SOME INEQUALITIES FOR RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

Assume that $0 < mA \leq B \leq MA$ for some constants M, m and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. In this paper we show among others that,

$$\begin{aligned} 1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2M^2}} \\ &\leq \frac{\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)}}{D_P(A|B)} \\ &\leq \quad \exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\Bigg)^{\frac{1}{2m^2}}. \end{aligned}$$

1. Introduction

In 1952, in the paper [10], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

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If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [16], [17], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [19].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*, then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \leq \|T\| \, \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| \, e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \ \text{and} \ \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\left\{\langle A, B\rangle_2 \ \mid B \in \mathcal{B}_2(H), \ \ \|B\|_2 \leq 1\right\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}(P^{1/2}TP^{1/2}) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [1]-[7] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [8]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(t1_H) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In [8], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by [9]

$$\eta_P(A) := \exp[-\operatorname{tr}(PA\ln A)] = \exp\{\operatorname{tr}[P\eta(A)]\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A) P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln(tA)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}(PA)\right)\exp\left(-t\operatorname{tr}(PA\ln A)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}(PA\ln A)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_P(1_H)=1 \text{ and } \eta_P(t1_H)=t^{-t} \tag{1.14}$$

for $t>0$.

Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If A, $B>0$, then we have the Ky Fan type inequality [9]

$$\eta_P((1-t)A+tB)\geq\left[\eta_P(A)\right]^{1-t}\left[\eta_P(B)\right]^{t} \tag{1.15}$$

for all $t\in[0,1]$.

Also we have the inequalities [9]:

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)}\leq\frac{\eta_P(A)}{\left[\operatorname{tr}(PA)\right]^{-\operatorname{tr}(PA)}}\leq 1$$

and if there exists the constants $0<m<M$ such that $m1_H\leq A\leq M1_H$, then [9]

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M}&\leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)}\leq\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)}\\
&\leq\frac{\eta_P(A)}{\left[\operatorname{tr}(PA)\right]^{-\operatorname{tr}(PA)}}\leq 1.
\end{aligned}$$

Kamei and Fujii [14], [15] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [23]. For various results on relative operator entropy see [11]-[24] and the references therein.

Definition 1. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}
D_P(A|B)&:=\exp\{\operatorname{tr}\left[PS(A|B)\right]\}\\
&=\exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.
\end{aligned}$$

We observe that for $A>0$,

$$D_P(A|1_H):=\exp\{\operatorname{tr}\left[PS(A|1_H)\right]\}=\exp\left\{\operatorname{tr}(-PA\ln A)\right\}=\eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B>0$,

$$D_P(1_H|B) := \exp\{\operatorname{tr}[PS(1_H|B)]\} = \exp\{\operatorname{tr}(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

Assume that $0 < mA \le B \le MA$ for some constants M, m and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. In this paper we show among others that,

$$\begin{aligned} 1 &\le \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2M^2}} \\ &\le \frac{\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)}}{D_P(A|B)} \\ &\le \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2m^2}}. \end{aligned}$$

2. Main Results

We start to the following logarithmic inequalities:

Lemma 1. *For any a, $b>0$ we have*

$$\begin{aligned} \frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 &= \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \\ &\le \frac{b-a}{a} - \ln b + \ln a \\ &\le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2. \end{aligned} \tag{2.1}$$

Proof. It is easy to see that

$$\int_a^b \frac{b-t}{t^2}dt = \frac{b-a}{a} - \ln b + \ln a \tag{2.2}$$

for any a, $b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.3}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \ge \int_b^a \frac{t-b}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.4}$$

Therefore, by (2.3) and (2.4) we have for any a, $b>0$ that

$$\int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (2.2) we then get the desired result (2.1). □

When some bounds for a, b are provided, then we have:

