SOME BOUNDS FOR THE RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA OSTROWSKI TYPE INEQUALITIES

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Abstract. For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle.$$

In this paper we show among others that, if the positive operators A, B satisfy the condition $0 < mA \le B \le MA$, then

$$\begin{aligned}
&\left(\frac{m}{M}\right)^{\langle Ax,x\rangle}\\
&\le \left(\frac{m}{M}\right)^{\frac{1}{2}\langle Ax,x\rangle+\frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|A^{1/2}x,x\right\rangle}\\
&\le \frac{D_x(A|B)}{[I(m,M)]^{\langle Ax,x\rangle}}\\
&\le \left(\frac{M}{m}\right)^{\frac{1}{2}\langle Ax,x\rangle+\frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|A^{1/2}x,x\right\rangle}\\
&\le \left(\frac{M}{m}\right)^{\langle Ax,x\rangle},
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [11], [12], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [11].

For each unit vector $x \in H$, see also [14], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;

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(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [11] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [21]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [12], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp\left(-\langle A\ln Ax, x\rangle\right) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\langle tA\ln(tA)x, x\rangle\right) \\
&= \exp\left(-\langle tA(\ln t + \ln A)x, x\rangle\right) = \exp\left(-\langle (tA\ln t + tA\ln A)x, x\rangle\right) \\
&= \exp\left(-\langle Ax, x\rangle t\ln t\right)\exp\left(-t\langle A\ln Ax, x\rangle\right) \\
&= \exp\ln\left(t^{-\langle Ax, x\rangle t}\right)\left[\exp\left(-\langle A\ln Ax, x\rangle\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle} [\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In the recent paper [4] we showed among others that, if A, $B > 0$, then for all $x \in H, \|x\| = 1$ and $t \in [0,1]$,

$$\eta_x((1-t)A + tB) \geq (\eta_x(A))^{1-t} (\eta_x(B))^t .$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A > 0$ and $x \in H, \|x\| = 1$.

Definition 1. *For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B) := \exp\langle S(A|B)x,x\rangle = \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle .$$

We observe that for $A > 0$,

$$D_x(A|1_H) = \exp\langle S(A|1_H)x,x\rangle = \exp(-\langle A\ln Ax,x\rangle) = \eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B > 0$,

$$D_x(1_H|B) := \exp\langle S(1_H|B)x,x\rangle = \exp\langle \ln Bx,x\rangle = \Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant.*

Motivated by the above results, in this paper we show among others that, if the positive operators A, B satisfy the condition $0 < mA \leq B \leq MA$, then

$$\begin{aligned}
&\left(\frac{m}{M}\right)^{\langle Ax,x\rangle} \\
&\leq \left(\frac{m}{M}\right)^{\frac{1}{2}\langle Ax,x\rangle + \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|A^{1/2}x,x\right\rangle} \\
&\leq \frac{D_x(A|B)}{[I(m,M)]^{\langle Ax,x\rangle}} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\langle Ax,x\rangle + \frac{1}{M-m}\left\langle A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|A^{1/2}x,x\right\rangle} \\
&\leq \left(\frac{M}{m}\right)^{\langle Ax,x\rangle},
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

2. Bounds Via Ostrowski's Inequality

Recall the *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b \neq a \\ a & \text{if } b = a \end{cases} ; \; a, \; b > 0.$$

It is easy to observe the connection between the integral mean of the logarithmic function and the logarithm of the identric mean,

$$\frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b)$$

for $a \neq b$ positive numbers.

Theorem 1. *Assume that the positive operators A_i, B_i satisfy the condition $0 < mA_i \leq B_i \leq MA_i$ for $i \in \{1,...,n\}$, , where m, M are positive numbers, then*

$$\begin{aligned} &\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\sum_{i=1}^n p_i \langle A_i x, x\rangle\right] \quad (2.1) \\ &\leq \exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^n p_i \langle A_i x, x\rangle \right.\right. \\ &\left.\left.+\frac{1}{(M-m)^2}\sum_{i=1}^n p_i \left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right]\right\} \\ &\leq \frac{\prod_{i=1}^n [D_x(A_i|B_i)]^{p_i}}{I(m,M)^{\sum_{i=1}^n p_i\langle A_i x, x\rangle}} \\ &\leq \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^n p_i \langle A_i x, x\rangle \right.\right. \\ &\left.\left.+\frac{1}{(M-m)^2}\sum_{i=1}^n p_i \left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right]\right\} \\ &\leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\sum_{i=1}^n p_i \langle A_i x, x\rangle\right] \end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

Proof. We use Ostrowski's inequality [20]:

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{s\in(a,b)} |f'(s)| < \infty$, then

$$\left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \quad (2.2)$$

for all $t \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $f(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$ in (2.2) and observe that

$$\|f'\|_\infty = \sup_{t\in[a,b]} t^{-1} = \frac{1}{a},$$

then we get

$$|\ln t - \ln I(a,b)| \leq \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a} - 1\right),$$

for all $t \in [a,b]$.

