QUASI MONOTONICITY FOR THE RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A, B in the Hilbert space H and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp \left\langle A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}} x, x \right\rangle .$$

In this paper we show among others that, if $0 < m1_H \leq A \leq M1_H$, $0 < \gamma 1_H \leq C \leq \Gamma 1_H$ and $0 < k1_H \leq B - C \leq K1_H$, then

$$\begin{aligned} 1 &\leq \left(1 + \frac{K}{\Gamma}\right)^{\frac{m^2 kM}{K}} \\ &\leq \left(1 + \frac{K}{\Gamma}\right)^{\frac{kM}{K}\langle A^2 x, x\rangle} \leq \frac{D_x(A|B)}{D_x(A|C)} \leq \left(1 + \frac{k}{\gamma}\right)^{\frac{Km}{k}\langle A^2 x, x\rangle} \\ &\leq \left(1 + \frac{k}{\gamma}\right)^{\frac{M^2 Km}{k}} \end{aligned}$$

for all $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [9].

For each unit vector $x \in H$, see also [13], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive operators, Normalized determinants, Inequalities.

RGMIA Res. Rep. Coll. **25** (2022), Art. 63, 14 pp. Received 07/06/22

(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [9] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < m1_H \leq A \leq M1_H$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [19]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [10], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < m1_H \leq A \leq M1_H$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA\ln(tA)x, x\rangle) \\ &= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\ &= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\ &= \exp\ln\left(t^{-\langle Ax, x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax, x\rangle}[\eta_x(A)]^{-t} \tag{1.6}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_x(1_H)=1 \text{ and } \eta_x(t1_H)=t^{-t} \tag{1.7}$$

for $t>0$.

In the recent paper [3] we showed among others that, if $A,\ B>0$, then for all $x\in H$, $\|x\|=1$ and $t\in[0,1]$,

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A>0$ and $x\in H$, $\|x\|=1$.

Definition 1. *For positive invertible operators $A,\ B$ and $x\in H$ with $\|x\|=1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B):=\exp\langle S(A|B)x,x\rangle=\exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle,$$

where the relative operator entropy $S(A|B)$, is defined by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}. \tag{1.9}$$

We observe that for $A>0$,

$$D_x(A|1_H)=\exp\langle S(A|1_H)x,x\rangle=\exp(-\langle A\ln Ax,x\rangle)=\eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B>0$,

$$D_x(1_H|B):=\exp\langle S(1_H|B)x,x\rangle=\exp\langle\ln Bx,x\rangle=\Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant.*

Motivated by the above results, in this paper we show among others that, if $0<m1_H\leq A\leq M1_H$, $0<\gamma1_H\leq C\leq\Gamma1_H$ and $0<k1_H\leq B-C\leq K1_H$, then

$$\begin{aligned}1&\leq\left(1+\frac{K}{\Gamma}\right)^{\frac{m^2kM}{K}}\\&\leq\left(1+\frac{K}{\Gamma}\right)^{\frac{kM}{K}\langle A^2x,x\rangle}\leq\frac{D_x(A|B)}{D_x(A|C)}\leq\left(1+\frac{k}{\gamma}\right)^{\frac{Km}{k}\langle A^2x,x\rangle}\\&\leq\left(1+\frac{k}{\gamma}\right)^{\frac{M^2Km}{k}}\end{aligned}$$

for all $x\in H$ with $\|x\|=1$.

2. Main Results

In order to simplify the notation, we write k instead of $k1_H$. We can state the following representation result that is of interest in itself:

Lemma 1. *For all U, $V>0$ we have*

$$\begin{aligned}\ln V-\ln U &= \int_0^\infty \left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]d\lambda \\ &= \int_0^\infty \left(\int_0^1 (\lambda+(1-t)U+tV)^{-1}(V-U)(\lambda+(1-t)U+tV)^{-1}dt\right)d\lambda. \end{aligned} \tag{2.1}$$

Proof. Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.2}$$

for all $t>0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T=\int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}d\lambda \tag{2.3}$$

for all operators $T>0$.

