BASIC PROPERTIES OF RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

ABSTRACT. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

Assume that A, $B > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. In this paper we show among others that,

$$\begin{aligned} &s^{\operatorname{tr}(PA)}\exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right) \\ &\leq D_P(A|B) \\ &\leq s^{\operatorname{tr}(PA)}\exp\left(\frac{\operatorname{tr}(PB) - s\operatorname{tr}(PA)}{s}\right) \end{aligned}$$

for any $s > 0$.

The best lower bound in the first inequality is

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PAB^{-1}A)}\right)^{\operatorname{tr}(PA)} \leq D_P(A|B),$$

while the best upper bound in the second inequality is

$$D_P(A|B) \leq \left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)}.$$

1. Introduction

In 1952, in the paper [10], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

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For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [16], [17], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [19].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*, then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [1]-[7] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [8]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In [8], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic* P-*determinant* of the positive invertible operator A by [9]

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA\ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_P(I)=1 \text{ and } \eta_P(tI)=t^{-t} \tag{1.14}$$

for $t>0$.

Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If $A,\ B>0$, then we have the Ky Fan type inequality [9]

$$\eta_P((1-t)\,A+tB)\geq\left[\eta_P\left(A\right)\right]^{1-t}\left[\eta_P\left(B\right)\right]^{t} \tag{1.15}$$

for all $t\in[0,1]$.

Also we have the inequalities [9]:

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1$$

and if there exists the constants $0<m<M$ such that $m\leq A\leq M$, then [9]

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M}&\leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)}\leq\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\\
&\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1.
\end{aligned}$$

2. Relative Entropic Normalized P-Determinant

Kamei and Fujii [14], [15] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S\left(A|B\right):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{2.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [23].

In general, we can define for positive operators A, B

$$S\left(A|B\right):=s\text{-}\lim_{\varepsilon\to 0+}S\left(A+\varepsilon 1_H|B\right)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t)=-t\ln t$, the *operator entropy* has the following expression:

$$\eta\left(A\right)=-A\ln A=S\left(A|1_H\right)\geq 0$$

for positive contraction A. This shows that the relative operator entropy (2.1) is a relative version of the operator entropy.

For $A=1_H$ in (2.1) we have

$$S(1_H|B)=\ln B$$

for positive contraction B.

Following [18, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B)=-A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2}=B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \tag{2.2}$$

(ii) We have the inequalities

$$S(A|B)\le A(\ln\|B\|-\ln A)\ \text{ and } S(A|B)\le B-A; \tag{2.3}$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D)\ge S(A|C)+S(B|D);$$

(iv) If $B\le C$ then

$$S(A|B)\le S(A|C);$$

(v) If $B_n\downarrow B$ then

$$S(A|B_n)\downarrow S(A|B);$$

(vi) For $\alpha>0$ we have

$$S(\alpha A|\alpha B)=\alpha S(A|B);$$

(vii) For every operator T we have

$$T^*S(A|B)T\le S(T^*AT|T^*BT).$$

(viii) The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA+(1-t)B|tC+(1-t)D)\ge tS(A|C)+(1-t)S(B|D)$$

for any $t\in[0,1]$.

For other results on the relative operator entropy see [11]-[24] and the references therein.

Observe that, if we replace in (2.2) B with A, then we get

$$\begin{aligned}S(B|A)&=A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\\&=A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2},\end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}=-S(B|A) \tag{2.4}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

In [26], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B)=s\text{-}\lim_{t\to0}\frac{A\sharp_tB-A}{t}, \tag{2.5}$$

where

$$A\sharp_\nu B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu\in[0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν.

For $B = 1_H$ we have

$$A\sharp_\nu 1_H = A^{1-\nu}$$

while for $A = 1_H$ we get

$$1_H\sharp_\nu B = B^\nu$$

for any real number ν.

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [11]) by

$$T_t\left(A|B\right) := \frac{A\sharp_t B - A}{t}.$$

We then have

$$T_t\left(A|1_H\right) := \frac{A\sharp_t 1_H - A}{t} = \frac{A^{1-t} - A}{t},\ t > 0$$

and

$$T_t\left(1_H|B\right) := \frac{B^t - 1_H}{t},\ t > 0$$

for A, $B > 0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t\left(\cdot|\cdot\right)$ has been obtained in [14] for $0 < t \leq 1$. However, it hods for any $t > 0$.