This inequality is equivalent to

$$\begin{aligned} & -\left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a} - 1\right) \\ & \leq \ln t - \ln I(a,b) \leq \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a} - 1\right), \end{aligned} \tag{2.3}$$

for all $t \in [a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.3) that

$$\begin{aligned} & -\frac{1}{2}\left(\frac{M}{m} - 1\right)1_H \\ & \leq -\left(\frac{M}{m} - 1\right)\left[\frac{1}{4}1_H + \frac{1}{(M-m)^2}\left(T - \frac{m+M}{2}1_H\right)^2\right] \\ & \leq \ln T - \ln I(m,M)\,1_H \\ & \leq \left(\frac{M}{m} - 1\right)\left[\frac{1}{4}1_H + \frac{1}{(M-m)^2}\left(T - \frac{m+M}{2}1_H\right)^2\right] \\ & \leq \frac{1}{2}\left(\frac{M}{m} - 1\right)1_H, \end{aligned}$$

where $0 < m1_H \leq T \leq M1_H$.

Since $0 < mA_i \leq B_i \leq MA_i$ for $i \in \{1,...,n\}$, then by multiplying both sides by $A_i^{-1/2}$ we obtain

$$0 < m1_H \leq A_i^{-1/2}B_iA_i^{-1/2} \leq M1_H \textit{ for } i \in \{1,...,n\}.$$

If we replace T with $A_i^{-1/2}B_iA_i^{-1/2}$, then we get

$$\begin{aligned} & -\frac{1}{2}\left(\frac{M}{m} - 1\right)1_H \\ & \leq -\left(\frac{M}{m} - 1\right)\left[\frac{1}{4}1_H + \frac{1}{(M-m)^2}\left(A_i^{-1/2}B_iA_i^{-1/2} - \frac{m+M}{2}1_H\right)^2\right] \end{aligned}$$

$$\leq \ln A_i^{-1/2} B_i A_i^{-1/2} - \ln I(m,M) 1_H$$
$$\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}1_H + \frac{1}{(M-m)^2}\left(A_i^{-1/2}B_iA_i^{-1/2} - \frac{m+M}{2}1_H\right)^2\right]$$
$$\leq \frac{1}{2}\left(\frac{M}{m}-1\right)1_H$$

for $i \in \{1, ..., n\}$.

If we multiply both sides by $A_i^{1/2} \geq 0$, then we get

$$-\frac{1}{2}\left(\frac{M}{m}-1\right)A_i$$
$$\leq -\left(\frac{M}{m}-1\right)$$
$$\times\left[\frac{1}{4}A_i + \frac{1}{(M-m)^2}A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2} - \frac{m+M}{2}1_H\right)^2 A_i^{1/2}\right]$$
$$\leq A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2} - \ln I(m,M) A_i$$
$$\leq \left(\frac{M}{m}-1\right)$$
$$\times\left[\frac{1}{4}A_i + \frac{1}{(M-m)^2}A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2} - \frac{m+M}{2}1_H\right)^2 A_i^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\frac{M}{m}-1\right)A_i$$

for $i \in \{1, ..., n\}$.

If we further multiply by $p_i \geq 0$ and sum over i from 1 to n, then we get

$$-\frac{1}{2}\left(\frac{M}{m}-1\right)\sum_{i=1}^n p_iA_i$$
$$\leq -\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^n p_iA_i\right.$$
$$\left.+\frac{1}{(M-m)^2}\sum_{i=1}^n p_iA_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2} - \frac{m+M}{2}1_H\right)^2 A_i^{1/2}\right]$$
$$\leq \sum_{i=1}^n p_iA_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2} - \ln I(m,M)\sum_{i=1}^n p_iA_i$$
$$\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^n p_iA_i\right.$$
$$\left.+\frac{1}{(M-m)^2}\sum_{i=1}^n p_iA_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2} - \frac{m+M}{2}1_H\right)^2 A_i^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\frac{M}{m}-1\right)\sum_{i=1}^n p_iA_i.$$

Now, if we take the inner product over $x \in H$, $\|x\| = 1$, then

$$-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle + \frac{1}{(M-m)^2}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right]$$

$$\leq \sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}x, x\right\rangle - \sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle \ln I(m,M)$$

$$\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle + \frac{1}{(M-m)^2}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right].$$

Further, if we take the exponential, then we get

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle\right] \\
&\leq \exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle + \frac{1}{(M-m)^2}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right]\right\} \\
&\leq \frac{\exp\left(\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}x, x\right\rangle\right)}{\left[I(m,M)\right]^{\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle}} \\
&\leq \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle + \frac{1}{(M-m)^2}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right]\right\} \\
&\leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\sum_{i=1}^{n} p_i \left\langle A_i x, x\right\rangle\right].
\end{aligned} \tag{2.4}$$