We have from (2.3) for U, $V>0$ that

$$\ln V-\ln U=\int_0^\infty \frac{1}{\lambda+1}\left[(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\right]d\lambda. \tag{2.4}$$

Since

$$\begin{aligned}&(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\\ &=V(\lambda+V)^{-1}-U(\lambda+U)^{-1}-\left((\lambda+V)^{-1}-(\lambda+U)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&V(\lambda+V)^{-1}-U(\lambda+U)^{-1}\\ &=(V+\lambda-\lambda)(\lambda+V)^{-1}-(U+\lambda-\lambda)(\lambda+U)^{-1}\\ &=1-\lambda(\lambda+V)^{-1}-1+\lambda(\lambda+U)^{-1}=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\\ &=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1}-\left((\lambda+V)^{-1}-(\lambda+U)^{-1}\right)\\ &=(\lambda+1)\left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]\end{aligned}$$

and by (2.4) we get

$$\ln V-\ln U=\int_0^\infty \left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]d\lambda, \tag{2.5}$$

we proves the first equality in (2.1).

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D]$: $\{(1-t)C + tD,\ t \in [0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C + tD),\ t \in [0,1].$$

Then we have, by the properties of the integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt.$$

If we write this equality for the function $f(t) = -t^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we take in (2.6) $C = \lambda + U$, $D = \lambda + V$, then

$$\begin{aligned} &(\lambda+U)^{-1} - (\lambda+V)^{-1} \\ &= \int_0^1 ((1-t)(\lambda+U) + t(\lambda+V))^{-1}(V-U) \\ &\times((1-t)(\lambda+U)+t(\lambda+V))^{-1}\,dt \\ &= \int_0^1 (\lambda + (1-t)U + tV)^{-1}(V-U)(\lambda+(1-t)U+tV)^{-1}\,dt. \end{aligned} \tag{2.7}$$

By employing (2.7) and (2.5) we derive the desired result (2.1). □

Lemma 2. *For all A, B, $C > 0$ we have*

$$\begin{aligned} S(A|B) - S(A|C) = \int_0^\infty \Big(&\int_0^1 A(\lambda A + (1-t)C + tB)^{-1}(B-C) \\ &\times(\lambda A + (1-t)C + tB)^{-1} A dt\Big) d\lambda. \end{aligned} \tag{2.8}$$

Proof. If we take in (2.1) $V = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $U = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$, then we get

$$\begin{aligned}
&\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \ln\left(A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right) \\
&= \int_0^\infty \left(\int_0^1 \left(\lambda A^{-\frac{1}{2}}AA^{-\frac{1}{2}} + (1-t)\, A^{-\frac{1}{2}}CA^{-\frac{1}{2}} + tA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-1}\right. \\
&\times \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right) \\
&\times \left.\left(\lambda A^{-\frac{1}{2}}AA^{-\frac{1}{2}} + (1-t)\, A^{-\frac{1}{2}}CA^{-\frac{1}{2}} + tA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-1} dt\right) d\lambda \\
&= \int_0^\infty \left(\int_0^1 A^{\frac{1}{2}} \left(\lambda A + (1-t)\, C + tB\right)^{-1} A^{\frac{1}{2}} A^{-\frac{1}{2}} (B-C)\, A^{-\frac{1}{2}}\right. \\
&\times A^{\frac{1}{2}} \left.\left(\lambda A + (1-t)\, C + tB\right)^{-1} A^{\frac{1}{2}} dt\right) d\lambda \\
&= \int_0^\infty \left(\int_0^1 A^{\frac{1}{2}} \left(\lambda A + (1-t)\, C + tB\right)^{-1} (B-C)\right. \\
&\times \left.\left(\lambda A + (1-t)\, C + tB\right)^{-1} A^{\frac{1}{2}} dt\right) d\lambda.
\end{aligned}$$

Now, if we multiply both sides by $A^{\frac{1}{2}} > 0$, then we get the desired result (2.8). □

We have the following representation result

Theorem 1. *For all A, B, $C > 0$ and $x \in H$ with $\|x\| = 1$, we have*

$$\begin{aligned}
\frac{D_x(A|B)}{D_x(A|C)} &= \exp\left\{\int_0^\infty \left(\int_0^1 \left\langle A\left(\lambda A + (1-t)\, C + tB\right)^{-1} (B-C)\right.\right.\right. \\
&\times \left.\left.\left.\left(\lambda A + (1-t)\, C + tB\right)^{-1} Ax, x\right\rangle dt\right) d\lambda\right\}.
\end{aligned} \tag{2.9}$$