Corollary 1. *Assume that a, $b \in [m,M] \subset (0,\infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{m^2} \tag{2.5}$$

and, by swapping a with b,

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{2.6}$$

Theorem 4. *Assume that $0 < mA \le B \le MA$ for some constants M and m. Then for all $a \in [m,M]$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$,*

$$\begin{aligned} 1 &\le \left(\exp\left[\operatorname{tr}\left(PBA^{-1}B\right) - 2a\operatorname{tr}(PB) + a^2\operatorname{tr}(PA)\right]\right)^{\frac{1}{2M^2}} \\ &\le \frac{\left(\frac{a}{e}\right)^{\operatorname{tr}(PA)}\left[\exp\left(\operatorname{tr}(PB)\right)\right]^{\frac{1}{a}}}{D_P(A|B)} \\ &\le \left(\exp\left[\operatorname{tr}\left(PBA^{-1}B\right) - 2a\operatorname{tr}(PB) + a^2\operatorname{tr}(PA)\right]\right)^{\frac{1}{2m^2}} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} 1 &\le \left(\exp\left[\operatorname{tr}\left(PBA^{-1}B\right) - 2a\operatorname{tr}(PB) + a^2\operatorname{tr}(PA)\right]\right)^{\frac{1}{2M^2}} \\ &\le \frac{D_P(A|B)}{(ae)^{\operatorname{tr}(PA)}\left[\exp\left(-\operatorname{tr}\left(PAB^{-1}A\right)\right)\right]^{a}} \\ &\le \left(\exp\left[\operatorname{tr}\left(PBA^{-1}B\right) - 2a\operatorname{tr}(PB) + a^2\operatorname{tr}(PA)\right]\right)^{\frac{1}{2m^2}}. \end{aligned} \tag{2.8}$$

Proof. If we use the continuous functional calculus for selfadjoint operator T with spectrum $\operatorname{Sp} T$ in $[m,M]$ and the inequality (2.5) we have

$$\frac{1}{2}\frac{(T-a1_H)^2}{M^2} \le \frac{T-a1_H}{a} - \ln T + (\ln a)1_H \le \frac{1}{2}\frac{(T-a1_H)^2}{m^2} \tag{2.9}$$

for all $a \in [m,M]$.

Since $0 < mA \le B \le MA$, hence by multiplying both sides by $A^{-1/2} > 0$ we get $0 < m \le A^{-1/2}BA^{-1/2} \le A$. By writing (2.9) for $T = A^{-1/2}BA^{-1/2}$ we get

$$\begin{aligned} &\frac{1}{2}\frac{\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2}{M^2} \\ &\le \frac{A^{-1/2}BA^{-1/2} - a1_H}{a} - \ln\left(A^{-1/2}BA^{-1/2}\right) + (\ln a)1_H \\ &\le \frac{1}{2}\frac{\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2}{m^2} \end{aligned}$$

and by multiplying both sides by $A^{1/2}>0$, we derive

$$\begin{aligned}
&\frac{1}{2M^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2A^{1/2} \\
&\le \frac{1}{a}B-A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}+(\ln a)A-A \\
&\le \frac{1}{2m^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2A^{1/2}
\end{aligned} \tag{2.10}$$

for all $a\in[m,M]$.

Observe that

$$\begin{aligned}
&A^{1/2}\left(A^{-1/2}BA^{-1/2}-a1_H\right)^2A^{1/2} \\
&=A^{1/2}\left(A^{-1/2}BA^{-1/2}A^{-1/2}BA^{-1/2}-2aA^{-1/2}BA^{-1/2}+a^2 1_H\right)A^{1/2} \\
&=BA^{-1}B-2B+a^2A
\end{aligned}$$

and by (2.10) we get

$$\begin{aligned}
&\frac{1}{2M^2}\left(BA^{-1}B-2B+a^2A\right) \\
&\le \frac{1}{a}B-A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}+(\ln a)A-A \\
&\le \frac{1}{2m^2}\left(BA^{-1}B-2B+a^2A\right).
\end{aligned}$$

Now, if we multiply both sides by $P^{1/2}\ge 0$, take the trace and use its properties, then we get

$$\begin{aligned}
&\frac{1}{2M^2}\operatorname{tr}\left[P\left(BA^{-1}B-2aB+a^2A\right)\right] \\
&\le \frac{1}{a}\operatorname{tr}(PB)-\operatorname{tr}\left(PA^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}\right)+\ln\left(\frac{a}{e}\right)^{\operatorname{tr}(PA)} \\
&\le \frac{1}{2m^2}\operatorname{tr}\left[P\left(BA^{-1}B-2aB+a^2A\right)\right],
\end{aligned} \tag{2.11}$$