Theorem 4. *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$T_t\left(A|B\right)\left(A\sharp_t B\right)^{-1} A \leq S\left(A|B\right) \leq T_t\left(A|B\right). \tag{2.6}$$

In particular, we have for $t = 1$ that

$$\left(1_H - AB^{-1}\right) A \leq S\left(A|B\right) \leq B - A,\ [14] \tag{2.7}$$

and for $t = 2$ that

$$\frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) A \leq S\left(A|B\right) \leq \frac{1}{2}\left(BA^{-1}B - A\right). \tag{2.8}$$

The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}\left(A|B\right) := 2\left(A\sharp B - A\right)$$

and

$$T_{1/2}\left(A|B\right)\left(A\sharp_{1/2}B\right)^{-1} A = 2\left(1_H - A\left(A\sharp B\right)^{-1}\right) A,$$

hence by (2.6) we get

$$2\left(1_H - A\left(A\sharp B\right)^{-1}\right) A \leq S\left(A|B\right) \leq 2\left(A\sharp B - A\right) \leq B - A. \tag{2.9}$$

Definition 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned} D_P\left(A|B\right) &:= \exp\{\operatorname{tr}\left[PS\left(A|B\right)\right]\} \\ &= \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}. \end{aligned}$$

We observe that for $A>0$,

$$D_P(A|1_H) := \exp\{\operatorname{tr}[PS(A|1_H)]\} = \exp\{\operatorname{tr}(-PA\ln A)\} = \eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B>0$,

$$D_P(1_H|B) := \exp\{\operatorname{tr}[PS(1_H|B)]\} = \exp\{\operatorname{tr}(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

We have the following fundamental properties for the relative entropic normalized determinant:

Proposition 2. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. Assume that A, $B>0$, then:*

(1) *We have the upper bound*
$$D_P(A|B) \leq \frac{\exp[\operatorname{tr}(PB)]}{\exp[\operatorname{tr}(PA)]};$$
(2) *For any C, D positive invertible operators we have that*
$$D_P(A+B|C+D) \geq D_P(A|C)\,D_P(B|D); \tag{2.10}$$
(3) *If $B\leq C$ then*
$$D_P(A|B) \leq D_P(A|C);$$
(4) *If $B_n\downarrow B$ then*
$$D_P(A|B_n) \downarrow D_P(A|B);$$
(5) *For $\alpha>0$ we have*
$$D_P(\alpha A|\alpha B) = [D_P(A|B)]^{\alpha}.$$

The proof follows by the properties "(ii)-(iii)" above.

Corollary 1. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. For A, $B>0$, α, $\beta>0$, we have*

$$\frac{\eta_P(A+B)}{\eta_P(A)\eta_P(B)} \geq \frac{\alpha^{\operatorname{tr}(PA)}\beta^{\operatorname{tr}(PB)}}{(\alpha+\beta)^{\operatorname{tr}[P(A+B)]}}. \tag{2.11}$$

In particular, for $\alpha=\beta=1$, we get

$$\frac{\eta_P(A+B)}{\eta_P(A)\eta_P(B)} \geq \frac{1}{2^{\operatorname{tr}[P(A+B)]}}. \tag{2.12}$$

Proof. Observe that

$$\begin{aligned}
D_P(A|\alpha 1_H) &= \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}\alpha 1_H A^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}\\
&= \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}(\ln\alpha 1_H - \ln A)A^{\frac{1}{2}}\right]\right\}\\
&= \exp(\operatorname{tr}(PA)\ln\alpha - \operatorname{tr}(PA\ln A)) = \alpha^{\operatorname{tr}(PA)}\eta_P(A).
\end{aligned}$$

