangle\right)^2}{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}\right) \tag{2.21}$$

and

$$1 \leq \frac{\prod_{i=1}^{n} [\Delta_x(B_i)]^{p_i}}{\frac{1}{\sum_{i=1}^{n} p_i\left\langle B_i^{-1}x, x\right\rangle}} \leq \exp\left(\sum_{i=1}^{n} p_i\left\langle B_i^{-1}x, x\right\rangle \sum_{i=1}^{n} p_i\langle B_i x, x\rangle - 1\right), \tag{2.22}$$

for all $x \in H$, $\|x\| = 1$.

The inequality (2.20) follows by the second inequality in (2.1) for

$$s = \frac{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}{\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x \right\rangle}$$

and performing the required calculations.

We observe that for $s = 1$ in (2.1) we get the simpler bounds

$$\text{(2.23)} \quad \exp\left(\sum_{i=1}^{n} p_i \langle (A_i - B_i)\, x, x \rangle\right) \leq \left(\prod_{i=1}^{n} \left[D_x \left(A_i | B_i\right)\right]^{p_i}\right)^{-1} \leq \exp\left(\sum_{i=1}^{n} p_i \left\langle A_i \left(B_i^{-1} - A_i^{-1}\right) A_i x, x \right\rangle\right),$$

for all $x \in H$, $\|x\| = 1$.

Proposition 1. *Assume that* $0 < A_i,\ B_i,\ i \in \{1, ..., n\}$, $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $x \in H$, $\|x\| = 1$. *The best lower bound for*

$$\left(\prod_{i=1}^{n} \left[D_x \left(A_i | B_i\right)\right]^{p_i}\right)^{-1}$$

out of the inequality (2.1) is obtained for

$$s = \frac{\sum_{i=1}^{n} p_i \langle B_i x, x \rangle}{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}$$

and is

$$\left(\frac{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}{\sum_{i=1}^{n} p_i \langle B_i x, x \rangle}\right)^{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}.$$

The best upper bound for the same quantity is obtained for

$$s = \frac{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}{\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x \right\rangle}$$

and is

$$\left(\frac{\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x \right\rangle}{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}\right)^{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}.$$

Proof. Consider the function

$$f(s) := \frac{\exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x \rangle - \frac{1}{s} \sum_{i=1}^{n} p_i \langle B_i x, x \rangle\right)}{s^{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle}}, \ s > 0.$$

Then

$$\begin{aligned} f'(s) &= \exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - \frac{1}{s}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right) \\ &\quad \times \frac{\frac{1}{s^2}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle \, s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}{s^{2\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}} \\ &\quad - \exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - \frac{1}{s}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right) \\ &\quad \times \frac{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle \, s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - 1}}{s^{2\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}} \\ &= \frac{\exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - \frac{1}{s}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right)}{s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle + 1}} \\ &\quad \times \left(\frac{1}{s}\sum_{i=1}^{n} p_i \langle B_i x, x\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \end{aligned}$$

for $s > 0$.

This shows that the function f is increasing on $\left(0, \frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}\right)$ and decreasing on $\left(\frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}, \infty\right)$ and

$$\sup_{s \in (0,\infty)} f(s) = f\left(\frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}\right) = \left(\frac{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}\right)^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}.$$

Further, consider the function

$$g(s) := \frac{\exp\left(s\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)}{s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}, \; s > 0.$$

Then

$$\begin{aligned} g'(s) &:= \frac{\exp\left(s\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)}{s^{2\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}} \\ &\quad \times \sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle} \\ &\quad - \frac{\exp\left(s\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)}{s^{2\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}} \\ &\quad \times \sum_{i=1}^{n} p_i \langle A_i x, x\rangle \, s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - 1} \\ &= \frac{\exp\left(s\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)}{s^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle + 1}} \\ &\quad \times \left(\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle s - \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \end{aligned}$$

for $s > 0$.