for all $a\in[m,M]$ and $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

If we take the exponential in (2.11), then we get

$$\begin{aligned}
1&\le \exp\left[\frac{1}{2M^2}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2a\operatorname{tr}(PB)+a^2\operatorname{tr}(PA)\right)\right] \\
&\le \exp\left[\frac{1}{a}\operatorname{tr}(PB)-\operatorname{tr}\left(PA^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2}\right)+\ln\left(\frac{a}{e}\right)^{\operatorname{tr}(PA)}\right] \\
&\le \exp\left[\frac{1}{2m^2}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2a\operatorname{tr}(PB)+a^2\operatorname{tr}(PA)\right)\right],
\end{aligned} \tag{2.12}$$

for all $a\in[m,M]$ and $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

This is equivalent to (2.7).

From (2.6) we get

$$\frac{1}{2}\frac{\left(T-a1_H\right)^2}{M^2}\le \ln T-\ln(ae)1_H+aT^{-1}\le\frac{1}{2}\frac{(T-a)^2}{m^2}$$

for selfadjoint operator T with spectrum in $[m,M]$ and for all $a\in[m,M]$.

If we take $T = A^{-1/2}BA^{-1/2}$, then we get

$$\frac{1}{2M^2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 \leq \ln A^{-1/2}BA^{-1/2} - \ln(ae)\,1_H + aA^{1/2}B^{-1}A^{1/2} \leq \frac{1}{2m^2}\left(A^{-1/2}BA^{-1/2} - a\right)^2$$

and by multiplying both sides by $A^{1/2} > 0$, we derive

$$\begin{aligned}&\frac{1}{2M^2}A^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 A^{1/2}\\ &\leq A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2} - \ln(ae)\,A + aAB^{-1}A\\ &\leq \frac{1}{2m^2}A^{1/2}\left(A^{-1/2}BA^{-1/2} - a\right)^2 A^{1/2}\end{aligned}$$

for all $a \in [m, M]$.

If we multiply both sides by $P^{1/2} \geq 0$, take the trace and use its properties, then we get

$$\begin{aligned}&\frac{1}{2M^2}\operatorname{tr}\left[PA^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 A^{1/2}\right]\\ &\leq \operatorname{tr}\left(PA^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2}\right) - \ln(ae)\operatorname{tr}(PA) + a\operatorname{tr}\left(PAB^{-1}A\right)\\ &\leq \frac{1}{2m^2}\operatorname{tr}\left[PA^{1/2}\left(A^{-1/2}BA^{-1/2} - a1_H\right)^2 A^{1/2}\right],\end{aligned}$$

which produces the inequality (2.8). □

Corollary 2. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}(2.13)\qquad 1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2M^2}}\\ &\leq \frac{\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)}}{D_P(A|B)}\\ &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2m^2}}\end{aligned}$$

and

$$\begin{aligned}(2.14)\qquad 1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2M^2}}\\ &\leq \frac{D_P(A|B)}{\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)^{\operatorname{tr}(PA)}\left[\exp\left(\frac{\operatorname{tr}(PA)^2 - \operatorname{tr}(PB)\operatorname{tr}(PAB^{-1}A)}{\operatorname{tr}(PB)}\right)\right]^{\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}}}\\ &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PAB^{-1}A\right)\operatorname{tr}(PA) - \operatorname{tr}(PB)^2}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2m^2}}\end{aligned}$$

for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

The proof follows by Theorem 4 for $a=\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}$, which, due to the condition $mA\leq B\leq MA$, belongs to the interval $[m,M]$ for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

Remark 1. *Assume that* $0<m1_H\leq B\leq M1_H$ *for some constants* M *and* m. *Then by Corollary 2 for* $A=1_H$ *we have*

$$
\begin{aligned}
1 &\leq \left(\exp\left[\operatorname{tr}\left(PB^2\right)-\operatorname{tr}\left(PB\right)^2\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\operatorname{tr}(PB)}{\eta_P(B)} \\
&\leq \left(\exp\left[\operatorname{tr}\left(PB^2\right)-\operatorname{tr}\left(PB\right)^2\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
1 &\leq \left(\exp\left[\operatorname{tr}\left(PB^2\right)-\operatorname{tr}\left(PB\right)^2\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\eta_P(B)}{\operatorname{tr}(PB)\left[\exp\left(\operatorname{tr}(PB)^{-1}-\operatorname{tr}\left(PB^{-1}\right)\right)\right]^{\operatorname{tr}(PB)}} \\
&\leq \left(\exp\left[\operatorname{tr}\left(PB^2\right)-\operatorname{tr}\left(PB\right)^2\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.16}
$$

for all $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$.