Proof. We take the inner product over $x \in H$ with $\|x\| = 1$ in (2.8) to get

$$\begin{aligned}
&\langle S(A|B)\, x, x\rangle - \langle S(A|C)\, x, x\rangle \\
&= \int_0^\infty \left(\int_0^1 \left\langle A\left(\lambda A + (1-t)\, C + tB\right)^{-1} (B-C)\right.\right. \\
&\times \left.\left(\lambda A + (1-t)\, C + tB\right)^{-1} Ax, x\right\rangle dt d\lambda
\end{aligned} \tag{2.10}$$

and by taking the exponential, we derive the desired result (2.9). □

Corollary 1. *For all B, $C > 0$ and $x \in H$ with $\|x\| = 1$, we have*

$$\begin{aligned}
\frac{\Delta_x(B)}{\Delta_x(C)} &= \exp\left\{\int_0^\infty \left(\int_0^1 \left\langle \left(\lambda + (1-t)\, C + tB\right)^{-1} (B-C)\right.\right.\right. \\
&\times \left.\left.\left.\left(\lambda + (1-t)\, C + tB\right)^{-1} x, x\right\rangle dt\right) d\lambda\right\}.
\end{aligned} \tag{2.11}$$

Follows by (2.9) for $A = 1$.

Theorem 2. *Assume that* $0 < m \leq A \leq M$, $B \geq m_2 > 0$ *and* $C \geq m_1 > 0$, *then for* $x \in H$ *with* $\|x\| = 1$,

$$\exp\left[-M^2 \|B - C\| \Phi(m_1, m_2)\right] \leq \frac{D_x(A|B)}{D_x(A|C)} \leq \exp\left[M^2 \|B - C\| \Phi(m_1, m_2)\right], \tag{2.12}$$

where

$$\Phi(m_1, m_2) := \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \text{ if } m_2 \neq m_1, \\ \\ \frac{1}{m_1} \text{ if } m_2 = m_1. \end{cases}$$

Proof. If we take the modulus in (2.10) then we get for $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned} &|\langle S(A|B)x, x\rangle - \langle S(A|C)x, x\rangle| \\ &\leq \int_0^\infty \left(\int_0^1 \left|\left\langle A(\lambda A + (1-t)C + tB)^{-1}(B - C)\right.\right.\right. \\ &\left.\left.\times (\lambda A + (1-t)C + tB)^{-1} Ax, x\right\rangle\right| dtd\lambda \\ &\leq \int_0^\infty \int_0^1 \left\|A(\lambda A + (1-t)C + tB)^{-1}(B - C)\right. \\ &\left.\times (\lambda A + (1-t)C + tB)^{-1} A\right\| dtd\lambda. \end{aligned}$$

Observe that

$$\begin{aligned} &\left\|A(\lambda A + (1-t)C + tB)^{-1}(B - C)(\lambda A + (1-t)C + tB)^{-1} A\right\| \\ &\leq \|A\|^2 \left\|(\lambda A + (1-t)C + tB)^{-1}\right\|^2 \|B - C\|. \end{aligned}$$

Assume that $m_2 > m_1$. Then

$$(1-t)C + tB + \lambda A \geq (1-t)m_1 + tm_2 + m\lambda,$$

which implies that

$$((1-t)C + tB + \lambda A)^{-1} \leq ((1-t)m_1 + tm_2 + m\lambda)^{-1}$$

and

$$\left\|((1-t)C + tB + \lambda A)^{-1}\right\|^2 \leq ((1-t)m_1 + tm_2 + m\lambda)^{-2}$$

for all $t \in [0, 1]$ and $\lambda \geq 0$.