Since

$$\begin{aligned}
&\exp\left(\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}x,x\right\rangle\right)\\
&=\prod_{i=1}^{n}\left[\exp\left\langle A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}x,x\right\rangle\right]^{p_i}\\
&=\prod_{i=1}^{n}\left[D_x(A|B)\right]^{p_i},
\end{aligned}$$

hence by (2.4) we get (2.1). □

Remark 1. *Assume that the positive operators* A, B *satisfy the condition* $0 < mA \le B \le MA$, *then*

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\langle Ax,x\rangle\right]\\
&\le \exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\langle Ax,x\rangle\right.\right.\\
&\left.\left.+\frac{1}{(M-m)^2}\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2A^{1/2}x,x\right\rangle\right]\right\}\\
&\le \frac{D_x(A|B)}{I(m,M)^{\langle Ax,x\rangle}}\\
&\le \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\langle Ax,x\rangle\right.\right.\\
&\left.\left.+\frac{1}{(M-m)^2}\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2A^{1/2}x,x\right\rangle\right]\right\}\\
&\le \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\langle Ax,x\rangle\right]
\end{aligned} \tag{2.5}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 1. *Assume that the positive operators* B_i *satisfy the condition* $0 < m1_H \le B_i \le M1_H$ *for* $i \in \{1,...,n\}$, , *where* m, M *are positive numbers, then*

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right]\\
&\le \exp\left\{-\left(\frac{M}{m}-1\right)\right.\\
&\left.\times\left[\frac{1}{4}+\frac{1}{(M-m)^2}\sum_{i=1}^{n}p_i\left\langle\left(B_i-\frac{m+M}{2}1_H\right)^2x,x\right\rangle\right]\right\}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\leq \frac{\prod_{i=1}^{n}\left[\Delta_x\left(B_i\right)\right]^{p_i}}{I\left(m,M\right)} \\
&\leq \exp\left\{\left(\frac{M}{m}-1\right)\right. \\
&\times\left.\left[\frac{1}{4}+\frac{1}{\left(M-m\right)^2}\sum_{i=1}^{n}p_i\left\langle\left(B_i-\frac{m+M}{2}1_H\right)^2x,x\right\rangle\right]\right\} \\
&\leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right]
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

In the case of one operator, namely, if $0 < m1_H \leq B \leq M1_H$, then we have

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right] \\
&\leq \exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{\left(M-m\right)^2}\left\langle\left(B-\frac{m+M}{2}1_H\right)^2x,x\right\rangle\right]\right\} \\
&\leq \frac{\Delta_x\left(B\right)}{I\left(m,M\right)} \\
&\leq \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{\left(M-m\right)^2}\left\langle\left(B-\frac{m+M}{2}1_H\right)^2x,x\right\rangle\right]\right\} \\
&\leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right]
\end{aligned} \tag{2.7}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 2. *Assume that the positive operators* A_i *satisfy the condition* $0 < p1_H \leq A_i \leq N1_H$ *for* $i \in \{1, ..., n\}$ *, , where* n*,* N *are positive numbers, then*

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\frac{N}{p}-1\right)p\right] \\
&\leq \exp\left[-\frac{1}{2}\left(\frac{N}{p}-1\right)\sum_{i=1}^{n}p_i\left\langle A_ix,x\right\rangle\right] \\
&\leq \exp\left\{-\left(\frac{N}{p}-1\right)\left[\frac{1}{4}\sum_{i=1}^{n}p_i\left\langle A_ix,x\right\rangle\right.\right. \\
&\left.\left.+\frac{1}{\left(p^{-1}-N^{-1}\right)^2}\sum_{i=1}^{n}p_i\left\langle A_i^{1/2}\left(A_i^{-1}-\frac{N^{-1}+p^{-1}}{2}1_H\right)^2A_i^{1/2}x,x\right\rangle\right]\right\}
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&\leq \frac{\prod_{i=1}^{n}\left[\eta_x\left(A_i\right)\right]^{p_i}}{I\left(N^{-1},p^{-1}\right)^{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}}\\
&\leq \exp\left\{\left(\frac{N}{p}-1\right)\left[\frac{1}{4}\sum_{i=1}^{n}p_i\left\langle A_ix,x\right\rangle\right.\right.\\
&\left.\left.+\frac{1}{\left(p^{-1}-N^{-1}\right)^2}\sum_{i=1}^{n}p_i\left\langle A_i^{1/2}\left(A_i^{-1}-\frac{N^{-1}+n^{-1}}{2}1_H\right)^2A_i^{1/2}x,x\right\rangle\right]\right\}\\
&\leq \exp\left[\frac{1}{2}\left(\frac{N}{p}-1\right)\sum_{i=1}^{n}p_i\left\langle A_ix,x\right\rangle\right]\leq \exp\left[\frac{1}{2}\left(\frac{N}{p}-1\right)N\right]
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$.