Then by (2.10) for $C=\alpha 1_H$ and $D=\beta 1_H$ we have

$$D_P(A+B|(\alpha+\beta)1_H) \geq D_P(A|\alpha 1_H)\,D_P(B|\beta 1_H),$$

namely

$$(\alpha+\beta)^{\operatorname{tr}[P(A+B)]}\eta_P(A+B) \geq \alpha^{\operatorname{tr}(PA)}\eta_P(A)\beta^{\operatorname{tr}(PB)}\eta_P(B)$$

and the inequality (2.11) is obtained. □

Also, we have:

Corollary 2. *For C, $D>0$, γ, $\delta>0$ and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we have*

$$\frac{[\Delta_P(C+D)]^{\gamma+\delta}}{[\Delta_P(C)]^{\gamma}[\Delta_P(D)]^{\delta}}\geq\frac{(\gamma+\delta)^{\gamma+\delta}}{\gamma^{\gamma}\delta^{\delta}}. \tag{2.13}$$

In particular, for $\gamma=\delta=1$, we get

$$\frac{[\Delta_P(C+D)]^{2}}{\Delta_P(C)\Delta_P(D)}\geq 4. \tag{2.14}$$

Proof. Observe that

$$\begin{aligned}
&D_P(\gamma 1_H|C)\\
&=\exp\left\{\operatorname{tr}\left[P(\gamma 1_H)^{\frac{1}{2}}\left(\ln\left((\gamma 1_H)^{-\frac{1}{2}}C(\gamma 1_H)^{-\frac{1}{2}}\right)\right)(\gamma 1_H)^{\frac{1}{2}}\right]\right\}\\
&=\exp\{\operatorname{tr}[P(\ln C-\ln\gamma)]\}=\exp(\gamma\operatorname{tr}(P\ln C)-\ln(\gamma^{\gamma}))\\
&=\frac{\exp(\gamma\operatorname{tr}(P\ln C))}{\exp\ln(\gamma^{\gamma})}=\left(\frac{\Delta_P(C)}{\gamma}\right)^{\gamma}.
\end{aligned}$$

By (2.10) we have

$$D_P((\gamma+\delta)1_H|C+D)\geq D_P(\gamma 1_H|C)\,D_P(\delta 1_H|D),$$

namely

$$\left(\frac{\Delta_P(C+D)}{\gamma+\delta}\right)^{\gamma+\delta}\geq\left(\frac{\Delta_P(C)}{\gamma}\right)^{\gamma}\left(\frac{\Delta_P(D)}{\delta}\right)^{\delta}.$$

□

We have:

Proposition 3. *Assume that A, $B>0$ and and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and* $\operatorname{tr}(P)=1$.

(a) *We have*

$$D_P(A|B)\leq\|B\|^{\operatorname{tr}(AP)}\eta_P(A) \tag{2.15}$$

(aa) *For every operator T with $TP\neq 0$, we have*

$$\left[D_{\frac{TPT^*}{\operatorname{tr}(P|T|^2)}}(A|B)\right]^{\operatorname{tr}\left(P|T|^2\right)}\leq D_P(T^*AT|T^*BT). \tag{2.16}$$

(aaa) *For every C, $D>0$*

$$D_P(tA+(1-t)B|tC+(1-t)D)\geq[D_P(A|C)]^{t}[D_P(B|D)]^{1-t} \tag{2.17}$$

for all $t\in[0,1]$.

Proof. a. By multiplying both sides by $P^{1/2}\geq 0$ and taking the trace in (ii) we get

$$\begin{aligned}
D_P(A|B)&=\exp\{\operatorname{tr}[PS(A|B)]\}=\exp\left\{\operatorname{tr}\left[P^{1/2}S(A|B)P^{1/2}\right]\right\}\\
&\leq\exp\{\operatorname{tr}[P(\ln\|B\|A-A\ln A)]\}\\
&=\exp(\ln\|B\|\operatorname{tr}(AP)-\operatorname{tr}(PA\ln A))\\
&=\exp\left(\ln\|B\|^{\operatorname{tr}(AP)}\right)\exp(-\operatorname{tr}(PA\ln A))\\
&=\|B\|^{\operatorname{tr}(AP)}\eta_P(A)
\end{aligned}$$

and the statement is proved.