Therefore

$$\begin{aligned} &|\langle S(A|B)x, x\rangle - \langle S(A|C)x, x\rangle| \\ &\leq M^2 \|B - C\| \int_0^\infty \int_0^1 \left\|(\lambda A + (1-t)C + tB)^{-1}\right\|^2 dtd\lambda \\ &\leq M^2 \|B - C\| \int_0^\infty \int_0^1 ((1-t)m_1 + tm_2 + m\lambda)^{-2} dtd\lambda \end{aligned}$$

If we use the identity (2.8) for $A = m$, $B = m_2$ and $C = m_1$ we get the scalar identity

$$\begin{aligned}
&\int_0^\infty \left(\int_0^1 m \left(\lambda m + (1-t) m_1 + t m_2\right)^{-1} (m_2 - m_1) \right. \\
&\left. \times \left(\lambda m + (1-t) m_1 + t m_2\right)^{-1} m dt \right) d\lambda \\
&= S(m|m_2) - S(m|m_1) \\
&= m^2 \ln\left(m^{-1} m_2\right) - m^2 \ln\left(m^{-1} m_1\right) = m^2 \ln\left(\frac{m_2}{m_1}\right),
\end{aligned}$$

which gives that

$$\int_0^\infty \int_0^1 \left(\lambda m + (1-t) m_1 + t m_2\right)^{-2} dt d\lambda = \frac{1}{m_2 - m_1} \ln\left(\frac{m_2}{m_1}\right).$$

Therefore

$$\begin{aligned}
-M^2 \|B - C\| \frac{1}{m_2 - m_1} \ln\left(\frac{m_2}{m_1}\right) &\le \langle S(A|B) x, x\rangle - \langle S(A|C) x, x\rangle \\
&\le M^2 \|B - C\| \frac{1}{m_2 - m_1} \ln\left(\frac{m_2}{m_1}\right)
\end{aligned}$$

and by taking the exponential, we derive (2.12).

The case $m_2 < m_1$ goes in a similar way.

Now, assume that B, $C \ge m_1 > 0$. Let $\epsilon > 0$, then $B + \epsilon \ge m_1 + \epsilon$. Put $m_2 = m_1 + \epsilon > m_1$. If we write the inequality (2.12) for $B + \epsilon$ and C, we get

$$\begin{aligned}
& - M^2 \|B + \epsilon - C\| \frac{1}{m_1 + \epsilon - m_1} \ln\left(\frac{m_1 + \epsilon}{m_1}\right) \\
&\le \langle S(A|B + \epsilon) x, x\rangle - \langle S(A|C) x, x\rangle \\
&\le M^2 \|B + \epsilon - C\| \frac{1}{m_1 + \epsilon - m_1} \ln\left(\frac{m_1 + \epsilon}{m_1}\right).
\end{aligned} \tag{2.13}$$

If we take the limit over $\epsilon \to 0+$ in (2.13) and observe that

$$\lim_{\epsilon \to 0+} \frac{\ln(m_1 + \epsilon) - \ln m_1}{\epsilon} = \frac{1}{m_1},$$

then we also get (2.12) for $m_2 = m_1$. □

Corollary 2. *Assume that* $B \ge m_2 > 0$ *and* $C \ge m_1 > 0$, *then for* $x \in H$ *with* $\|x\| = 1$,

$$\begin{aligned}
\exp\left[-\|B - C\| \Phi(m_1, m_2)\right] &\le \frac{\Delta_x(B)}{\Delta_x(C)} \\
&\le \exp\left[\|B - C\| \Phi(m_1, m_2)\right].
\end{aligned} \tag{2.14}$$

Further on, we also have

Theorem 3. *Assume that* $0 < m \le A \le M$, $0 < \gamma \le C \le \Gamma$ *and* $0 < k \le B - C \le K$, *then*

$$1 \le \left(1 + \frac{K}{\Gamma}\right)^{\frac{m^2 kM}{K}} \le \left(1 + \frac{K}{\Gamma}\right)^{\frac{kM}{K}\left\langle A^2 x, x\right\rangle} \le \frac{D_x(A|B)}{D_x(A|C)} \le \left(1 + \frac{k}{\gamma}\right)^{\frac{Km}{k}\left\langle A^2 x, x\right\rangle} \le \left(1 + \frac{k}{\gamma}\right)^{\frac{M^2 Km}{k}} \tag{2.15}$$

for $x \in H$ *with* $\|x\| = 1$.