The case of one operator, namely, if $0<p1_H\leq A\leq N1_H$, is as follows:

$$
\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\frac{N}{p}-1\right)p\right]\\
&\leq \exp\left[-\frac{1}{2}\left(\frac{N}{p}-1\right)\langle Ax,x\rangle\right]\\
&\leq \exp\left\{-\left(\frac{N}{p}-1\right)\left[\frac{1}{4}\langle Ax,x\rangle\right.\right.\\
&\left.\left.+\frac{1}{\left(p^{-1}-N^{-1}\right)^2}\left\langle A^{1/2}\left(A^{-1}-\frac{N^{-1}+p^{-1}}{2}1_H\right)^2A^{1/2}x,x\right\rangle\right]\right\}\\
&\leq \frac{\eta_x\left(A\right)}{I\left(N^{-1},p^{-1}\right)^{\langle Ax,x\rangle}}\\
&\leq \exp\left\{\left(\frac{N}{p}-1\right)\left[\frac{1}{4}\langle Ax,x\rangle\right.\right.\\
&\left.\left.+\frac{1}{\left(p^{-1}-N^{-1}\right)^2}\left\langle A^{1/2}\left(A^{-1}-\frac{N^{-1}+p^{-1}}{2}1_H\right)^2A^{1/2}x,x\right\rangle\right]\right\}\\
&\leq \exp\left[\frac{1}{2}\left(\frac{N}{p}-1\right)\langle Ax,x\rangle\right]\leq \exp\left[\frac{1}{2}\left(\frac{N}{p}-1\right)N\right]
\end{aligned}
\tag{2.9}
$$

for all $x\in H$, $\|x\|=1$.

3. Bounds Via L_1-Type Ostrowski Inequality

We also have the following result that is a consequence of an L_1-Ostrowski Type Inequality:

Theorem 2. *Assume that the positive operators* A_i, B_i *satisfy the condition* $0<mA_i\leq B_i\leq MA_i$ *for* $i\in\{1,...,n\}$, *where* m, M *are positive numbers, then*

$$\left(\frac{M}{m}\right)^{-\sum_{i=1}^{n}p_i\langle A_ix,x\rangle} \tag{3.1}$$
$$\leq \left(\frac{M}{m}\right)^{-\left[\frac{1}{2}\sum_{i=1}^{n}p_i\langle A_ix,x\rangle+\frac{1}{M-m}\sum_{i=1}^{n}p_i\left\langle A_i^{1/2}\left|A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right|A_i^{1/2}x,x\right\rangle\right]}$$
$$\leq \frac{\prod_{i=1}^{n}\left[D_x\left(A_i|B_i\right)\right]^{p_i}}{\left[I\left(m,M\right)\right]^{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle}}$$
$$\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\sum_{i=1}^{n}p_i\langle A_ix,x\rangle+\frac{1}{M-m}\sum_{i=1}^{n}p_i\left\langle A_i^{1/2}\left|A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right|A_i^{1/2}x,x\right\rangle}$$
$$\leq \left(\frac{M}{m}\right)^{\sum_{i=1}^{n}p_i\langle A_ix,x\rangle},$$

for all $x\in H$, $\|x\|=1$.

Proof. In 1997, Dragomir and Wang proved the following Ostrowski type inequality [5]:

Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$, then

$$\left|f\left(t\right)-\frac{1}{b-a}\int_a^b f\left(s\right)ds\right|\leq\left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]\left\|f'\right\|_{[a,b],1}, \tag{3.2}$$

for all $t\in[a,b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],1}:=\int_a^b\left|g\left(t\right)\right|dt.$$

The constant $\frac{1}{2}$ is best possible.

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (3.2) and observe that

$$\|f'\|_{[a,b],1}=\ln b-\ln a,$$

then we get

$$\left|\ln t-\ln I\left(a,b\right)\right|\leq\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right]\left(\ln b-\ln a\right),$$

for all $t\in[a,b]$.

This inequality is equivalent to

$$-\left(\ln b-\ln a\right) \tag{3.3}$$
$$\leq-\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right]\left(\ln b-\ln a\right)$$
$$\leq\ln t-\ln I\left(a,b\right)$$
$$\leq\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right]\left(\ln b-\ln a\right)$$
$$\leq\ln b-\ln a$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (3.3) that

$$\begin{aligned}
&-(\ln M-\ln m)\,1_H \\
&\leq -(\ln M-\ln m)\left[\frac{1}{2}1_H+\frac{1}{M-m}\left|T-\frac{m+M}{2}1_H\right|\right] \\
&\leq \ln T-\ln I(m,M)\,1_H \\
&\leq (\ln M-\ln m)\left[\frac{1}{2}1_H+\frac{1}{M-m}\left|T-\frac{m+M}{2}1_H\right|\right] \\
&\leq (\ln M-\ln m)\,1_H.
\end{aligned}$$