aa. By multiplying both sides by $P^{1/2} \geq 0$ and taking the trace in (vii) then we get

$$\begin{aligned}\exp\{\operatorname{tr}[PT^*S(A|B)T]\} &\leq \exp\{\operatorname{tr}[PS(T^*AT|T^*BT)]\}\\ &= D_P(T^*AT|T^*BT).\end{aligned}$$

Also, if $TP \neq 0$,

$$\begin{aligned}\exp\{\operatorname{tr}[PT^*S(A|B)T]\} &= \exp\{\operatorname{tr}[TPT^*S(A|B)]\}\\ &= \exp\{\operatorname{tr}[TPT^*S(A|B)]\}\\ &= \exp\left\{\operatorname{tr}\left(P|T|^2\right)\operatorname{tr}\left[\frac{TPT^*}{\operatorname{tr}\left(P|T|^2\right)}S(A|B)\right]\right\}\\ &= \left(\exp\left\{\operatorname{tr}\left[\frac{TPT^*}{\operatorname{tr}\left(P|T|^2\right)}S(A|B)\right]\right\}\right)^{\operatorname{tr}\left(P|T|^2\right)}\\ &= \left[D_{\frac{TPT^*}{\operatorname{tr}\left(P|T|^2\right)}}(A|B)\right]^{\operatorname{tr}\left(P|T|^2\right)},\end{aligned}$$

which proves the statement.

aaa. By multiplying both sides by $P^{1/2} \geq 0$ and taking the trace in (viii), then we get for all $t \in [0,1]$ that

$$\begin{aligned}&D_P(tA+(1-t)B|tC+(1-t)D)\\ &= \exp\{\operatorname{tr}[PS(tA+(1-t)B|tC+(1-t)D)]\}\\ &\geq \exp\{\operatorname{tr}[tP(S(A|C)+(1-t)S(B|D))]\}\\ &= \exp[t\operatorname{tr}(PS(A|C))+(1-t)\operatorname{tr}(PS(B|D))]\\ &= (\exp\{\operatorname{tr}[PS(A|C)]\})^t(\exp\{\operatorname{tr}[PS(B|D)]\})^{1-t}\\ &= [D_P(A|C)]^t[D_P(B|D)]^{1-t}\end{aligned}$$

and the statement is proved. □

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 3. *With the assumptions of Proposition 3,*

$$\int_0^1 D_P(tA+(1-t)B|tC+(1-t)D)dt \geq L(D_P(A|B), D_P(C|D)). \tag{2.18}$$

and

$$\begin{aligned}D_P\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq \int_0^1 &[D_P((1-t)A+tB|(1-t)C+tD)]^{1/2}\\ &\times[D_P(tA+(1-t)B|tC+(1-t)D)]^{1/2}dt.\end{aligned} \tag{2.19}$$

Proof. If we take the integral over $t \in [0,1]$ in (2.17), then we get

$$\int_0^1 D_P(tA+(1-t)B|tC+(1-t)D)dt \geq \int_0^1 [D_P(A|C)]^t [D_P(B|D)]^{1-t} dt = L(D_P(A|C), D_P(B|D))$$

for all A, B, C, $D>0$, which proves (2.18).

We get from (2.17) for $t=1/2$ that

$$D_P\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq [D_P(A|C)]^{1/2} [D_P(B|D)]^{1/2}.$$

If we replace A by $(1-t)A+tB$, B by $tA+(1-t)B$, C by $(1-t)C+tD$ and D by $tC+(1-t)D$ we obtain

$$\begin{aligned} &D_P\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \\ &\geq [D_P((1-t)A+tB|(1-t)C+tD)]^{1/2} \\ &\times [D_P(tA+(1-t)B|tC+(1-t)D)]^{1/2}. \end{aligned}$$

By taking the integral, we derive the desired result (2.19). □

By the use of Theorem 1 we can also state:

Proposition 4. *Assume that* A, $B>0$ *and* $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$. *Then for any* $t>0$ *we have*

$$\exp\left\{\operatorname{tr}\left[PT_t(A|B)(A\sharp_t B)^{-1}A\right]\right\} \leq D_P(A|B) \leq \exp\{\operatorname{tr}[PT_t(A|B)]\}. \tag{2.20}$$