Assume that $0<mA\leq 1\leq MA$ *for some constants* M *and* m. *Then by Corollary 2 for* $B=1_H$ *we have*

$$
\begin{aligned}
1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}}{\Delta_P(A)} \\
&\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2M^2}} \\
&\leq \frac{\Delta_P(A)}{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}\left[\exp\left[-\left(\operatorname{tr}\left(PA^2\right)-\operatorname{tr}(PA)^2\right)\right]\right]^{\frac{1}{\operatorname{tr}(PA)}}} \\
&\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1}{\operatorname{tr}(PA)}\right]\right)^{\frac{1}{2m^2}}
\end{aligned}
\tag{2.18}
$$

for all $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$.

If $0 < n1_H \leq A \leq N1_H$, *then* $0 < \frac{1}{N}A \leq 1_H \leq \frac{1}{n}A$ *and by taking* $m = \frac{1}{N}$ *and* $M = \frac{1}{n}$ *in (2.17) and (2.18), then we get*

$$
\begin{aligned}
1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1}{\operatorname{tr}(PA)}\right]\right)^{\frac{n^2}{2}} \\
&\leq \frac{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}}{\Delta_P(A)} \\
&\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1}{\operatorname{tr}(PA)}\right]\right)^{\frac{N^2}{2}}
\end{aligned}
\tag{2.19}
$$

and

$$
\begin{aligned}
1 &\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1}{\operatorname{tr}(PA)}\right]\right)^{\frac{n^2}{2}} \\
&\leq \frac{\Delta_P(A)}{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}\left[\exp\left[-\left(\operatorname{tr}\left(PA^2\right) - \operatorname{tr}(PA)^2\right)\right]\right]^{\frac{1}{\operatorname{tr}(PA)}}} \\
&\leq \left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1}{\operatorname{tr}(PA)}\right]\right)^{\frac{N^2}{2}}
\end{aligned}
\tag{2.20}
$$

for all $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$.

We have the following reverses of Schwarz's inequality, see for instance [7]

$$
0 \leq \operatorname{tr}\left(PB^2\right) - \operatorname{tr}(PB)^2 \leq \frac{1}{4}(M - m)^2,
$$

and

$$
0 \leq \operatorname{tr}\left(PB^2\right) - \operatorname{tr}(PB)^2 \leq \frac{(M - m)^2}{4mM}\operatorname{tr}(PB)^2,
$$

for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, where $0 < m1_H \leq B \leq M1_H$ for some constants M and m.

By (2.15) we get

$$
\begin{aligned}
\frac{\operatorname{tr}(PB)}{\eta_P(B)} &\leq \left(\exp\left[\operatorname{tr}\left(PB^2\right) - \operatorname{tr}(PB)^2\right]\right)^{\frac{1}{2m^2}} \\
&\leq \begin{cases} \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right] \\ \exp\left[\frac{1}{8mM}\left(\frac{M}{m} - 1\right)^2\operatorname{tr}(PB)^2\right] \end{cases}
\end{aligned}
\tag{2.21}
$$

while by (2.16) we derive

$$\frac{\eta_P(B)}{\operatorname{tr}(PB)\left[\exp\left(\operatorname{tr}(PB)^{-1}-\operatorname{tr}(PB^{-1})\right)\right]^{\operatorname{tr}(PB)}} \tag{2.22}$$
$$\leq\left(\exp\left[\operatorname{tr}\left(PB^2\right)-\operatorname{tr}(PB)^2\right]\right)^{\frac{1}{2m^2}}$$
$$\leq\begin{cases}\exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right]\\ \\ \exp\left[\frac{1}{8mM}\left(\frac{M}{m}-1\right)^2\operatorname{tr}(PB)^2\right]\end{cases}$$

for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, where $0<m1_H\leq B\leq M1_H$ for some constants M and m.

