A SUB-MULTIPLICATIVE PROPERTY FOR THE RELATIVE ENTROPIC NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp \left\langle A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}} x, x \right\rangle .$$

In this paper we show, among others, that, if A, B, $C > 0$, $x \in H$, $\|x\| = 1$ and $BA^{-1}C + CA^{-1}B \geq 0$, then

$$D_x(A|B + C + A) \leq D_x(A|B + A) D_x(A|C + A).$$

Some examples for normalized determinant are also provided.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [9].

For each unit vector $x \in H$, see also [13], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;

(ii) *bounds*: $\langle A^{-1}x, x \rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x \rangle$;

(iii) *continuous mean*: $\langle A^p x, x \rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x \rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;

(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(t1_H) = t$ for all $t > 0$;

(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;

(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

(viii) *Ky Fan type inequality*: $\Delta_x((1 - \alpha) A + \alpha B) \geq \Delta_x(A)^{1-\alpha} \Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

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We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [9] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < m1_H \leq A \leq M1_H$, where m, M are positive numbers,

$$0 \leq \langle Ax, x \rangle - \Delta_x(A) \leq L(m,M) \left[\ln L(m,M) + \frac{M \ln m - m \ln M}{M-m} - 1 \right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [19]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln \left(h^{\frac{1}{h-1}} \right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [10], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x \rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < m1_H \leq A \leq M1_H$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A \ln Ax, x \rangle) = \exp \langle \eta(A) x, x \rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA \ln(tA) x, x \rangle) \\ &= \exp(-\langle tA(\ln t + \ln A) x, x \rangle) = \exp(-\langle (tA \ln t + tA \ln A) x, x \rangle) \\ &= \exp(-\langle Ax, x \rangle t \ln t) \exp(-t \langle A \ln Ax, x \rangle) \\ &= \exp \ln \left(t^{-\langle Ax, x \rangle t} \right) [\exp(-\langle A \ln Ax, x \rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax, x \rangle} [\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(1_H) = 1 \text{ and } \eta_x(t1_H) = t^{-t} \tag{1.7}$$

for $t > 0$.

In the recent paper [3] we showed among others that, if A, $B>0$, then for all $x\in H, \|x\|=1$ and $t\in[0,1]$,

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A>0$ and $x\in H, \|x\|=1$.

Definition 1. *For positive invertible operators A, B and $x\in H$ with $\|x\|=1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B):=\exp\langle S(A|B)x,x\rangle=\exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x,x\right\rangle,$$

where the relative operator entropy $S(A|B)$, is defined by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}. \tag{1.9}$$

We observe that for $A>0$,

$$D_x(A|1_H)=\exp\langle S(A|1_H)x,x\rangle=\exp(-\langle A\ln Ax,x\rangle)=\eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B>0$,

$$D_x(1_H|B):=\exp\langle S(1_H|B)x,x\rangle=\exp\langle\ln Bx,x\rangle=\Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant.*

Motivated by the above results, in this paper we show, among others, that, if A, B, $C>0$, $x\in H$, $\|x\|=1$ and $BA^{-1}C+CA^{-1}B\geq 0$, then

$$D_x(A|B+C+A)\leq D_x(A|B+A)\,D_x(A|C+A).$$

Some examples for normalized determinant are also provided.

2. Main Results

Further on, in order to simplify notations, instead of $k1_H$ with k a real number, we write k.

The following representation result holds:

Lemma 1. *For all U, $V\geq 0$ and $a>0$ we have*

$$\begin{aligned}&\ln(U+a)+\ln(V+a)-\ln(U+V+a)-\ln a\\&=\int_0^\infty(a+\lambda)^{-1}S(\lambda,a,U,V)\,d\lambda+\int_0^\infty(a+\lambda)^{-1}Q(\lambda,a,U,V)\,d\lambda,\end{aligned} \tag{2.1}$$

where

$$S(\lambda,a,U,V):=(U+V+a+\lambda)^{-1}(UV+VU)(U+V+a+\lambda)^{-1}$$

and

$$\begin{aligned}Q(\lambda,a,U,V):=&(U+V+a+\lambda)^{-1}\\&\times\left[V(U+a+\lambda)^{-1}UV+U(V+a+\lambda)^{-1}VU\right]\\&\times(U+V+a+\lambda)^{-1}\end{aligned}$$

for $\lambda>0$.