This shows that the function g is decreasing on $\left(0,\frac{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}{\sum_{i=1}^{n}p_i\left\langle A_iB_i^{-1}A_ix,x\right\rangle}\right)$ and increasing on $\left(\frac{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}{\sum_{i=1}^{n}p_i\left\langle A_iB_i^{-1}A_ix,x\right\rangle},\infty\right)$ and

$$\begin{aligned}\inf_{s\in(0,\infty)}g(s) &= \frac{1}{\left(\frac{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}{\sum_{i=1}^{n}p_i\left\langle A_iB_i^{-1}A_ix,x\right\rangle}\right)^{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}} \\ &= \left(\frac{\sum_{i=1}^{n}p_i\left\langle A_iB_i^{-1}A_ix,x\right\rangle}{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}\right)^{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}\end{aligned}$$

and the proposition is proved. □

3. Related Results

We start to the following inequality which is of interest in itself:

Lemma 1. *Assume that the positive invertible operators A_i, B_i satisfy the condition*

$$0<mA_i\le B_i\le MA_i \text{ for } i\in\{1,...,n\}, \tag{3.1}$$

then for $p_i\ge 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$ we have

$$\sum_{i=1}^{n}p_i\left\langle A_iB_i^{-1}A_ix,x\right\rangle\sum_{i=1}^{n}p_i\langle B_ix,x\rangle\le\frac{(M+m)^2}{4mM}\left(\sum_{i=1}^{n}p_i\langle A_ix,x\rangle\right)^2. \tag{3.2}$$

Proof. Since $0<mA_i\le B_i\le MA_i$ for $i\in\{1,...,n\}$, then by multiplying both sides by $A_i^{-1/2}$ we obtain

$$0<m1_H\le A_i^{-1/2}B_iA_i^{-1/2}\le M1_H \text{ for } i\in\{1,...,n\}$$

and by taking the inverse, we also have

$$0<M^{-1}1_H\le A_i^{1/2}B_i^{-1}A_i^{1/2}\le m^{-1}1_H \text{ for } i\in\{1,...,n\}.$$

This implies that

$$\left(M1_H-A_i^{-1/2}B_iA_i^{-1/2}\right)\left(m^{-1}1_H-A_i^{1/2}B_i^{-1}A_i^{1/2}\right)\ge 0,$$

namely

$$\begin{aligned}&m^{-1}M1_H+A_i^{-1/2}B_iA_i^{-1/2}A_i^{1/2}B_i^{-1}A_i^{1/2}\\ &\ge MA_i^{1/2}B_i^{-1}A_i^{1/2}+m^{-1}A_i^{-1/2}B_iA_i^{-1/2}\end{aligned}$$

that is

$$m^{-1}M1_H+1_H\ge MA_i^{1/2}B_i^{-1}A_i^{1/2}+m^{-1}A_i^{-1/2}B_iA_i^{-1/2}$$

for $i\in\{1,...,n\}$.

Now, if we multiply by $m>0$ and both sides by $A_i^{1/2}>0$, then we get

$$(M+m)A_i\ge MmA_iB_i^{-1}A_i+B_i$$

for $i\in\{1,...,n\}$ and if we multiply by $p_i\ge 0$ and sum over i from 1 to n, then we get the operator inequality of interest

$$(M+m)\sum_{i=1}^{n}p_iA_i\ge Mm\sum_{i=1}^{n}p_iA_iB_i^{-1}A_i+\sum_{i=1}^{n}p_iB_i. \tag{3.3}$$

Now, if we take the inner product over $x \in H$, $\|x\| = 1$, then we get

$$(M+m)\sum_{i=1}^{n} p_i \langle A_i x, x\rangle \geq Mm\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle + \sum_{i=1}^{n} p_i \langle B_i x, x\rangle . \tag{3.4}$$

Since $\langle A_i x, x\rangle$, $\left\langle A_i B_i^{-1} A_i x, x\right\rangle$, $\langle B_i x, x\rangle > 0$ for $i \in \{1, ..., n\}$, then by arithmetic mean-geometric mean inequality $a + b \geq 2\sqrt{ab}$ for $a, b > 0$, we get

$$\begin{aligned} & Mm\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle + \sum_{i=1}^{n} p_i \langle B_i x, x\rangle \\ & \geq 2\sqrt{Mm}\left(\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle\right)^{1/2}\left(\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right)^{1/2}. \end{aligned} \tag{3.5}$$

By (3.4) and (3.5) we get

$$\begin{aligned} & (M+m)\sum_{i=1}^{n} p_i \langle A_i x, x\rangle \\ & \geq 2\sqrt{Mm}\left(\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle\right)^{1/2}\left(\sum_{i=1}^{n} p_i \langle B_i x, x\rangle\right)^{1/2} \end{aligned}$$

and by taking the square, we derive (3.2). □

Remark 2. *The case of one pair of operators is as follows, if* $0 < mA \leq B \leq MA$ *for positive constants* m, M, *then*

$$\left\langle AB^{-1}Ax, x\right\rangle \langle Bx, x\rangle \leq \frac{(M+m)^2}{4mM}\langle Ax, x\rangle^2 \tag{3.6}$$

for all $x \in H$ *with* $\|x\| = 1$.