We know the following reverse inequalities hold as well, see for instance [7]

$$\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)\leq\frac{(N+v)^2}{4nN},$$

and

$$\operatorname{tr}\left(PA^{-1}\right)-\operatorname{tr}(PA)^{-1}\leq\frac{\left(\sqrt{N}-\sqrt{n}\right)^2}{nN},$$

for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, where $0<n1_H\leq A\leq N1_H$ for some constants N and n.

From (2.19) we then get

$$\frac{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}}{\Delta_P(A)}\leq\left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1}{\operatorname{tr}(PA)}\right]\right)^{\frac{N^2}{2}} \tag{2.23}$$
$$\leq\exp\left[\frac{N(N-n)^2}{8n}\operatorname{tr}(PA)^{-1}\right],$$

while from (2.20) we obtain

$$\frac{\Delta_P(A)}{\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}\left[\exp\left[-\left(\operatorname{tr}(PA^2)-\operatorname{tr}(PA)^2\right)\right]\right]^{\frac{1}{\operatorname{tr}(PA)}}}$$
$$\leq\left(\exp\left[\frac{\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1}{\operatorname{tr}(PA)}\right]\right)^{\frac{N^2}{2}}$$
$$\leq\exp\left[\frac{N\left(\sqrt{N}-\sqrt{n}\right)^2}{2n}\right]$$

for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, where $0<n1_H\leq A\leq N1_H$ for some constants N and n.

3. Some Related Results

If we take in (2.1) $a=1$ and $b=u\in(0,\infty)$, then we get

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\min\{1,u\}}{\max\{1,u\}}\right)^2 &= \frac{1}{2}\frac{(u-1)^2}{\max^2\{1,u\}} \\ &\le u-1-\ln u \\ &\le \frac{1}{2}\frac{(u-1)^2}{\min^2\{1,u\}} = \frac{1}{2}\left(\frac{\max\{1,u\}}{\min\{1,u\}}-1\right)^2\end{aligned} \tag{3.1}$$

and if we take $a=u$ and $b=1$, then we also get

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\min\{1,u\}}{\max\{1,u\}}\right)^2 &= \frac{1}{2}\frac{(u-1)^2}{\max^2\{1,u\}} \\ &\le \ln u-\frac{u-1}{u} \\ &\le \frac{1}{2}\frac{(u-1)^2}{\min^2\{1,u\}} = \frac{1}{2}\left(\frac{\max\{1,u\}}{\min\{1,u\}}-1\right)^2.\end{aligned} \tag{3.2}$$

If $u\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\}\le\min\{1,u\}\le\min\{1,K\}$$

and

$$\max\{1,k\}\le\max\{1,u\}\le\max\{1,K\}.$$

By (3.1) and (3.2) we get the *local bounds*

$$\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,K\}}\le u-1-\ln u\le\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,k\}} \tag{3.3}$$

and

$$\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,K\}}\le \ln u-\frac{u-1}{u}\le\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,k\}} \tag{3.4}$$

for any $u\in[k,K]\subset(0,\infty)$.

Theorem 5. *Assume that $0<mA\le B\le MA$ for some constants M and m. Then*

$$\begin{aligned}1 &\le \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2\operatorname{tr}(PB)+\operatorname{tr}(PA)\right)\right] \\ &\le \frac{\exp\left(\operatorname{tr}(PB)-\operatorname{tr}(PA)\right)}{D_P(A|B)} \\ &\le \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2\operatorname{tr}(PB)+\operatorname{tr}(PA)\right)\right],\end{aligned} \tag{3.5}$$

and

$$1\le \exp\left[\frac{1}{2\max^{2}\{1,M\}}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2\operatorname{tr}(PB)+\operatorname{tr}(PA)\right)\right] \tag{3.6}$$
$$\le \frac{D_P(A|B)}{\exp\left(\operatorname{tr}(PA)-\operatorname{tr}\left(PAB^{-1}A\right)\right)}$$
$$\le \exp\left[\frac{1}{2\min^{2}\{1,m\}}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2\operatorname{tr}(PB)+\operatorname{tr}(PA)\right)\right]$$

for all $P\ge 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$.