Proof. Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.2}$$

for all $t>0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T=\int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}d\lambda \tag{2.3}$$

for all operators $T>0$.

Observe that

$$\begin{aligned}\int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}d\lambda&=\int_0^\infty \frac{1}{\lambda+1}(T+\lambda-\lambda-1)(\lambda+T)^{-1}d\lambda\\&=\int_0^\infty \left[(\lambda+1)^{-1}-(\lambda+T)^{-1}\right]d\lambda\end{aligned}$$

and then

$$\ln T=\int_0^\infty \left[(\lambda+1)^{-1}-(\lambda+T)^{-1}\right]d\lambda.$$

Therefore

$$\ln(U+a)+\ln(V+a)-\ln(U+V+a)-\ln a=\int_0^\infty K_\lambda d\lambda, \tag{2.4}$$

where

$$K_\lambda:=(U+V+a+\lambda)^{-1}+(a+\lambda)^{-1}-(U+a+\lambda)^{-1}-(V+a+\lambda)^{-1}.$$

To simplify calculations, consider $\delta:=a+\lambda$ and set

$$L_\delta:=(U+V+\delta)^{-1}+\delta^{-1}-(U+\delta)^{-1}-(V+\delta)^{-1}.$$

If we multiply both sides by $U+V+\delta$ we get

$$\begin{aligned}
W_\delta &:= (U+V+\delta)\, L_\delta\, (U+V+\delta) \\
&= (U+V+\delta) + \delta^{-1} (U+V+\delta)^2 \\
&- (U+V+\delta)(U+\delta)^{-1}(U+V+\delta) \\
&- (U+V+\delta)(V+\delta)^{-1}(U+V+\delta) \\
&= (U+V+\delta) + \delta^{-1} (U+V+\delta)^2 \\
&- (U+V+\delta) - V(U+\delta)^{-1}(U+V+\delta) \\
&- U(V+\delta)^{-1}(U+V+\delta) - (U+V+\delta) \\
&= \delta^{-1}(U+V+\delta)^2 - V(U+\delta)^{-1}V - V \\
&- U(V+\delta)^{-1}U - U - (U+V+\delta)
\end{aligned}$$

$$\begin{aligned}
&= \delta^{-1}\left(U^2 + UV + \delta U + VU + V^2 + \delta V + \delta U + \delta V + \delta^2\right) \\
&- V(U+\delta)^{-1}V - 2V - U(V+\delta)^{-1}U - 2U - \delta \\
&= \delta^{-1}\left(U^2 + UV + VU + V^2\right) + 2V + 2U + \delta \\
&- V(U+\delta)^{-1}V - U(V+\delta)^{-1}U - 2U - 2V - \delta \\
&= \delta^{-1}\left(U^2 + UV + VU + V^2\right) - V(U+\delta)^{-1}V - U(V+\delta)^{-1}U \\
&= \delta^{-1}\left[U^2 + UV + VU + V^2 - \delta V(U+\delta)^{-1}V - \delta U(V+\delta)^{-1}U\right] \\
&= \delta^{-1}\left[U^2 + UV + VU + V^2 - V\left(\delta^{-1}U+1\right)^{-1}V - U\left(\delta^{-1}V+1\right)^{-1}U\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&V^2 - V\left(\delta^{-1}U+1\right)^{-1}V \\
&= V\left(\delta^{-1}U+1\right)^{-1}\left(\delta^{-1}U+1\right)V - V\left(\delta^{-1}U+1\right)^{-1}V \\
&= V\left(\delta^{-1}U+1\right)^{-1}\left(\delta^{-1}U+1-1\right)V \\
&= \delta^{-1}V\left(\delta^{-1}U+1\right)^{-1}UV = V(U+\delta)^{-1}UV
\end{aligned}$$