In particular, we have for $t=1$ *that*

$$\frac{\exp[\operatorname{tr}(PA)]}{\exp[\operatorname{tr}(PAB^{-1}A)]} \leq D_P(A|B) \leq \frac{\exp(PB)}{\exp(PA)} \tag{2.21}$$

and for $t=2$ *that*

$$\left(\frac{\exp[\operatorname{tr}(PA)]}{\operatorname{tr}\left[P(AB^{-1})^2 A\right]}\right)^{\frac{1}{2}} \leq D_P(A|B) \leq \left(\frac{\exp\left[\operatorname{tr}\left(PBA^{-1}B\right)\right]}{\exp[\operatorname{tr}(PA)]}\right)^{\frac{1}{2}}. \tag{2.22}$$

We have the following bounds for the *normalized entropic determinant.*

Corollary 4. *Assume that* $A>0$ *and* $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$. *If* α, $t>0$, *then*

$$\begin{aligned} &\alpha^{-\operatorname{tr}(PA)} \exp\left[\operatorname{tr}\left(P\frac{A-\alpha^{-t}A^{t+1}}{t}\right)\right] \\ &\leq \eta_P(A) \\ &\leq \alpha^{-\operatorname{tr}(PA)} \exp\left[\operatorname{tr}\left(P\frac{\alpha^t A^{1-t}-A}{t}\right)\right]. \end{aligned} \tag{2.23}$$

In particular, for $\alpha=1$, *we get*

$$\exp\left[\operatorname{tr}\left(P\frac{A-A^{t+1}}{t}\right)\right] \leq \eta_P(A) \leq \exp\left[\operatorname{tr}\left(P\frac{A^{1-t}-A}{t}\right)\right] \tag{2.24}$$

for all $t>0$.

For $t=1$, *we get*

$$\alpha^{-\operatorname{tr}(PA)}\exp\left\{\operatorname{tr}\left[P\left(A-\alpha^{-1}A^{2}\right)\right]\right\}\leq\eta_{P}(A)\leq\alpha^{-\operatorname{tr}(PA)}\exp\left\{\operatorname{tr}\left[P\left(\alpha 1_{H}-A\right)\right]\right\}, \tag{2.25}$$

for all $\alpha>0$.

Also, for $\alpha=t=1$, *we obtain*

$$\exp\left\{\operatorname{tr}\left[P\left(A-A^{2}\right)\right]\right\}\leq\eta_{P}(A)\leq\exp\left\{\operatorname{tr}\left[P\left(1_{H}-A\right)\right]\right\}. \tag{2.26}$$

Proof. If we take $B=\alpha 1_{H}$ in (2.20), we get

$$\exp\left\{\operatorname{tr}\left[PT_{t}\left(A|\alpha 1_{H}\right)\left(A\sharp_{t}\left(\alpha 1_{H}\right)\right)^{-1}A\right]\right\}\leq D_{P}\left(A|\alpha 1_{H}\right)\leq\exp\left\{\operatorname{tr}\left[PT_{t}\left(A|\alpha 1_{H}\right)\right]\right\}. \tag{2.27}$$

Observe that

$$A\sharp_{t}\left(\alpha 1_{H}\right)=A^{1/2}\left(A^{-1/2}\left(\alpha 1_{H}\right)A^{-1/2}\right)^{t}A^{1/2}=\alpha^{t}A^{1-t}$$

and

$$T_{t}\left(A|\alpha 1_{H}\right)=\frac{A\sharp_{t}\left(\alpha 1_{H}\right)-A}{t}=\frac{\alpha^{t}A^{1-t}-A}{t}.$$

Also

$$\begin{aligned}T_{t}\left(A|\alpha 1_{H}\right)\left(A\sharp_{t}\left(\alpha 1_{H}\right)\right)^{-1}A&=\frac{\alpha^{t}A^{1-t}-A}{t}\left(\alpha^{t}A^{1-t}\right)^{-1}A\\&=\frac{A-A\left(\alpha^{t}A^{1-t}\right)^{-1}A}{t}\\&=\frac{A-\alpha^{-t}A^{t+1}}{t}.\end{aligned}$$

Then by (2.27) we get

$$\exp\left[\left(\operatorname{tr}\left(P\frac{A-\alpha^{-t}A^{t+1}}{t}\right)\right)\right]\leq\alpha^{\operatorname{tr}(PA)}\eta_{P}(A)\leq\exp\left[\operatorname{tr}\left(P\frac{\alpha^{t}A^{1-t}-A}{t}\right)\right]$$

and the inequality (2.23) is obtained. □

We also have the following bounds for the *normalized determinant*.