If we replace T with $A_i^{-1/2}B_iA_i^{-1/2}$, then we get

$$\begin{aligned}
&-(\ln M-\ln m)\,1_H \\
&\leq -(\ln M-\ln m)\left[\frac{1}{2}1_H+\frac{1}{M-m}\left|A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right|\right] \\
&\leq \ln A_i^{-1/2}B_iA_i^{-1/2}-\ln I(m,M)\,1_H \\
&\leq (\ln M-\ln m)\left[\frac{1}{2}1_H+\frac{1}{M-m}\left|A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right|\right] \\
&\leq (\ln M-\ln m)\,1_H
\end{aligned}$$

for $i\in\{1,...,n\}$.

If we multiply both sides by $A_i^{1/2}\geq 0$, then we get

$$\begin{aligned}
&-(\ln M-\ln m)\,A_i \\
&\leq -(\ln M-\ln m) \\
&\times\left[\frac{1}{2}A_i+\frac{1}{M-m}A_i^{1/2}\left|A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right|A_i^{1/2}\right] \\
&\leq A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}-\ln I(m,M)\,A_i \\
&\leq (\ln M-\ln m) \\
&\times\left[\frac{1}{2}A_i+\frac{1}{M-m}A_i^{1/2}\left|A_i^{-1/2}B_iA_i^{-1/2}-\frac{m+M}{2}1_H\right|A_i^{1/2}\right] \\
&\leq (\ln M-\ln m)\,A_i
\end{aligned}$$

for $i\in\{1,...,n\}$.

If we further multiply by $p_i \geq 0$ and sum over i from 1 to n, then we get

$$\begin{aligned}
& -(\ln M - \ln m) \sum_{i=1}^{n} p_i A_i \\
& \leq -(\ln M - \ln m) \\
& \times \left[\frac{1}{2} \sum_{i=1}^{n} p_i A_i + \frac{1}{M-m} \sum_{i=1}^{n} p_i A_i^{1/2} \left| A_i^{-1/2} B_i A_i^{-1/2} - \frac{m+M}{2} 1_H \right| A_i^{1/2} \right] \\
& \leq \sum_{i=1}^{n} p_i A_i^{1/2} \left(\ln \left(A_i^{-1/2} B_i A_i^{-1/2} \right) \right) A_i^{1/2} - \ln I(m, M) \sum_{i=1}^{n} p_i A_i \\
& \leq (\ln M - \ln m) \\
& \times \left[\frac{1}{2} \sum_{i=1}^{n} p_i A_i + \frac{1}{M-m} \sum_{i=1}^{n} p_i A_i^{1/2} \left| A_i^{-1/2} B_i A_i^{-1/2} - \frac{m+M}{2} 1_H \right| A_i^{1/2} \right] \\
& \leq (\ln M - \ln m) \sum_{i=1}^{n} p_i A_i .
\end{aligned}$$

Now, if we take the inner product over $x \in H$, $\|x\| = 1$, then

$$\begin{aligned}
& \ln \left(\frac{M}{m} \right)^{-\sum_{i=1}^{n} p_i \langle A_i x, x \rangle} \\
& \leq \ln \left(\frac{M}{m} \right)^{-\left[\frac{1}{2} \sum_{i=1}^{n} p_i \langle A_i x, x \rangle + \frac{1}{M-m} \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \left| A_i^{-1/2} B_i A_i^{-1/2} - \frac{m+M}{2} 1_H \right| A_i^{1/2} x, x \right\rangle \right]} \\
& \leq \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \left(\ln \left(A_i^{-1/2} B_i A_i^{-1/2} \right) \right) A_i^{1/2} x, x \right\rangle - \ln \left[I(m, M) \right]^{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle} \\
& \leq \ln \left(\frac{M}{m} \right)^{\left[\frac{1}{2} \sum_{i=1}^{n} p_i \langle A_i x, x \rangle + \frac{1}{M-m} \sum_{i=1}^{n} p_i \left\langle A_i^{1/2} \left| A_i^{-1/2} B_i A_i^{-1/2} - \frac{m+M}{2} 1_H \right| A_i^{1/2} x, x \right\rangle \right]} \\
& \leq \ln \left(\frac{M}{m} \right)^{\sum_{i=1}^{n} p_i \langle A_i x, x \rangle} .
\end{aligned}$$