and

$$\begin{aligned}
&U^2 - U\left(\delta^{-1}V+1\right)^{-1}U \\
&= U\left(\delta^{-1}V+1\right)^{-1}\left(\delta^{-1}V+1\right)U - U\left(\delta^{-1}V+1\right)^{-1}U \\
&= U\left(\delta^{-1}V+1\right)^{-1}\left(\delta^{-1}V+1-1\right)U \\
&= \delta^{-1}U\left(\delta^{-1}V+1\right)^{-1}VU = U(V+\delta)^{-1}VU.
\end{aligned}$$

Therefore

$$W_\delta = \delta^{-1}\left[UV + VU + V(U+\delta)^{-1}UV + U(V+\delta)^{-1}VU\right],$$

which gives that

$$L_\delta := (U+V+\delta)^{-1}\, W_\delta\, (U+V+\delta)^{-1}.$$

We obtain then the following representation

$$K_\lambda = (a+\lambda)^{-1} (U+V+a+\lambda)^{-1} (UV+VU) (U+V+a+\lambda)^{-1} + (a+\lambda)^{-1} (U+V+a+\lambda)^{-1} \times \left[V (U+a+\lambda)^{-1} UV + U (V+a+\lambda)^{-1} VU\right] (U+V+a+\lambda)^{-1} = (a+\lambda)^{-1} S(\lambda,a,U,V) + (a+\lambda)^{-1} P(\lambda,a,U,V) \tag{2.5}$$

for a, $\lambda > 0$.

By utilizing (2.4) and (2.5) we derive the representation (2.1). □

Corollary 1. *For all U, $V \geq 0$ we have*

$$\ln(U+1) + \ln(V+1) - \ln(U+V+1) = \int_0^\infty (1+\lambda)^{-1} S(\lambda,U,V)\, d\lambda + \int_0^\infty (1+\lambda)^{-1} Q(\lambda,U,V)\, d\lambda, \tag{2.6}$$

where

$$S(\lambda,U,V) := (U+V+1+\lambda)^{-1} (UV+VU) (U+V+1+\lambda)^{-1}$$

and

$$Q(\lambda,U,V) := (U+V+1+\lambda)^{-1} \times \left[V (U+1+\lambda)^{-1} UV + U (V+1+\lambda)^{-1} VU\right] \times (U+V+1+\lambda)^{-1}.$$

We have the following operator inequalities

Theorem 1. *For all U, $V > 0$ and $a > 0$ we have*

$$\int_0^\infty (a+\lambda)^{-1} S(\lambda,a,U,V)\, d\lambda \leq \ln(U+a) + \ln(V+a) - \ln(U+V+a) - \ln a \leq \int_0^\infty (a+\lambda)^{-1} R(\lambda,a,U,V)\, d\lambda, \tag{2.7}$$

where

$$R(\lambda,a,U,V) = (U+V+a+\lambda)^{-1} (U+V)^2 (U+V+a+\lambda)^{-1}$$

for $\lambda \geq 0$.

In particular,

$$\int_0^\infty (1+\lambda)^{-1} S(\lambda,U,V)\, d\lambda \leq \ln(U+1) + \ln(V+1) - \ln(U+V+1) \leq \int_0^\infty (1+\lambda)^{-1} R(\lambda,U,V)\, d\lambda, \tag{2.8}$$

where

$$R(\lambda,U,V) = (U+V+1+\lambda)^{-1} (U+V)^2 (U+V+1+\lambda)^{-1}$$

for $\lambda \geq 0$.