Corollary 4. *Assume that the positive invertible operators* A_i, B_i *satisfy the condition (3.1), then for* $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *we have*

$$1 \leq \frac{\left(\frac{\sum_{i=1}^{n} p_i\langle B_i x, x\rangle}{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}\right)^{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}}{\prod_{i=1}^{n}\left[D_x\left(A_i|B_i\right)\right]^{p_i}} \leq \exp\left(\frac{(M-m)^2}{4mM}\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \tag{3.7}$$

and

$$\begin{aligned} 1 & \leq \frac{\prod_{i=1}^{n}\left[D_x\left(A_i|B_i\right)\right]^{p_i}}{\left(\frac{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}{\sum_{i=1}^{n} p_i\left\langle A_i B_i^{-1} A_i x, x\right\rangle}\right)^{\sum_{i=1}^{n} p_i\langle A_i x, x\rangle}} \\ & \leq \exp\left(\frac{(M-m)^2}{4mM}\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \end{aligned} \tag{3.8}$$

for all $x \in H$ *with* $\|x\| = 1$.

The proof follows by the inequalities (2.2), (2.20) and (3.2).

Remark 3. *If* $0 < m1_H \leq B_i \leq M1_H$ *for* $i \in \{1, ..., n\}$, *then for* $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *we have*

$$1 \leq \frac{\sum_{i=1}^{n} p_i \langle B_i x, x\rangle}{\prod_{i=1}^{n} [\Delta_x(B_i)]^{p_i}} \leq \exp\left(\frac{(M-m)^2}{4mM}\right) \tag{3.9}$$

and

$$1 \leq \frac{\prod_{i=1}^{n} [\Delta_x(B_i)]^{p_i}}{\left(\sum_{i=1}^{n} p_i \left\langle B_i^{-1} x, x\right\rangle\right)^{-1}} \leq \exp\left(\frac{(M-m)^2}{4mM}\right) \tag{3.10}$$

for all $x \in H$ *with* $\|x\| = 1$.

If $0 < n1_H \leq A_i \leq N1_H$ *for* $i \in \{1, ..., n\}$, *then for* $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *we have*

$$1 \leq \frac{\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right)^{-\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}}{\prod_{i=1}^{n} [\eta_x(A_i)]^{p_i}} \leq \exp\left(\frac{(N-n)^2}{4nN} \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \leq \exp\left(\frac{(N-n)^2}{4n}\right) \tag{3.11}$$

and

$$1 \leq \frac{\prod_{i=1}^{n} [\eta_x(A_i)]^{p_i}}{\left(\frac{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}{\sum_{i=1}^{n} p_i \langle A_i^2 x, x\rangle}\right)^{\sum_{i=1}^{n} p_i \langle A_i x, x\rangle}} \leq \exp\left(\frac{(N-n)^2}{4nN} \sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \leq \exp\left(\frac{(N-n)^2}{4n}\right) \tag{3.12}$$

We finally observe that, if $0 < m1_H \leq B \leq M1_H$, then

$$1 \leq \frac{\langle Bx, x\rangle}{\Delta_x(B)} \leq \exp\left(\frac{(M-m)^2}{4mM}\right) \tag{3.13}$$

and

$$1 \leq \frac{\Delta_x(B)}{\langle B^{-1} x, x\rangle^{-1}} \leq \exp\left(\frac{(M-m)^2}{4mM}\right) \tag{3.14}$$

for all $x \in H$ with $\|x\| = 1$.

Also, if $0 < n1_H \leq A_i \leq N1_H$, then

$$1 \leq \frac{\langle Ax, x\rangle^{-\langle Ax, x\rangle}}{\eta_x(A)} \leq \exp\left(\frac{(N-n)^2}{4nN} \langle Ax, x\rangle\right) \leq \exp\left(\frac{(N-n)^2}{4n}\right) \tag{3.15}$$

and

$$1 \leq \frac{\eta_x(A)}{\left(\frac{\langle Ax, x\rangle}{\langle A^2 x, x\rangle}\right)^{\langle Ax, x\rangle}} \leq \exp\left(\frac{(N-n)^2}{4nN} \langle Ax, x\rangle\right) \leq \exp\left(\frac{(N-n)^2}{4n}\right) \tag{3.16}$$

for all $x \in H$ with $\|x\| = 1$.

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