By taking the exponential, we derive the desired result (3.1). □

Remark 2. *Assume that the positive operators A, B satisfy the condition $0 < mA \leq B \leq MA$, then*

$$\begin{aligned}
\left(\frac{M}{m} \right)^{-\langle Ax, x \rangle} & \leq \left(\frac{M}{m} \right)^{-\left[\frac{1}{2} \langle Ax, x \rangle + \frac{1}{M-m} \left\langle A^{1/2} \left| A^{-1/2} B A^{-1/2} - \frac{m+M}{2} 1_H \right| A^{1/2} x, x \right\rangle \right]} \\
& \leq \frac{D_x (A|B)}{\left[I(m, M) \right]^{\langle Ax, x \rangle}} \\
& \leq \left(\frac{M}{m} \right)^{\frac{1}{2} \langle Ax, x \rangle + \frac{1}{M-m} \left\langle A^{1/2} \left| A^{-1/2} B A^{-1/2} - \frac{m+M}{2} 1_H \right| A^{1/2} x, x \right\rangle} \\
& \leq \left(\frac{M}{m} \right)^{\langle Ax, x \rangle} ,
\end{aligned} \tag{3.4}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 3. *Assume that the positive operators B_i satisfy the condition $0 < m1_H \leq B_i \leq M1_H$ for $i \in \{1, ..., n\}$, , where m, M are positive numbers, then*

$$\frac{m}{M} \leq \left(\frac{M}{m}\right)^{-\left[\frac{1}{2}+\frac{1}{M-m}\sum_{i=1}^{n} p_i \left\langle \left|B_i - \frac{m+M}{2}1_H\right| x,x\right\rangle\right]} \leq \frac{\prod_{i=1}^{n}\left[\Delta_x\left(B_i\right)\right]^{p_i}}{I\left(m,M\right)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\sum_{i=1}^{n} p_i \left\langle \left|B_i - \frac{m+M}{2}1_H\right| x,x\right\rangle} \leq \frac{M}{m}, \tag{3.5}$$

for all $x \in H$, $\|x\| = 1$.

In the case of one operator, namely, if $0 < m1_H \leq B \leq M1_H$, then we have

$$\frac{m}{M} \leq \left(\frac{M}{m}\right)^{-\left[\frac{1}{2}+\frac{1}{M-m}\left\langle \left|B - \frac{m+M}{2}1_H\right| x,x\right\rangle\right]} \leq \frac{\Delta_x\left(B\right)}{I\left(m,M\right)} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\left\langle \left|B - \frac{m+M}{2}1_H\right| x,x\right\rangle} \leq \frac{M}{m}, \tag{3.6}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 4. *Assume that the positive operators A_i satisfy the condition $0 < p1_H \leq A_i \leq N1_H$ for $i \in \{1, ..., n\}$, , where p, N are positive numbers, then*

$$\begin{aligned} &\left(\frac{N}{p}\right)^{-\sum_{i=1}^{n} p_i \langle A_i x,x\rangle} \\ &\leq \left(\frac{N}{p}\right)^{-\left[\frac{1}{2}\sum_{i=1}^{n} p_i \langle A_i x,x\rangle + \frac{1}{p^{-1}-N^{-1}}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left|A_i^{-1} - \frac{N^{-1}+p^{-1}}{2}1_H\right| A_i^{1/2}x,x\right\rangle\right]} \\ &\leq \frac{\prod_{i=1}^{n}\left[\eta_x\left(A_i\right)\right]^{p_i}}{\left[I\left(p^{-1},N^{-1}\right)\right]^{\sum_{i=1}^{n} p_i \langle A_i x,x\rangle}} \\ &\leq \left(\frac{N}{p}\right)^{\frac{1}{2}\sum_{i=1}^{n} p_i \langle A_i x,x\rangle + \frac{1}{p^{-1}-N^{-1}}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left|A_i^{-1} - \frac{N^{-1}+p^{-1}}{2}1_H\right| A_i^{1/2}x,x\right\rangle} \\ &\leq \left(\frac{N}{p}\right)^{\sum_{i=1}^{n} p_i \langle A_i x,x\rangle}, \end{aligned} \tag{3.7}$$

for all $x \in H$, $\|x\| = 1$.

The case of one operator, namely, if $0 < p1_H \leq A \leq N1_H$, is as follows:

$$\left(\frac{p}{N}\right)^p \leq \left(\frac{p}{N}\right)^{\langle Ax,x\rangle} \tag{3.8}$$
$$\leq \left(\frac{N}{p}\right)^{-\left[\frac{1}{2}\langle Ax,x\rangle+\frac{1}{p^{-1}-N^{-1}}\left\langle A^{1/2}\left|A^{-1}-\frac{N^{-1}+p^{-1}}{2}1_H\right|A^{1/2}x,x\right\rangle\right]}$$
$$\leq \frac{\eta_x(A)}{\left[I\left(p^{-1},N^{-1}\right)\right]^{\langle Ax,x\rangle}}$$
$$\leq \left(\frac{N}{p}\right)^{\frac{1}{2}\langle Ax,x\rangle+\frac{1}{p^{-1}-N^{-1}}\left\langle A^{1/2}\left|A^{-1}-\frac{N^{-1}+p^{-1}}{2}1_H\right|A^{1/2}x,x\right\rangle}$$
$$\leq \left(\frac{N}{p}\right)^{\langle Ax,x\rangle} \leq \left(\frac{N}{p}\right)^N$$

for all $x \in H$, $\|x\| = 1$.