Proof. Assume that U, $V \geq 0$. Observe that for a, $\lambda > 0$

$$\begin{aligned}(U+a+\lambda)^{-1} U &= (U+a+\lambda)^{-1} (U+a+\lambda-a-\lambda) \\ &= 1-(a+\lambda)(U+a+\lambda)^{-1},\end{aligned}$$

which shows that

$$0 \leq (U+a+\lambda)^{-1} U \leq 1.$$

If we multiply this inequality both sides by V, then we get

$$0 \leq V (U+a+\lambda)^{-1} UV \leq V^2.$$

Similarly,

$$0 \leq U (V+a+\lambda)^{-1} VU \leq U^2.$$

Therefore

$$0 \leq V (U+a+\lambda)^{-1} UV + U (V+a+\lambda)^{-1} VU \leq U^2 + V^2$$

and by multiplying both sides by $(U+V+1+\lambda)^{-1}$ we deduce

$$0 \leq Q(\lambda, a, U, V) \leq (U+V+a+\lambda)^{-1} \left(U^2+V^2\right) (U+V+a+\lambda)^{-1}$$

for a, $\lambda > 0$.

Now, if to this inequality we add $S(\lambda, a, U, V)$, then we obtain

$$\begin{aligned} S(\lambda, a, U, V) &\leq Q(\lambda, a, U, V) + S(\lambda, a, U, V) \\ &\leq (U+V+a+\lambda)^{-1} \left(U^2+V^2\right)(U+V+a+\lambda)^{-1} \\ &+ (U+V+a+\lambda)^{-1} (UV+VU)(U+V+a+\lambda)^{-1} \\ &= (U+V+a+\lambda)^{-1} (U+V)^2 (U+V+a+\lambda)^{-1} \\ &= R(\lambda, a, U, V) \end{aligned} \tag{2.9}$$

for a, $\lambda > 0$.

If we multiply (2.9) by $(1+\lambda)^{-1} > 0$, integrate over λ on $[0, \infty)$ and use representation (2.1) we derive (2.7). □

Corollary 2. *Let U, $V > 0$ and $a > 0$.*

(i) *If $UV + VU \geq 0$ for the positive number ω, then*

$$\ln(U+V+a) + \ln a \leq \ln(U+a) + \ln(V+a). \tag{2.10}$$

In particular,

$$\ln(U+V+1) \leq \ln(U+1) + \ln(V+1). \tag{2.11}$$

(ii) *If $U+V \leq \Omega$, with Ω a positive constant, then*

$$\ln(U+a) + \ln(V+a) - \ln(U+V+a) - \ln a \leq \frac{\Omega^2}{a} (U+V+a)^{-1}. \tag{2.12}$$

In particular,

$$\ln(U+1) + \ln(V+1) - \ln(U+V+1) \leq \Omega^2 (U+V+1)^{-1}. \tag{2.13}$$

Proof. (i) If $UV+VU\geq 0$, then by multiplying both sides by $(U+V+a+\lambda)^{-1}$ we get

$$0\leq (U+V+a+\lambda)^{-1}(UV+VU)(U+V+a+\lambda)^{-1}$$

for a, $\lambda>0$, which implies that

$$\begin{aligned}0&\leq \int_0^\infty (a+\lambda)^{-1}(U+V+a+\lambda)^{-1}(UV+VU)(U+V+a+\lambda)^{-1}d\lambda\\&=\int_0^\infty (a+\lambda)^{-1}S(\lambda,a,U,V)\,d\lambda\end{aligned}$$

and by (2.7) we get (2.10).

(ii) If $U+V\leq\Omega$, then

$$(U+V+a+\lambda)^{-1}(U+V)^2(U+V+a+\lambda)^{-1}\leq \Omega^2(U+V+a+\lambda)^{-2}$$

for a, $\lambda>0$. This implies that

$$\begin{aligned}&\int_0^\infty (a+\lambda)^{-1}(U+V+a+\lambda)^{-1}(U+V)^2(U+V+a+\lambda)^{-1}d\lambda\\&\leq\Omega^2\int_0^\infty (a+\lambda)^{-1}(U+V+a+\lambda)^{-2}d\lambda\\&\leq\frac{\Omega^2}{a}\int_0^\infty (U+V+a+\lambda)^{-2}d\lambda.\end{aligned}\tag{2.14}$$