Corollary 5. *Assume that* $B>0$ *and* $P\geq 0$ *with* $P\in\mathcal{B}_{1}(H)$ *and* $\operatorname{tr}(P)=1$. *If* β, $t>0$, *then*

$$\beta\exp\left[\operatorname{tr}\left(P\frac{1_{H}-\beta^{t}B^{-t}}{t}\right)\right]\leq\Delta_{P}(B)\leq\beta\exp\left[\operatorname{tr}\left(P\frac{\beta^{-t}B^{t}-1_{H}}{t}\right)\right]. \tag{2.28}$$

In particular, for $\beta=1$, *we get*

$$\exp\left[\operatorname{tr}\left(P\frac{1_{H}-B^{-t}}{t}\right)\right]\leq\Delta_{P}(B)\leq\exp\left[\operatorname{tr}\left(P\frac{B^{t}-1_{H}}{t}\right)\right], \tag{2.29}$$

for all $t>0$.

For $t=1$, *we get*

$$\beta\exp\left[\operatorname{tr}\left(P\left(1_{H}-\beta B^{-1}\right)\right)\right]\leq\Delta_{P}(B)\leq\beta\exp\left[\operatorname{tr}\left(P\left(\beta^{-1}B-1_{H}\right)\right)\right], \tag{2.30}$$

for all $\beta>0$.

Also, for $\beta=t=1$, *we obtain*

$$\exp\left[\operatorname{tr}\left(P\left(1_{H}-B^{-1}\right)\right)\right]\leq\Delta_{P}(B)\leq\exp\left[\operatorname{tr}\left(P\left(B-1_{H}\right)\right)\right]. \tag{2.31}$$

Proof. We have from (2.20) for $A = \beta 1_H$ that

$$\begin{aligned}(2.32)\qquad & \exp\left\{\operatorname{tr}\left[PT_t\left(\beta 1_H|B\right)\left(\left(\beta 1_H\right)\sharp_t B\right)^{-1}\left(\beta 1_H\right)\right]\right\} \\ & \le D_P\left(\beta 1_H|B\right) \\ & \le \exp\left\{\operatorname{tr}\left[PT_t\left(\beta 1_H|B\right)\right]\right\}.\end{aligned}$$

Observe that

$$\left(\beta 1_H\right)\sharp_t B = \left(\beta 1_H\right)^{1/2}\left(\left(\beta 1_H\right)^{-1/2}B\left(\beta 1_H\right)^{-1/2}\right)^t\left(\beta 1_H\right)^{1/2} = \beta^{1-t}B^t,$$

and

$$T_t\left(\left(\beta 1_H\right)|B\right) := \frac{\left(\beta 1_H\right)\sharp_t B - \beta 1_H}{t} = \frac{\beta^{1-t}B^t - \beta 1_H}{t}.$$

Also,

$$\begin{aligned}T_t\left(\beta 1_H|B\right)\left(\left(\beta 1_H\right)\sharp_t B\right)^{-1}\left(\beta 1_H\right) &= \frac{\beta^{1-t}B^t - \beta 1_H}{t}\left(\beta^{1-t}B^t\right)^{-1}\beta \\ &= \frac{\beta - \beta\left(\beta^{1-t}B^t\right)^{-1}\beta}{t} \\ &= \frac{\beta - \beta^{t+1}B^{-t}}{t}.\end{aligned}$$

Then by (2.32) we get

$$\begin{aligned}\exp\left[\operatorname{tr}\left(P\frac{\beta 1_H - \beta^{t+1}B^{-t}}{t}\right)\right] &\le \left(\frac{\Delta_P(B)}{\beta}\right)^{\beta} \\ &\le \exp\left[\operatorname{tr}\left(P\frac{\beta^{1-t}B^t - \beta 1_H}{t}\right)\right].\end{aligned}$$