Proof. Since $0 < k \le B - C \le K$ then by multiplying both sides by $(\lambda A + (1-t) C + tB)^{-1} > 0$ and then by $A > 0$, we get

$$\begin{aligned} &kA(\lambda A + (1-t) C + tB)^{-2} A \\ &\le A(\lambda A + (1-t) C + tB)^{-1} (B - C)(\lambda A + (1-t) C + tB)^{-1} A \\ &\le KA(\lambda A + (1-t) C + tB)^{-2} A \end{aligned}$$

for all $t \in [0,1]$ and $\lambda > 0$.

If we take the integrals over t and λ and use the identity (2.8), then we derive

$$\begin{aligned} &k\int_0^\infty \int_0^1 A(\lambda A + (1-t) C + tB)^{-2} A dt d\lambda \\ &\le S(A|B) - S(A|C) \\ &\le K\int_0^\infty \int_0^1 A(\lambda A + (1-t) C + tB)^{-2} A dt d\lambda. \end{aligned} \tag{2.16}$$

Observe that

$$\lambda A + (1-t) C + tB = \lambda A + C + t(B - C).$$

Then

$$\lambda m + \gamma + tk \le \lambda A + (1-t) C + tB \le \lambda M + \Gamma + tK$$

for all $t \in [0,1]$ and $\lambda > 0$, which implies that

$$(\lambda M + \Gamma + tK)^{-1} \le (\lambda A + (1-t) C + tB)^{-1} \le (\lambda m + \gamma + tk)^{-1},$$

which gives that

$$(\lambda M + \Gamma + tK)^{-2} \le (\lambda A + (1-t) C + tB)^{-2} \le (\lambda m + \gamma + tk)^{-2},$$

for all $t \in [0,1]$ and $\lambda > 0$.

If we multiply both sides by $A > 0$ we get

$$A(\lambda M + \Gamma + tK)^{-2} A \le A(\lambda A + (1-t) C + tB)^{-2} A \le A(\lambda m + \gamma + tk)^{-2} A,$$

for all $t \in [0,1]$ and $\lambda > 0$.

If we take the double integral over $t \in [0,1]$ and $\lambda > 0$, then we get

$$A\left(\int_0^\infty \int_0^1 (\lambda M + \Gamma + tK)^{-2}\, dt d\lambda\right) A \leq \int_0^\infty \int_0^1 A\,(\lambda A + (1-t)\,C + tB)^{-2}\, A dt d\lambda \leq A\left(\int_0^\infty \int_0^1 (\lambda m + \gamma + tk)^{-2}\, dt d\lambda\right) A. \tag{2.17}$$

Observe that

$$\int_0^1 (\lambda m + \gamma + tk)^{-2}\, dt = -\frac{1}{k}(\lambda m + \gamma + k)^{-1} + \frac{1}{k}(\lambda m + \gamma)^{-1} = \frac{1}{k}\left((\lambda m + \gamma)^{-1} - (\lambda m + \gamma + k)^{-1}\right),$$

which gives

$$\int_0^\infty \left(\int_0^1 (\lambda m + \gamma + tk)^{-2}\, dt\right) d\lambda = \frac{1}{k}\int_0^\infty \left((\lambda m + \gamma)^{-1} - (\lambda m + \gamma + k)^{-1}\right) d\lambda = \frac{m}{k}\int_0^\infty \left(\left(\lambda + \frac{\gamma}{m}\right)^{-1} - \left(\lambda + \frac{\gamma}{m} + \frac{k}{m}\right)^{-1}\right) d\lambda.$$

By the first identity in (2.1) in the scalar case, we have

$$\ln\left(\frac{\gamma}{m} + \frac{k}{m}\right) - \ln\frac{\gamma}{m} = \int_0^\infty \left[\left(\lambda + \frac{\gamma}{m}\right)^{-1} - \left(\lambda + \frac{\gamma}{m} + \frac{k}{m}\right)^{-1}\right] d\lambda,$$

namely

$$\int_0^\infty \left[\left(\lambda + \frac{\gamma}{m}\right)^{-1} - \left(\lambda + \frac{\gamma}{m} + \frac{k}{m}\right)^{-1}\right] d\lambda = \ln(\gamma + k) - \ln\gamma$$

which gives that

$$\int_0^\infty \left(\int_0^1 (\lambda m + \gamma + tk)^{-2}\, dt\right) d\lambda = \frac{m}{k}\left[\ln(\gamma + k) - \ln\gamma\right] = \ln\left(1 + \frac{k}{\gamma}\right)^{\frac{m}{k}}.$$