Now, if we take the derivative over t in (2.2), then we get

$$\begin{aligned}t^{-1}&=\int_0^\infty(\lambda+1)^{-1}\left(\frac{t-1}{\lambda+t}\right)'d\lambda\\&=\int_0^\infty(\lambda+1)^{-1}\frac{\lambda+1}{(\lambda+t)^2}d\lambda=\int_0^\infty(\lambda+t)^{-2}d\lambda.\end{aligned}$$

This gives that

$$\int_0^\infty (U+V+a+\lambda)^{-2}d\lambda=(U+V+a)^{-1}$$

and by (2.14) and (2.7) we obtain (2.12). □

We have the following representation result:

Theorem 2. *For all A, B, $C>0$ we have the representation*

$$\begin{aligned}&S(A|B+A)+S(A|C+A)-S(A|B+C+A)\\&=\int_0^\infty(1+\lambda)^{-1}\Phi(\lambda,A,B,C)\,d\lambda+\int_0^\infty(1+\lambda)^{-1}\Psi(\lambda,A,B,C)\,d\lambda,\end{aligned}\tag{2.15}$$

where

$$\begin{aligned}\Phi(\lambda,A,B,C)&:=A(B+C+(1+\lambda)A)^{-1}\left(BA^{-1}C+CA^{-1}B\right)\\&\quad\times(B+C+(1+\lambda)A)^{-1}A\end{aligned}$$

and

$$\begin{aligned}&\Psi(\lambda,A,B,C)\\&:=A(B+C+(1+\lambda)A)^{-1}\\&\times\left[C(B+(1+\lambda)A)^{-1}BA^{-1}C+B(C+(1+\lambda)A)^{-1}CA^{-1}B\right]\\&\times(B+C+(1+\lambda)A)^{-1}A\end{aligned}$$

for $\lambda \geq 0$.

Proof. Consider $U = A^{-1/2}BA^{-1/2}$ and $V = A^{-1/2}CA^{-1/2}$, then

$$\begin{aligned}&\ln(U+1)+\ln(V+1)-\ln(U+V+1)\\&=\ln\left(A^{-1/2}BA^{-1/2}+1\right)+\ln\left(A^{-1/2}CA^{-1/2}+1\right)\\&-\ln\left(A^{-1/2}BA^{-1/2}+A^{-1/2}CA^{-1/2}+1\right)\\&=\ln\left(A^{-1/2}(B+A)A^{-1/2}\right)+\ln\left(A^{-1/2}(C+A)A^{-1/2}\right)\\&-\ln\left(A^{-1/2}(B+C+A)A^{-1/2}\right),\end{aligned}$$

$$\begin{aligned}&S(\lambda,U,V)\\&=\left(A^{-1/2}BA^{-1/2}+A^{-1/2}CA^{-1/2}+1+\lambda\right)^{-1}\\&\times\left(A^{-1/2}BA^{-1/2}A^{-1/2}CA^{-1/2}+A^{-1/2}CA^{-1/2}A^{-1/2}BA^{-1/2}\right)\\&\times\left(A^{-1/2}BA^{-1/2}+A^{-1/2}CA^{-1/2}+1+\lambda\right)^{-1}\\&=A^{1/2}(B+C+(1+\lambda)A)^{-1}A^{1/2}A^{-1/2}\left(BA^{-1}CA^{-1/2}+CA^{-1}B\right)A^{-1/2}\\&\times A^{1/2}(B+C+(1+\lambda)A)^{-1}A^{1/2}\\&=A^{1/2}(B+C+(1+\lambda)A)^{-1}\left(BA^{-1}C+CA^{-1}B\right)\\&\times(B+C+(1+\lambda)A)^{-1}A^{1/2}\end{aligned}$$