4. Related Results

The following results of Ostrowski type holds, see [1]:

Lemma 1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $t \in [a,b]$ one has the inequality*

$$\frac{1}{2}\left[(b-t)^2 f'_+(t) - (t-a)^2 f'_-(t)\right] \tag{4.1}$$
$$\leq \int_a^b f(s)\,ds - (b-a) f(t)$$
$$\leq \frac{1}{2}\left[(b-t)^2 f'_-(b) - (t-a)^2 f'_+(a)\right].$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $t = a$ or $t = b$.

If the function is differentiable in $t \in (a,b)$ then the first inequality in (3.2) becomes

$$\left(\frac{a+b}{2} - t\right) f'(t) \leq \frac{1}{b-a}\int_a^b f(s)\,ds - f(t). \tag{4.2}$$

Theorem 3. *Assume that the positive operators A_i, B_i satisfy the condition $0 < mA_i \leq B_i \leq MA_i$ for $i \in \{1,...,n\}$, where m, M are positive numbers, then*

$$\exp\left(\sum_{i=1}^n p_i \langle A_i x, x\rangle - \frac{m+M}{2}\sum_{i=1}^n p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle\right) \tag{4.3}$$
$$\leq \frac{\prod_{i=1}^n \left[D_x(A_i|B_i)\right]^{p_i}}{\left[I(m,M)\right]^{\sum_{i=1}^n p_i\langle A_i x,x\rangle}}$$

$$\leq \exp\left[\frac{1}{m}\sum_{i=1}^{n}p_i\left\langle A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-m1_H\right)^2A_i^{1/2}x,x\right\rangle\right.$$
$$\left.-\frac{1}{M}\sum_{i=1}^{n}p_i\left\langle A_i^{1/2}\left(M1_H-A_i^{-1/2}B_iA_i^{-1/2}\right)^2A_i^{1/2}x,x\right\rangle\right]$$

for all $x\in H$, $\|x\|=1$.

Proof. Writing (3.2) and (3.3) for the convex function $f(t)=-\ln t$, then we get

$$1-\frac{a+b}{2}t^{-1}\leq \ln t-\ln I(a,b)\leq \frac{(t-a)^2}{a}-\frac{(b-t)^2}{b},$$

for all $t\in[a,b]\subset(0,\infty)$.

If we use the functional calculus, we get

$$1_H-\frac{m+M}{2}T^{-1}\leq \ln T-\ln I(m,M)\leq \frac{(T-m1_H)^2}{m}-\frac{(M1_H-T)^2}{M}.$$

If we replace T with $A_i^{-1/2}B_iA_i^{-1/2}$, then we get

$$1_H-\frac{m+M}{2}A_i^{1/2}B_i^{-1}A_i^{1/2}$$
$$\leq \ln A_i^{-1/2}B_iA_i^{-1/2}-\ln I(m,M)$$
$$\leq \frac{\left(A_i^{-1/2}B_iA_i^{-1/2}-m1_H\right)^2}{m}-\frac{\left(M1_H-A_i^{-1/2}B_iA_i^{-1/2}\right)^2}{M}$$

for $i\in\{1,...,n\}$.

If we multiply both sides by $A_i^{1/2}\geq 0$, then we get

$$A_i-\frac{m+M}{2}A_iB_i^{-1}A_i$$
$$\leq A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}-\ln I(m,M)A_i$$
$$\leq \frac{A_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-m1_H\right)^2A_i^{1/2}}{m}-\frac{A_i^{1/2}\left(M1_H-A_i^{-1/2}B_iA_i^{-1/2}\right)^2A_i^{1/2}}{M}$$

for $i\in\{1,...,n\}$.

If we further multiply by $p_i\geq 0$ and sum over i from 1 to n, then we get

$$\sum_{i=1}^{n}p_iA_i-\frac{m+M}{2}\sum_{i=1}^{n}p_iA_iB_i^{-1}A_i$$
$$\leq \sum_{i=1}^{n}p_iA_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{1/2}-\ln I(m,M)\sum_{i=1}^{n}p_iA_i$$
$$\leq \frac{1}{m}\sum_{i=1}^{n}p_iA_i^{1/2}\left(A_i^{-1/2}B_iA_i^{-1/2}-m1_H\right)^2A_i^{1/2}$$
$$-\frac{1}{M}\sum_{i=1}^{n}p_iA_i^{1/2}\left(M1_H-A_i^{-1/2}B_iA_i^{-1/2}\right)^2A_i^{1/2}.$$