Proof. From (3.3) we have for the selfadjoint operator T with $0<m1_H\le T\le M1_H$ that

$$\frac{1}{2\max^{2}\{1,M\}}(T-1_H)^2\le T-1_H-\ln T\le\frac{1}{2\min^{2}\{1,m\}}(T-1_H)^2.$$

If we write this inequality for $T=A^{-1/2}BA^{-1/2}$, then we get

$$\frac{1}{2\max^{2}\{1,M\}}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2$$
$$\le A^{-1/2}BA^{-1/2}-1_H-\ln\left(A^{-1/2}BA^{-1/2}\right)$$
$$\le\frac{1}{2\min^{2}\{1,m\}}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2.$$

If we multiply this inequality both sides by $A^{1/2}>0$, then we get

$$\frac{1}{2\max^{2}\{1,M\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2} \tag{3.7}$$
$$\le B-A-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}$$
$$\le\frac{1}{2\min^{2}\{1,m\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2}.$$

Observe that

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2}$$
$$=A^{1/2}\left(A^{-1/2}BA^{-1}BA^{-1/2}-2A^{-1/2}BA^{-1/2}+1_H\right)A^{1/2}$$
$$=BA^{-1}B-2B+A$$

and by (3.7) we get

$$0\le\frac{1}{2\max^{2}\{1,M\}}\left(BA^{-1}B-2B+A\right)$$
$$\le B-A-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}$$
$$\le\frac{1}{2\min^{2}\{1,m\}}\left(BA^{-1}B-2B+A\right),$$

which gives, by multiplying both sides by $P^{1/2}$ and taking the trace that

$$0 \leq \frac{1}{2\max^2\{1,M\}}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2\operatorname{tr}(PB)+\operatorname{tr}(PA)\right) \tag{3.8}$$
$$\leq \operatorname{tr}(PB)-\operatorname{tr}(PA)-\operatorname{tr}\left[PA^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\right]$$
$$\leq \frac{1}{2\min^2\{1,m\}}\left(\operatorname{tr}\left(PBA^{-1}B\right)-2\operatorname{tr}(PB)+\operatorname{tr}(PA)\right)$$

for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

If we take the exponential in (3.8), then we get the desired result (3.5).

By (3.4) we obtain

$$\frac{1}{2\max^2\{1,M\}}(T-1_H)^2 \leq \ln T - 1_H + T^{-1}$$
$$\leq \frac{1}{2\min^2\{1,m\}}(T-1_H)^2,$$

which implies that

$$\frac{1}{2\max^2\{1,M\}}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2$$
$$\leq \ln\left(A^{-1/2}BA^{-1/2}\right)-1_H+A^{1/2}B^{-1}A^{1/2}$$
$$\leq \frac{1}{2\min^2\{1,m\}}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2.$$

If we multiply this inequality both sides by $A^{1/2}>0$, then we get

$$\frac{1}{2\max^2\{1,M\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2}$$
$$\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}-A+AB^{-1}A$$
$$\leq \frac{1}{2\min^2\{1,m\}}A^{1/2}\left(A^{-1/2}BA^{-1/2}-1_H\right)^2A^{1/2}.$$

By multiplying both sides by $P^{1/2}$ and taking the trace, we deduce (3.6). $\square$

Remark 2. *Assume that $0 < m1_H \leq B \leq M1_H$ for some constants M and m. Then by Theorem 5 we get*

$$1 \leq \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\operatorname{tr}\left(PB^2\right)-2\operatorname{tr}(PB)+1\right)\right] \tag{3.9}$$
$$\leq \frac{\exp\left(\operatorname{tr}(PB)-1\right)}{\eta_P(B)}$$
$$\leq \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\operatorname{tr}\left(PB^2\right)-2\operatorname{tr}(PB)+1\right)\right]$$

and

$$1 \le \exp\left[\frac{1}{2\max^2\{1,M\}}\left(\operatorname{tr}\left(PB^2\right)-2\operatorname{tr}(PB)+1\right)\right] \le \frac{\eta_P(B)}{\exp\left(1-\operatorname{tr}\left(PB^{-1}\right)\right)} \le \exp\left[\frac{1}{2\min^2\{1,m\}}\left(\operatorname{tr}\left(PB^2\right)-2\operatorname{tr}(PB)+1\right)\right] \tag{3.10}$$

for all $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$.