and

$$\begin{aligned}&Q(\lambda,U,V)\\&=A^{1/2}(B+C+(1+\lambda)A)^{-1}A^{1/2}\\&\times\left[A^{-1/2}CA^{-1/2}A^{1/2}(B+(1+\lambda)A)^{-1}A^{1/2}A^{-1/2}BA^{-1/2}A^{-1/2}CA^{-1/2}\right.\\&\left.+A^{-1/2}BA^{-1/2}A^{1/2}(C+(1+\lambda)A)^{-1}A^{1/2}A^{-1/2}CA^{-1/2}A^{-1/2}BA^{-1/2}\right]\\&\times A^{1/2}(B+C+(1+\lambda)A)^{-1}A^{1/2}\end{aligned}$$

$$= A^{1/2}\left(B+C+(1+\lambda)A\right)^{-1}A^{1/2}$$
$$\times A^{-1/2}\left[C\left(B+(1+\lambda)A\right)^{-1}BA^{-1}C+B\left(C+(1+\lambda)A\right)^{-1}CA^{-1}B\right]A^{-1/2}$$
$$\times A^{1/2}\left(B+C+(1+\lambda)A\right)^{-1}A^{1/2}$$
$$= A^{1/2}\left(B+C+(1+\lambda)A\right)^{-1}$$
$$\times\left[C\left(B+(1+\lambda)A\right)^{-1}BA^{-1}C+B\left(C+(1+\lambda)A\right)^{-1}CA^{-1}B\right]$$
$$\times\left(B+C+(1+\lambda)A\right)^{-1}A^{1/2}.$$

Now, if we use the identity (2.6) multiplied both sides by $A^{1/2}>0$, we obtain the desired representation (2.15). □

Corollary 3. *Assume that* A, B, $C>0$.

(i) *If* $BA^{-1}C+CA^{-1}B\geq 0$, *then*

$$S\left(A|B+C+A\right)\leq S\left(A|B+A\right)+S\left(A|C+A\right). \tag{2.16}$$

(ii) *If* $B+C\leq\Theta A$, *with* Θ *a positive constant, then*

$$\begin{aligned} &S\left(A|B+A\right)+S\left(A|C+A\right)-S\left(A|B+C+A\right)\\ &\leq\Theta^{2}A^{1/2}\left(B+C+A\right)^{-1}A^{1/2}. \end{aligned} \tag{2.17}$$

Corollary 4. *For all* A, B, $C>0$ *we have the representation*

$$\begin{aligned} &\frac{D_{x}\left(A|B+A\right)D_{x}\left(A|C+A\right)}{D_{x}\left(A|B+C+A\right)}\\ &=\exp\left(\int_{0}^{\infty}(1+\lambda)^{-1}\left\langle\Phi\left(\lambda,A,B,C\right)x,x\right\rangle d\lambda\right)\\ &\times\exp\left(\int_{0}^{\infty}(1+\lambda)^{-1}\left\langle\Psi\left(\lambda,A,B,C\right)x,x\right\rangle d\lambda\right). \end{aligned} \tag{2.18}$$

Proof. If we take the inner product over $x\in H$, $\|x\|=1$ in the identity (2.15) then we get

$$\begin{aligned} &\left\langle S\left(A|B+A\right)x,x\right\rangle+\left\langle S\left(A|C+A\right)x,x\right\rangle-\left\langle S\left(A|B+C+A\right)x,x\right\rangle\\ &=\int_{0}^{\infty}(1+\lambda)^{-1}\left\langle\Phi\left(\lambda,A,B,C\right)x,x\right\rangle d\lambda+\int_{0}^{\infty}(1+\lambda)^{-1}\left\langle\Psi\left(\lambda,A,B,C\right)x,x\right\rangle d\lambda \end{aligned}$$

and by taking the exponential we derive the desired result (2.18). □

Corollary 5. *Assume that* A, B, $C>0$ *and* $x\in H$, $\|x\|=1$.