Further, if we take the inner product over $x \in H$, $\|x\| = 1$ and then take the exponential, we get

$$\exp\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle - \frac{m+M}{2}\sum_{i=1}^{n} p_i \left\langle A_i B_i^{-1} A_i x, x\right\rangle\right)$$
$$\leq \sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(\ln\left(A_i^{-1/2} B_i A_i^{-1/2}\right)\right) A_i^{1/2} x, x\right\rangle - \ln I(m,M)\sum_{i=1}^{n} p_i \langle A_i x, x\rangle$$
$$\leq \frac{1}{m}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(A_i^{-1/2} B_i A_i^{-1/2} - m1_H\right)^2 A_i^{1/2} x, x\right\rangle$$
$$-\frac{1}{M}\sum_{i=1}^{n} p_i \left\langle A_i^{1/2}\left(M1_H - A_i^{-1/2} B_i A_i^{-1/2}\right)^2 A_i^{1/2} x, x\right\rangle$$

and the inequality (4.3) is obtained. □

Remark 3. *Assume that the positive operators* A, B *satisfy the condition* $0 < mA \leq B \leq MA$, *then*

$$\exp\left(\langle Ax, x\rangle - \frac{m+M}{2}\left\langle AB^{-1}Ax, x\right\rangle\right) \tag{4.4}$$
$$\leq \frac{D_x(A|B)}{[I(m,M)]^{\langle Ax,x\rangle}}$$
$$\leq \exp\left[\frac{1}{m}\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2} - m1_H\right)^2 A^{1/2}x, x\right\rangle\right.$$
$$\left.-\frac{1}{M}\left\langle A^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)^2 A^{1/2}x, x\right\rangle\right]$$

for all $x \in H$, $\|x\| = 1$.

Corollary 5. *Assume that the positive operators* B_i *satisfy the condition* $0 < m1_H \leq B_i \leq M1_H$ *for* $i \in \{1, ..., n\}$ *, , where* m, M *are positive numbers, then*

$$\exp\left(1 - \frac{m+M}{2}\sum_{i=1}^{n} p_i \left\langle B_i^{-1}x, x\right\rangle\right) \tag{4.5}$$
$$\leq \frac{\prod_{i=1}^{n}[\Delta_x(B_i)]^{p_i}}{I(m,M)}$$
$$\leq \exp\left[\frac{1}{m}\sum_{i=1}^{n} p_i \left\langle (B_i - m1_H)^2 x, x\right\rangle - \frac{1}{M}\sum_{i=1}^{n} p_i \left\langle (M1_H - B_i)^2 x, x\right\rangle\right]$$

for all $x \in H$, $\|x\| = 1$.

In particular, if $0 < m1_H \le B \le M1_H$, where m, M are positive numbers, then

$$\exp\left(1 - \frac{m+M}{2}\left\langle B^{-1}x, x\right\rangle\right) \le \frac{\Delta_x(B)}{I(m,M)} \le \exp\left[\frac{1}{m}\left\langle (B - m1_H)^2 x, x\right\rangle - \frac{1}{M}\left\langle (M1_H - B)^2 x, x\right\rangle\right] \tag{4.6}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 6. *Assume that the positive operators A_i satisfy the condition $0 < p1_H \le A_i \le N1_H$ for $i \in \{1, ..., n\}$, , where p, N are positive numbers, then*

$$\begin{aligned}
&\exp\left(\sum_{i=1}^n p_i \langle A_i x, x\rangle - \frac{N^{-1}+p^{-1}}{2}\sum_{i=1}^n p_i \left\langle A_i^2 x, x\right\rangle\right) \\
&\le \frac{\prod_{i=1}^n [\eta_x(A_i)]^{p_i}}{[I(N^{-1}, p^{-1})]^{\sum_{i=1}^n p_i \langle A_i x, x\rangle}} \\
&\le \exp\left[N\sum_{i=1}^n p_i \left\langle A_i^{1/2}\left(A_i^{-1} - N^{-1}1_H\right)^2 A_i^{1/2}x, x\right\rangle\right. \\
&\left.-p\sum_{i=1}^n p_i \left\langle A_i^{1/2}\left(p^{-1}1_H - A_i^{-1}\right)^2 A_i^{1/2}x, x\right\rangle\right]
\end{aligned} \tag{4.7}$$

for all $x \in H$, $\|x\| = 1$.

The case of one operator, namely, if $0 < p1_H \le A \le N1_H$, is as follows:

$$\begin{aligned}
&\exp\left(\langle Ax, x\rangle - \frac{N^{-1}+p^{-1}}{2}\left\langle A^2 x, x\right\rangle\right) \\
&\le \frac{\eta_x(A)}{[I(N^{-1}, p^{-1})]^{\langle Ax, x\rangle}} \\
&\le \exp\left[N\left\langle A^{1/2}\left(A^{-1} - N^{-1}1_H\right)^2 A^{1/2}x, x\right\rangle\right. \\
&\left.-p\left\langle A^{1/2}\left(p^{-1}1_H - A^{-1}\right)^2 A^{1/2}x, x\right\rangle\right]
\end{aligned} \tag{4.8}$$

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