Similarly,

$$\int_0^\infty \int_0^1 (\lambda M + \Gamma + tK)^{-2}\, dt d\lambda = \ln\left(1 + \frac{K}{\Gamma}\right)^{\frac{M}{K}}.$$

By (2.16) and (2.17) we then obtain

$$0 \leq \ln\left(1 + \frac{K}{\Gamma}\right)^{\frac{kM}{K}} A^2 \leq k\int_0^\infty \int_0^1 A\,(\lambda A + (1-t)\,C + tB)^{-2}\, A dt d\lambda \leq S(A|B) - S(A|C) \leq K\int_0^\infty \int_0^1 A\,(\lambda A + (1-t)\,C + tB)^{-2}\, A dt d\lambda \leq \ln\left(1 + \frac{k}{\gamma}\right)^{\frac{Km}{k}} A^2, \tag{2.18}$$

which is an operator inequality of interest in itself.

If we take the inner product over $x \in H$, $\|x\| = 1$, then we get

$$
\begin{aligned}
0 &\leq \ln \left(1 + \frac{K}{\Gamma}\right)^{\frac{kM}{K}} \left\langle A^2 x, x\right\rangle \\
&\leq k \int_0^\infty \int_0^1 \left\langle A \left(\lambda A + (1-t)\, C + tB\right)^{-2} Ax, x\right\rangle dt d\lambda \\
&\leq \left\langle S\left(A|B\right) x, x\right\rangle - \left\langle S\left(A|C\right) x, x\right\rangle \\
&\leq K \int_0^\infty \int_0^1 \left\langle A \left(\lambda A + (1-t)\, C + tB\right)^{-2} Ax, x\right\rangle dt d\lambda \\
&\leq \ln \left(1 + \frac{k}{\gamma}\right)^{\frac{Km}{k}} \left\langle A^2 x, x\right\rangle,
\end{aligned}
$$

and by taking the exponential, we derive (2.15). □

Corollary 3. *Assume that* $0 < \gamma \leq C \leq \Gamma$ *and* $0 < k \leq B - C \leq K$, *then*

$$1 \leq \left(1 + \frac{K}{\Gamma}\right)^{\frac{k}{K}} \leq \frac{\Delta_x (B)}{\Delta_x (C)} \leq \left(1 + \frac{k}{\gamma}\right)^{\frac{K}{k}} \tag{2.19}$$

for $x \in H$ *with* $\|x\| = 1$.

It follows by (2.15) for $A = 1$.

3. Related Results

Let U and V be strictly positive operators on a Hilbert space H such that $V - U \geq m > 0$. In 2015, [11], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0, \infty)$

$$f(V) - f(U) \geq f(\|U\| + m) - f(\|U\|) \geq f(\|V\|) - f(\|V\| - m) > 0. \tag{3.1}$$

If $V > U > 0$, then [11]

$$
\begin{aligned}
f(V) - f(U) &\geq f\left(\|U\| + \frac{1}{\left\|(V-U)^{-1}\right\|}\right) - f(\|U\|) \\
&\geq f(\|V\|) - f\left(\|V\| - \frac{1}{\left\|(V-U)^{-1}\right\|}\right) > 0.
\end{aligned} \tag{3.2}
$$

The inequality between the first and third term in (3.2) was obtained earlier by H. Zuo and G. Duan in [21].