(i) *If* $BA^{-1}C+CA^{-1}B\geq 0$, *then*

$$D_{x}\left(A|B+C+A\right)\leq D_{x}\left(A|B+A\right)D_{x}\left(A|C+A\right). \tag{2.19}$$

(ii) *If* $B+C\leq\Theta A$, *with* Θ *a positive constant, then*

$$\begin{aligned} &\frac{D_{x}\left(A|B+A\right)D_{x}\left(A|C+A\right)}{D_{x}\left(A|B+C+A\right)}\\ &\leq\exp\left[\Theta^{2}\left\langle A^{1/2}\left(B+C+A\right)^{-1}A^{1/2}x,x\right\rangle\right]. \end{aligned} \tag{2.20}$$

3. Some Related Results

In the case of $A = 1$ we derive by (2.15) the representation

$$\begin{aligned} &\frac{\Delta_x (B+1) \Delta_x (C+1)}{\Delta_x (B+C+1)} \\ &= \exp \left(\int_0^\infty (1+\lambda)^{-1} \Phi(\lambda, B, C) \, d\lambda \right) \\ &\times \exp \left(\int_0^\infty (1+\lambda)^{-1} \Psi(\lambda, B, C) \, d\lambda \right), \end{aligned} \tag{3.1}$$

where B, $C > 0$ and $x \in H$, $\|x\| = 1$

$$\begin{aligned} \Phi(\lambda, B, C) &:= (B + C + (1+\lambda) 1)^{-1} (BC + CB) \\ &\times (B + C + (1+\lambda) 1)^{-1} \end{aligned}$$

and

$$\begin{aligned} \Psi(\lambda, B, C) &:= (B + C + (1+\lambda) 1)^{-1} \\ &\times \left[C (B + (1+\lambda) 1)^{-1} BC + B (C + (1+\lambda) 1)^{-1} CB \right] \\ &\times (B + C + (1+\lambda) 1)^{-1} \end{aligned}$$

for $\lambda \geq 0$.

If B, $C > 0$ with $BC + CB \geq 0$, then

$$\Delta_x (B + C + 1) \leq \Delta_x (B+1) \Delta_x (C+1), \tag{3.2}$$

for all $x \in H$, $\|x\| = 1$.

Also, if B, $C > 0$ with $B + C \leq \Theta$, where Θ is a positive constant, then

$$\frac{\Delta_x (B+1) \Delta_x (C+1)}{\Delta_x (B+C+1)} \leq \exp \left[\Theta^2 \left\langle (B + C + 1)^{-1} x, x \right\rangle \right], \tag{3.3}$$

for all $x \in H$, $\|x\| = 1$.

The symmetrized product of two operators $C, B \in B(H)$ is defined by $S(C, B) = CB + BC$. In general, the symmetrized product of two operators C, B is not positive. Also Gustafson [12] showed that if $0 \leq m \leq C \leq M$ and $0 \leq n \leq B \leq N$, then we have the lower bound

$$S(A, B) \geq 2mn - \frac{1}{4} (M - m)(N - n), \tag{3.4}$$

which can take positive or negative values depending on the parameters m, M, n, N.

So, if $0 \leq m \leq C \leq M$ and $0 \leq n \leq B \leq N$ with

$$8mn \geq (M - m)(N - n),$$

then by (3.2) we get that

$$\Delta_x (B + C + 1) \leq \Delta_x (B+1) \Delta_x (C+1), \tag{3.5}$$

for all $x \in H$, $\|x\| = 1$.

If $0 \leq m \leq C \leq M$ and $0 \leq n \leq B \leq N$, then $C + B \leq (M + N) 1$ and $(B + C + 1)^{-1} \leq (m + n + 1)^{-1} 1$ and by (3.3) we also obtain that

$$\frac{\Delta_x (B + 1) \Delta_x (C + 1)}{\Delta_x (B + C + 1)} \leq \exp\left(\frac{(M + N)^2}{(m + n + 1)}\right), \tag{3.6}$$

for all $x \in H$, $\|x\| = 1$.

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