If $0 < n1_H \le A \le N1_H$ *for some constants* N *and* n, *then by Theorem 5 we also obtain*

$$1 \le \exp\left[\frac{1}{2}\min\left\{1,n^2\right\}\left(\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}(PA)-2\right)\right] \le \frac{\exp(1-\operatorname{tr}(PA))}{\Delta_P(A)} \le \exp\left[\frac{1}{2}\max\left\{1,N^2\right\}\left(\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}(PA)-2\right)\right] \tag{3.11}$$

and

$$1 \le \exp\left[\frac{1}{2}\min\left\{1,n^2\right\}\left(\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}(PA)-2\right)\right] \le \frac{\Delta_P(A)}{\exp\left(\operatorname{tr}(PA)-\operatorname{tr}\left(PA^2\right)\right)} \le \exp\left[\frac{1}{2}\max\left\{1,N^2\right\}\left(\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}(PA)-2\right)\right] \tag{3.12}$$

for all $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$.

Observe also that for $u \in [k,K] \subset (0,\infty)$ we have

$$1-\frac{\min\{1,u\}}{\max\{1,u\}} \ge 1-\frac{\min\{1,K\}}{\max\{1,k\}} \ge 0$$

and

$$0 \le \frac{\max\{1,u\}}{\min\{1,u\}}-1 \le \frac{\max\{1,K\}}{\min\{1,k\}}-1.$$

Now, by (3.1) and (3.2) we get the *global bounds*

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \le u-1-\ln u \le \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{3.13}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \le \ln u-\frac{u-1}{u} \le \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{3.14}$$

for all $u \in [k,K] \subset (0,\infty)$.

Theorem 6. *Assume that* $0 < mA \leq B \leq MA$ *for some constants* M *and* m. *Then*

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2 \operatorname{tr}(PA)\right] \leq \frac{\exp(\operatorname{tr}(PB) - \operatorname{tr}(PA))}{D_P(A|B)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2 \operatorname{tr}(PA)\right] \tag{3.15}$$

and

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2 \operatorname{tr}(PA)\right] \leq \frac{D_P(A|B)}{\exp(\operatorname{tr}(PA) - \operatorname{tr}(PAB^{-1}A))} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2 \operatorname{tr}(PA)\right] \tag{3.16}$$

for all $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$.

Proof. From (3.13) we have for the selfadjoint operator T with $0 < m1_H \leq T \leq M1_H$ that

$$\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2 \leq T - 1_H - \ln T \leq \frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2.$$

If we write this inequality for $T = A^{-1/2}BA^{-1/2}$, then we get

$$\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2 \leq A^{-1/2}BA^{-1/2} - 1_H - \ln\left(A^{-1/2}BA^{-1/2}\right) \leq \frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2.$$

If we multiply this inequality both sides by $A^{1/2} > 0$, then we get

$$\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2 A \leq B - A - A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \leq \frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2 A.$$

By employing now a similar argument to the one in the proof of Theorem 5 we derive (3.15).

The inequality (3.16) follows in a similar way from (3.14). □

Remark 3. *Assume that* $0 < m1_H \leq B \leq M1_H$ *for some constants* M *and* m. *Then by Theorem 6 we get*

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right] \leq \frac{\exp(\operatorname{tr}(PB) - 1)}{\eta_P(B)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right], \tag{3.17}$$

and

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right] \leq \frac{\eta_P(B)}{\exp(1 - \operatorname{tr}(PB^{-1}))} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right] \tag{3.18}$$

for all $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$.

If $0 < n1_H \leq A \leq N1_H$ *for some constants* N *and* n, *then by Theorem 5 we also obtain*

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, N\}}{\max\{1, n\}}\right)^2 \operatorname{tr}(PA)\right] \leq \frac{\exp(1 - \operatorname{tr}(PA))}{\Delta_P(A)} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, N\}}{\min\{1, n\}} - 1\right)^2 \operatorname{tr}(PA)\right] \tag{3.19}$$

and

$$1 \leq \exp\left[\frac{1}{2}\left(1 - \frac{\min\{1, N\}}{\max\{1, n\}}\right)^2 \operatorname{tr}(PA)\right] \leq \frac{\Delta_P(A)}{\exp(\operatorname{tr}(PA) - \operatorname{tr}(PA^2))} \leq \exp\left[\frac{1}{2}\left(\frac{\max\{1, N\}}{\min\{1, n\}} - 1\right)^2 \operatorname{tr}(PA)\right] \tag{3.20}$$

for all $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$.

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