By taking the power $1/\beta$ we get

$$\begin{aligned}\exp\left[\operatorname{tr}\left(P\frac{\beta 1_H - \beta^{t+1}B^{-t}}{\beta t}\right)\right] &\le \frac{\Delta_P(B)}{\beta} \\ &\le \exp\left[\operatorname{tr}\left(P\frac{\beta^{1-t}B^t - \beta 1_H}{\beta t}\right)\right],\end{aligned}$$

which is equivalent to (2.28). □

3. Several Bounds

We have the following bounds for the relative entropic normalized determinant:

Theorem 5. *Assume that A, $B > 0$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Then for any $s > 0$ we have*

$$\begin{aligned}(3.1)\qquad & s^{\operatorname{tr}(PA)}\exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right) \\ & \le D_P(A|B) \\ & \le s^{\operatorname{tr}(PA)}\exp\left(\frac{\operatorname{tr}(PB) - s\operatorname{tr}(PA)}{s}\right).\end{aligned}$$

The best lower bound in the first inequality is

$$(3.2)\qquad \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}\left(PAB^{-1}A\right)}\right)^{\operatorname{tr}(PA)} \le D_P(A|B),$$

while the best upper bound in the second inequality is

$$D_P(A|B) \le \left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)}. \tag{3.3}$$

Proof. We use the gradient inequality for differentiable convex functions f on the open interval

$$f'(s)(t-s) \ge f(t) - f(s) \ge f'(t)(t-s)$$

for all $t,\ s \in I$.

If we write this inequality for the function ln on $(0,\infty)$, then we get

$$\frac{t}{s} - 1 \ge \ln t - \ln s \ge 1 - \frac{s}{t}$$

for all $t,\ s \in (0,\infty)$.

Using the functional calculus for positive operator $T > 0$, we get

$$\frac{1}{s}T - 1_H \ge \ln T - \ln s 1_H \ge 1_H - sT^{-1}.$$

for all $s \in (0,\infty)$.

If we take $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} > 0$, then we get

$$\frac{1}{s}A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_H \ge \ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \ln s1_H \ge 1_H - sA^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$$

for all $s \in (0,\infty)$.

If we multiply both sides by $A^{\frac{1}{2}} > 0$, then we get

$$\frac{1}{s}B - A \ge A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}} - (\ln s)A \ge A - sAB^{-1}A$$

for all $s \in (0,\infty)$.

Now, by multiplying both sides by $P^{1/2} \ge 0$ and taking the trace, then we get

$$\begin{aligned}\frac{1}{s}\operatorname{tr}(PB) - \operatorname{tr}(PA) &\ge \operatorname{tr}\left(PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right) - (\ln s)\operatorname{tr}(PA)\\ &\ge \operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\end{aligned}$$

for all $s \in (0,\infty)$.

By taking the exponential, we derive

$$\begin{aligned}\exp\left(\frac{\operatorname{tr}(PB) - s\operatorname{tr}(PA)}{s}\right) &\ge \frac{\exp\operatorname{tr}\left(PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right)}{\exp\left[(\ln s)\operatorname{tr}(PA)\right]}\\ &\ge \exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right)\end{aligned}$$

for all $s \in (0,\infty)$, which is equivalent to (3.1).

Now, consider the function

$$f(s) := s^{\operatorname{tr}(PA)}\exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right),\ s \in (0,\infty).$$

We have

$$\begin{aligned}f'(s) &= \operatorname{tr}(PA)\, s^{\operatorname{tr}(PA)-1}\exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right)\\ &\quad - \left\langle AB^{-1}Ax, x\right\rangle s^{\operatorname{tr}(PA)}\exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right)\\ &= s^{\operatorname{tr}(PA)-1}\exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PAB^{-1}A\right)\right)\\ &\quad \times \left(\operatorname{tr}(PA) - \operatorname{tr}\left(PAB^{-1}A\right)s\right).\end{aligned}$$

We observe that the function f is increasing on $\left(0, \frac{\operatorname{tr}(PA)}{\operatorname{tr}(PAB^{-1}A)}\right)$ and decreasing on $\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PAB^{-1}A)}, \infty\right)$. Therefore

$$\sup_{s\in(0,\infty)} f(s) = f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PAB^{-1}A)}\right) = \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PAB^{-1}A)}\right)^{\operatorname{tr}(PA)},$$

which gives the best lower bound in (3.1).