If we write the inequality (3.1) for $f(t) = \ln t$, then we get for $V - U \geq m > 0$ that

$$\ln V - \ln U \geq \ln \left(\frac{\|U\| + m}{\|U\|}\right) \geq \ln \left(\frac{\|V\|}{\|V\| - m}\right) > 0. \tag{3.3}$$

If $V > U > 0$, then by (3.2) written for for $f(t) = \ln t$, we get that

$$\ln V - \ln U \geq \ln\left(1 + \frac{1}{\|U\| \left\|(V-U)^{-1}\right\|}\right) \geq \ln\left(\frac{\|V\| \left\|(V-U)^{-1}\right\|}{\|V\| \left\|(V-U)^{-1}\right\| - 1}\right) > 0. \tag{3.4}$$

Proposition 1. *Assume that* $B - C \geq mA > 0$ *for the positive constant* m, *then*

$$\frac{D_x(A|B)}{D_x(A|C)} \geq \left(\frac{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| + m}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\|}\right)^{\langle Ax,x\rangle} \geq \left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| - m}\right)^{\langle Ax,x\rangle} > 1 \tag{3.5}$$

for all $x \in H$ *with* $\|x\| = 1$.

Proof. Since $B - C \geq mA > 0$, then by multiplying both sides by $A^{-\frac{1}{2}} > 0$, we get $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - A^{-\frac{1}{2}}CA^{-\frac{1}{2}} \geq m$ and by (3.3) for $V = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $U = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ we get

$$\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \ln\left(A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right) \geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| + m}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\|}\right) \geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| - m}\right) > 0.$$

If we multiply both sides by $A^{\frac{1}{2}} > 0$, then we get the operator inequality of interest

$$S(A|B) - S(A|C) \geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| + m}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\|}\right) A \geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| - m}\right) A > 0.$$

If we take the inner product over $x \in H$ with $\|x\| = 1$, then we get

$$\langle S(A|B)x, x\rangle - \langle S(A|C)x, x\rangle \geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| + m}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\|}\right)^{\langle Ax,x\rangle} \geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| - m}\right)^{\langle Ax,x\rangle} > 0.$$

If we take the exponential, then we derive the desired result (3.5). □

Corollary 4. *Assume that* $B - C \geq m > 0$ *for the positive constant* m, *then*

$$\frac{\Delta_x(B)}{\Delta_x(C)} \geq \frac{\|C\| + m}{\|C\|} \geq \frac{\|B\|}{\|B\| - m} > 1 \tag{3.6}$$

for all $x \in H$ *with* $\|x\| = 1$.

Proposition 2. *Assume that $B > C > 0$, then for $A > 0$,*

$$\frac{D_x(A|B)}{D_x(A|C)} \geq \left(1 + \frac{1}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}\right)^{\langle Ax,x\rangle} \geq \left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\| - 1}\right)^{\langle Ax,x\rangle} > 1 \tag{3.7}$$

for all $x \in H$ with $\|x\| = 1$.

In particular,

$$\frac{\Delta_x(B)}{\Delta_x(C)} \geq \frac{\|C\| \left\|(B-C)^{-1}\right\| + 1}{\|C\| \left\|(B-C)^{-1}\right\|} \geq \frac{\|B\| \left\|(B-C)^{-1}\right\|}{\|B\| \left\|(B-C)^{-1}\right\| - 1} > 1 \tag{3.8}$$

for all $x \in H$ with $\|x\| = 1$.

Proof. If we take $V = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $U = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ in (3.4), then we get

$$\begin{aligned} &\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \ln\left(A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right) \\ &\geq \ln\left(1 + \frac{1}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}\right) \\ &\geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\| - 1}\right) > 0. \end{aligned} \tag{3.9}$$

If we multiply both sides by $A^{\frac{1}{2}} > 0$, then we get the operator inequality

$$\begin{aligned} &S(A|B) - S(A|C) \\ &\geq \ln\left(1 + \frac{1}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}\right) A \\ &\geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\| - 1}\right) A > 0. \end{aligned} \tag{3.10}$$

If we take the inner product over $x \in H$ with $\|x\| = 1$, then we get

$$\begin{aligned} &\langle S(A|B)x, x\rangle - \langle S(A|C)x, x\rangle \\ &\geq \ln\left(1 + \frac{1}{\left\|A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}\right)^{\langle Ax,x\rangle} \\ &\geq \ln\left(\frac{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\|}{\left\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}(B-C)^{-1}A^{\frac{1}{2}}\right\| - 1}\right)^{\langle Ax,x\rangle} \end{aligned}$$

and by taking the exponential, we derive the desired result (3.7). □

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