Now, consider the function

$$g(s) := s^{\operatorname{tr}(PA)} \exp\left(\frac{\operatorname{tr}(PB)}{s} - \operatorname{tr}(PA)\right),\ s \in (0,\infty).$$

We have

$$\begin{aligned} g'(s) &:= \operatorname{tr}(PA)\, s^{\operatorname{tr}(PA)-1} \exp\left(\frac{\operatorname{tr}(PB)}{s} - \operatorname{tr}(PA)\right) \\ &\quad + s^{\operatorname{tr}(PA)} \exp\left(\frac{\operatorname{tr}(PB)}{s} - \operatorname{tr}(PA)\right)\left(-\frac{\operatorname{tr}(PB)}{s^2}\right) \\ &= s^{\operatorname{tr}(PA)-1} \exp\left(\frac{\operatorname{tr}(PB)}{s} - \operatorname{tr}(PA)\right)\left(\operatorname{tr}(PA) - \frac{\operatorname{tr}(PB)}{s}\right) \\ &= s^{\operatorname{tr}(PA)-2} \exp\left(\frac{\operatorname{tr}(PB)}{s} - \operatorname{tr}(PA)\right)\left(\operatorname{tr}(PA)\, s - \operatorname{tr}(PB)\right). \end{aligned}$$

We observe that the function g is decreasing on $\left(0, \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)$ and increasing on $\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}, \infty\right)$. Therefore

$$\inf_{s\in(0,\infty)} g(s) = g\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right) = \left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)},$$

which gives the best upper bound in (3.1). □

Corollary 6. *Assume that $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Then for any $s > 0$ we have*

$$\begin{aligned} & s^{\operatorname{tr}(PA)} \exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PA^2\right)\right) \\ & \leq \eta_P(A) \leq s^{\operatorname{tr}(PA)} \exp\left(\frac{1}{s} - \operatorname{tr}(PA)\right). \end{aligned} \tag{3.4}$$

The best lower bound for $\eta_P(A)$ is obtained for $s = \frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}$, namely

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)^{\operatorname{tr}(PA)} \leq \eta_P(A).$$

The best upper bound for $\eta_P(A)$ is obtained for $s = \operatorname{tr}(PA)^{-1}$, namely

$$\eta_P(A) \leq \operatorname{tr}(PA)^{-\operatorname{tr}(PA)}.$$

Proof. If we take $B = 1_H$ in (3.1), then we get

$$\begin{aligned} s^{\operatorname{tr}(PA)} \exp\left(\operatorname{tr}(PA) - s\operatorname{tr}\left(PA^2\right)\right) &\leq \eta_P(A) \\ &\leq s^{\operatorname{tr}(PA)} \exp\left(\frac{1 - s\operatorname{tr}(PA)}{s}\right), \end{aligned}$$

which is equivalent to (3.4). □

Corollary 7. *Assume that* $B>0$ *and* $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$. *Then for any* $s>0$ *we have*

$$s\exp\left(1-s\operatorname{tr}\left(PB^{-1}\right)\right)\leq\Delta_P(B)\leq s\exp\left(\frac{\operatorname{tr}(PB)-s}{s}\right). \tag{3.5}$$

The best lower bound for $\Delta_P(B)$ *is obtained for* $s=\operatorname{tr}\left(PB^{-1}\right)^{-1}$, *namely*

$$\operatorname{tr}\left(PB^{-1}\right)^{-1}\leq\Delta_P(B).$$

The best upper bound for $\Delta_P(B)$ *is obtained for* $s=\operatorname{tr}(PB)$, *namely*

$$\Delta_P(A)\leq\operatorname{tr}(PB).$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa