UPPER AND LOWER BOUNDS FOR RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES IN TERMS OF KANTOROVICH CONSTANT

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

Assume that A, $B > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. In this paper we show among others that, If A, B are operators satisfying the condition $0 < mA \leq B \leq MA$, then

$$\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M^2}\operatorname{tr}\left[P(MA-B)A^{-1}(B-mA)\right]\right] \\
&\leq \frac{D_P(A|B)}{m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}} \\
&\leq \exp\left[\frac{1}{2m^2}\operatorname{tr}\left[P(MA-B)A^{-1}(B-mA)\right]\right] \\
&\leq \exp\left[\frac{\operatorname{tr}(PA)}{2m^2}\left(M-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}-m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\operatorname{tr}(PA)\right].
\end{aligned}$$

1. Introduction

In 1952, in the paper [12], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive operators, Trace class operators, Determinants, Inequalities.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [18], [19], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [21].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \ \text{and} \ \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \ \mid B \in \mathcal{B}_2(H), \ \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)};\tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|;\tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P\ge0$ and $P\in\mathcal{B}_1(H)$, then for all $T\in\mathcal{B}(H)$, PT, $TP\in\mathcal{B}_1(H)$ and $\operatorname{tr}(PT)=\operatorname{tr}(TP)$. Also, since $P^{1/2}\in\mathcal{B}_2(H)$, $TP^{1/2}\in\mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2}=TP\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)=\operatorname{tr}(TP)$. Therefore, if $P\ge0$ and $P\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(PT)=\operatorname{tr}(TP)=\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T\in\mathcal{B}(H)$.

If $T\ge0$, then $P^{1/2}TP^{1/2}\ge0$, which implies that $\operatorname{tr}(PT)\ge0$ that shows that the functional $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n\to T$ for $n\to\infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty}\operatorname{tr}(PT_n)=\operatorname{tr}(PT)$, namely $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [1]-[9] and the references therein.

Now, for a given $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A)=\exp\operatorname{tr}((\ln A)P)=\exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right).\tag{1.12}$$

Assume that $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. We observe that we have the following elementary properties [10]:

(i) *continuity*: the map $A\to\Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t)=\Delta_P(A)^t$ for all $t>0$;
(iii) *homogeneity*: $\Delta_P(tA)=t\Delta_P(A)$ and $\Delta_P(t1_H)=t$ for all $t>0$;
(iv) *monotonicity*: $0<A\le B$ implies $\Delta_P(A)\le\Delta_P(B)$.

In [10], we presented some fundamental properties of this determinant. Among others we showed that

$$1\le\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\le\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]$$

and

$$1\le\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\le\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right],$$

for $A>0$ and $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

For the entropy function $\eta(t)=-t\ln t$, $t>0$, the operator entropy has the following expression:

$$\eta(A)=-A\ln A$$

for positive A.

Now, for a given $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *entropic P-determinant* of the positive invertible operator A by [11]

$$\eta_P(A):=\exp\left[-\operatorname{tr}(PA\ln A)\right]=\exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\}=\exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right) \\
&= \exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t + \ln A\right)\right]\right\}\right) = \exp\left(-\operatorname{tr}\left\{P\left(tA\ln t + tA\ln A\right)\right\}\right) \\
&= \exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right) \\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(1_H) = 1 \text{ and } \eta_P(t1_H) = t^{-t} \tag{1.14}$$

for $t > 0$.

Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, $B > 0$, then we have the Ky Fan type inequality [11]

$$\eta_P((1-t)A + tB) \geq \left[\eta_P(A)\right]^{1-t}\left[\eta_P(B)\right]^t \tag{1.15}$$

for all $t \in [0,1]$.

Also we have the inequalities [11]:

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)} \leq \frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}} \leq 1$$

and if there exists the constants $0 < m < M$ such that $m1_H \leq A \leq M1_H$, then [11]

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)} \leq \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)} \\
&\leq \frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}} \leq 1.
\end{aligned}$$

Kamei and Fujii [16], [17] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [25]. For various results on relative operator entropy see [13]-[26] and the references therein.

Definition 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}
D_P(A|B) &:= \exp\{\operatorname{tr}[PS(A|B)]\} \\
&= \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.
\end{aligned}$$

We observe that for $A>0$,

$$D_P(A|1_H) := \exp\{\operatorname{tr}[PS(A|1_H)]\} = \exp\{\operatorname{tr}(-PA\ln A)\} = \eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B>0$,

$$D_P(1_H|B) := \exp\{\operatorname{tr}[PS(1_H|B)]\} = \exp\{\operatorname{tr}(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

In this paper we show among others that, if A, B are operators satisfying the condition $0<mA\le B\le MA$, then

$$\begin{aligned}
1 &\le \exp\left[\frac{1}{2M^2}\operatorname{tr}\left[P(MA-B)A^{-1}(B-mA)\right]\right] \\
&\le \frac{D_P(A|B)}{m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}} \\
&\le \exp\left[\frac{1}{2m^2}\operatorname{tr}\left[P(MA-B)A^{-1}(B-mA)\right]\right] \\
&\le \exp\left[\frac{\operatorname{tr}(PA)}{2m^2}\left(M-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}-m\right)\right] \\
&\le \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\operatorname{tr}(PA)\right].
\end{aligned}$$

2. Main Results

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{2.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu}\le\right)K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \le (1-\nu)a+\nu b \le K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{2.2}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (2.2) was obtained by Zou et al. in [30] while the second by Liao et al. [29].

We start with the following main result:

Theorem 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, B are operators satisfying the condition $0 < mA \leq B \leq MA$, then*

$$
\begin{aligned}
1 &\leq K\left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(PA)-\frac{1}{M-m}\operatorname{tr}\left(PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right)} \\
&\leq \frac{D_P(A|B)}{m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}} \\
&\leq K\left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right)} \\
&\leq K\left(\frac{M}{m}\right)^{\operatorname{tr}(PA)}.
\end{aligned}
\tag{2.3}
$$

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0,1]$. Then

$$
\begin{aligned}
\min\{1-\nu,\nu\} &= \frac{1}{2}-\left|\nu-\frac{1}{2}\right| = \frac{1}{2}-\left|\frac{t-m}{M-m}-\frac{1}{2}\right| \\
&= \frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|,
\end{aligned}
$$

$$
\begin{aligned}
\max\{1-\nu,\nu\} &= \frac{1}{2}+\left|\nu-\frac{1}{2}\right| = \frac{1}{2}+\left|\frac{t-m}{M-m}-\frac{1}{2}\right| \\
&= \frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|,
\end{aligned}
$$

$$
(1-\nu)m+\nu M = \frac{M-t}{M-m}m+\frac{t-m}{M-m}M = t
$$

and

$$
m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.
$$

By using (2.2) we get

$$
\begin{aligned}
m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} &\leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \\
&\leq t \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}
\end{aligned}
\tag{2.4}
$$

for $t \in [m, M]$.

By taking the log in (2.4) we get

$$\begin{aligned}(2.5)\qquad & \frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\ & \leq \left[\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\ & +\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\ & \leq \ln t\leq \left[\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\ & +\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\ & \leq \ln K\left(\frac{M}{m}\right)+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M\end{aligned}$$

for $t\in[m,M]$.

If $0<m1_H\leq T\leq M1_H$, then by using the continuous functional calculus for selfadjoint operators we get from (2.5) that

$$\begin{aligned}(2.6)\qquad & \ln m\frac{M1_H-T}{M-m}+\ln M\frac{T-m1_H}{M-m} \\ & \leq \left[\frac{1}{2}1_H-\frac{1}{M-m}\left|T-\frac{1}{2}(m+M)1_H\right|\right]\ln K\left(\frac{M}{m}\right) \\ & +\ln m\frac{M1_H-T}{M-m}+\ln M\frac{T-m1_H}{M-m} \\ & \leq \ln T\leq \left[\frac{1}{2}1_H+\frac{1}{M-m}\left|T-\frac{1}{2}(m+M)1_H\right|\right]\ln K\left(\frac{M}{m}\right) \\ & +\ln m\frac{M1_H-T}{M-m}+\ln M\frac{T-m1_H}{M-m} \\ & \leq \ln K\left(\frac{M}{m}\right)1_H+\ln m\frac{M1_H-T}{M-m}+\ln M\frac{T-m1_H}{M-m}.\end{aligned}$$

Since $0<mA\leq B\leq MA$, hence by multiplying both sides by $A^{-1/2}>0$, we get $0<m1_H\leq A^{-1/2}BA^{-1/2}\leq M1_H$. Now, if we take $T=A^{-1/2}BA^{-1/2}$ in (2.6), then we get

$$\begin{aligned} & \ln m\frac{M1_H-A^{-1/2}BA^{-1/2}}{M-m}+\ln M\frac{A^{-1/2}BA^{-1/2}-m1_H}{M-m} \\ & \leq \left[\frac{1}{2}1_H-\frac{1}{M-m}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|\right]\ln K\left(\frac{M}{m}\right) \\ & +\ln m\frac{M1_H-A^{-1/2}BA^{-1/2}}{M-m}+\ln M\frac{A^{-1/2}BA^{-1/2}-m1_H}{M-m}\end{aligned}$$

$$\leq \ln\left(A^{-1/2}BA^{-1/2}\right)$$
$$\leq \left[\frac{1}{2}1_H + \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|\right]\ln K\left(\frac{M}{m}\right)$$
$$+\ln m\frac{M1_H - A^{-1/2}BA^{-1/2}}{M-m} + \ln M\frac{A^{-1/2}BA^{-1/2} - m1_H}{M-m}$$
$$\leq \ln K\left(\frac{M}{m}\right)1_H$$
$$+\ln m\frac{M1_H - A^{-1/2}BA^{-1/2}}{M-m} + \ln M\frac{A^{-1/2}BA^{-1/2} - m1_H}{M-m}.$$

If we multiply both sides by $A^{1/2} > 0$, then we get

$$\ln m\frac{MA-B}{M-m} + \ln M\frac{B-mA}{M-m}$$
$$\leq \left[\frac{1}{2}A - \frac{1}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}\right]\ln K\left(\frac{M}{m}\right)$$
$$+\ln m\frac{MA-B}{M-m} + \ln M\frac{B-mA}{M-m}$$
$$\leq A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$$
$$\leq \left[\frac{1}{2}A + \frac{1}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}\right]\ln K\left(\frac{M}{m}\right)$$
$$+\ln m\frac{MA-B}{M-m} + \ln M\frac{B-mA}{M-m}$$
$$\leq \ln K\left(\frac{M}{m}\right)A$$
$$+\ln m\frac{MA-B}{M-m} + \ln M\frac{B-mA}{M-m}.$$

Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If we multiply both sides by $P^{1/2} \geq 0$, take the trace and uses its properties, then we get

$$\ln m\frac{M\operatorname{tr}(PA) - \operatorname{tr}(PB)}{M-m} + \ln M\frac{\operatorname{tr}(PB) - m\operatorname{tr}(PA)}{M-m}$$
$$\leq \left[\frac{1}{2}\operatorname{tr}(PA) - \frac{1}{M-m}\operatorname{tr}\left(PA^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{1}{2}(m+M)1_H\right|A^{1/2}\right)\right]$$
$$\times\ln K\left(\frac{M}{m}\right) + \ln m\frac{M\operatorname{tr}(PA) - \operatorname{tr}(PB)}{M-m} + \ln M\frac{\operatorname{tr}(PB) - m\operatorname{tr}(PA)}{M-m}$$

$$\le \operatorname{tr}\left(PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\right)$$
$$\le \left[\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right]$$
$$\times \ln K\left(\frac{M}{m}\right)\operatorname{tr}(PA)+\ln m\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}+\ln M\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}$$
$$\le \ln K\left(\frac{M}{m}\right)\operatorname{tr}(PA)$$
$$+\ln m\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}+\ln M\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m},$$

namely

$$\ln\left(m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}\right) \tag{2.7}$$
$$\le \ln K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA)-\frac{1}{M-m}\operatorname{tr}\left(PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right)\right]}$$
$$+\ln\left(m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}\right)$$
$$\le \operatorname{tr}\left(PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\right)$$
$$\le \ln K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right]}$$
$$+\ln\left(m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}\right)$$
$$\le \ln K\left(\frac{M}{m}\right)^{\operatorname{tr}(PA)}$$
$$+\ln\left(m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}\right).$$

By taking the exponential in (2.7), we derive

$$m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}$$
$$\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA)-\frac{1}{M-m}\operatorname{tr}\left(PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right)\right]}$$
$$\times m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}$$
$$\le \exp\left[\operatorname{tr}\left(PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\right)\right]$$
$$\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}(m+M)1_H\right|A^{1/2}\right)\right]}$$
$$\times\left(m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}\right)$$
$$\le K\left(\frac{M}{m}\right)^{\operatorname{tr}(PA)}\left(m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}\right),$$

which is equivalent to (2.3). □

Corollary 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If B is an operator satisfying the condition $0 < m1_H \leq B \leq M1_H$ then*

$$1 \leq K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\operatorname{tr}\left(P\left|B-\frac{1}{2}(m+M)1_H\right|\right)} \tag{2.8}$$
$$\leq \frac{\Delta_P(B)}{m^{\frac{M-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m}{M-m}}}$$
$$\leq K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|B-\frac{1}{2}(m+M)1_H\right|\right)} \leq K\left(\frac{M}{m}\right).$$

The proof follows by (2.3) for $A = 1_H$.

Corollary 2. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A is an operator satisfying the condition $0 < n1_H \leq A \leq N1_H$, then*

$$1 \leq K\left(\frac{N}{n}\right)^{\frac{1}{2}\operatorname{tr}(PA)-\frac{1}{n^{-1}-N^{-1}}\operatorname{tr}\left(PA^{1/2}\left|A^{-1}-\frac{1}{2}\left(n^{-1}+N^{-1}\right)1_H\right|A^{1/2}\right)} \tag{2.9}$$
$$\leq \frac{\eta_P(A)}{M^{\frac{-N[\operatorname{tr}(PA)-n]}{N-n}}n^{\frac{-n[N-\operatorname{tr}(PA)]}{N-n}}}$$
$$\leq K\left(\frac{N}{n}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{n^{-1}-N^{-1}}\operatorname{tr}\left(PA^{1/2}\left|A^{-1}-\frac{1}{2}\left(n^{-1}+N^{-1}\right)1_H\right|A^{1/2}\right)}$$
$$\leq K\left(\frac{N}{n}\right)^{\operatorname{tr}(PA)}.$$

Proof. Since $0 < n1_H \leq A \leq N1_H$, hence $\frac{1}{N}A \leq 1_H \leq \frac{1}{n}A$ and by taking $B = 1_H$, $m = \frac{1}{N}$ and $M = \frac{1}{n}$ in (2.3), we get (2.9). □

We also have:

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, B are operators satisfying the condition $0 < mA \leq B \leq MA$, then*

$$1 \leq \frac{D_P(A|B)}{m^{\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}}} \tag{2.10}$$
$$\leq \exp\left[\frac{\operatorname{tr}\left[P(MA-B)A^{-1}(B-mA)\right]}{Mm}\right]$$
$$\leq \exp\left[\frac{\operatorname{tr}(PA)}{Mm}\left(M-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}-m\right)\right]$$
$$\leq \exp\left[\frac{1}{4mM}(M-m)^2\operatorname{tr}(PA)\right].$$

Proof. In [6] we obtained the following reverses of Young's inequality:

$$1 \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right],$$

where a, $b > 0$, $\nu \in [0,1]$.

This is equivalent, by taking the logarithm, with

$$0 \leq \ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b \leq \nu(1-\nu)\frac{(b-a)^2}{ba}$$

where $a,\ b>0,\ \nu\in[0,1]$.

If we take $a=m,\ b=M,\ t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$0\leq\ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M\leq\frac{(M-t)(t-m)}{(M-m)^2}\frac{(M-m)^2}{Mm}$$
$$=\frac{(M-t)(t-m)}{Mm}\leq\frac{1}{4mM}(M-m)^2$$

Using the continuous functional calculus for selfadjoint operators, we have

$$0\leq\ln T-\frac{M1_H-T}{M-m}\ln m-\frac{T-m1_H}{M-m}\ln M\leq\frac{(M1_H-T)(T-m1_H)}{Mm}\tag{2.11}$$
$$\leq\frac{1}{4mM}(M-m)^2 1_H,$$

provided that $0<m1_H\leq T\leq M1_H$.

Now, if we take $T=A^{-1/2}BA^{-1/2}$ in (2.11), then we get

$$0\leq\ln\left(A^{-1/2}BA^{-1/2}\right)$$
$$-\ln m\frac{M1_H-A^{-1/2}BA^{-1/2}}{M-m}-\ln M\frac{A^{-1/2}BA^{-1/2}-m1_H}{M-m}$$
$$\leq\frac{\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)}{Mm}$$

and if we multiply both sides by $A^{1/2}>0$, then

$$0\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}-\frac{MA-B}{M-m}\ln m-\frac{B-mA}{M-m}\ln M$$
$$\leq\frac{A^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)A^{1/2}}{Mm}$$

Now, if we multiply both sides by $P^{1/2}\geq 0$ and take the trace, then we get

$$0\leq\operatorname{tr}\left(PA^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\right)$$
$$-\frac{M\operatorname{tr}(PA)-\operatorname{tr}(PB)}{M-m}\ln m-\frac{\operatorname{tr}(PB)-m\operatorname{tr}(PA)}{M-m}\ln M$$
$$\leq\frac{\operatorname{tr}\left[PA^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)A^{1/2}\right]}{Mm}$$
$$=\frac{\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)\right]}{Mm}.$$

The function $g(t)=(M-t)(t-m)$ is concave on $[m,M]$ and by Jensen's inequality for trace

$$\operatorname{tr}(Qg(C))\leq g(\operatorname{tr}(QC)),$$

where $Q \geq 0$ with $\operatorname{tr} Q = 1$ and $0 < m1_H \leq C \leq M1_H$, we conclude that

$$\begin{aligned}
&\frac{\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)\right]}{\operatorname{tr}\left(A^{1/2}PA^{1/2}\right)} \\
&\leq \left(M - \frac{\operatorname{tr}\left(A^{1/2}PA^{1/2}A^{-1/2}BA^{-1/2}\right)}{\operatorname{tr}\left(A^{1/2}PA^{1/2}\right)}\right) \\
&\times \left(\frac{\operatorname{tr}\left(A^{1/2}PA^{1/2}A^{-1/2}BA^{-1/2}\right)}{\operatorname{tr}\left(A^{1/2}PA^{1/2}\right)} - m\right),
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)\right]}{\operatorname{tr}(PA)} \\
&\leq \left(M - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)} - m\right) \leq \frac{1}{4}(M-m)^2.
\end{aligned}$$

Therefore we have the chain of inequalities

$$\begin{aligned}
0 &\leq \operatorname{tr}\left(PA^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\right) \\
&- \ln\left(m^{\frac{M \operatorname{tr}(PA) - \operatorname{tr}(PB)}{M-m}} M^{\frac{\operatorname{tr}(PB) - m \operatorname{tr}(PA)}{M-m}}\right) \\
&\leq \frac{\operatorname{tr}\left[PA^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)A^{1/2}\right]}{Mm} \\
&\leq \frac{\operatorname{tr}(PA)}{Mm}\left(M - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)} - m\right) \\
&\leq \frac{1}{4mM}(M-m)^2 \operatorname{tr}(PA)
\end{aligned}$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

By taking the exponential, we derive the desired result (2.10). □

Corollary 3. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If B is an operator satisfying the condition $0 < m1_H \leq B \leq M1_H$ then*

$$\begin{aligned}
1 &\leq \frac{\Delta_P(B)}{m^{\frac{M - \operatorname{tr}(PB)}{M-m}} M^{\frac{\operatorname{tr}(PB) - m}{M-m}}} \\
&\leq \exp\left[\frac{\operatorname{tr}\left[P\left(M1_H - B\right)\left(B - m1_H\right)\right]}{Mm}\right] \\
&\leq \exp\left[\frac{1}{Mm}\left(M - \operatorname{tr}(PB)\right)\left(\operatorname{tr}(PB) - m\right)\right] \\
&\leq \exp\left[\frac{1}{4mM}(M-m)^2\right].
\end{aligned} \tag{2.12}$$

It follows by (2.10) on taking $A = 1_H$.

Corollary 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A is an operator satisfying the condition $0 < n1_H \leq A \leq N1_H$, then*

$$
\begin{aligned}
(2.13) \qquad 1 &\leq \frac{\eta_P(A)}{M^{\frac{-N[\operatorname{tr}(PA)-n]}{N-n}} n^{\frac{-n[N-\operatorname{tr}(PA)]}{N-n}}} \\
&\leq \exp\left[\frac{\operatorname{tr}\left[PA^{1/2}\left(n^{-1}1_H - A^{-1}\right)\left(A^{-1} - N^{-1}1_H\right)A^{1/2}\right]}{M^{-1}n^{-1}}\right] \\
&\leq \exp\left[\frac{\operatorname{tr}(PA)}{Mm}\left(n^{-1} - [\operatorname{tr}(PA)]^{-1}\right)\left([\operatorname{tr}(PA)]^{-1} - N^{-1}\right)\right] \\
&\leq \exp\left[\frac{1}{4nN}(N-n)^2 \operatorname{tr}(PA)\right].
\end{aligned}
$$

It follows by (2.10) on taking $B = 1_H$.

3. More Results

In [7] we obtained the following refinement and reverse of Young's inequality:

$$
\begin{aligned}
(3.1) \qquad & \exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\
&\leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right],
\end{aligned}
$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Theorem 6. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, B are operators satisfying the condition $0 < mA \leq B \leq MA$, then*

$$
\begin{aligned}
(3.2) \qquad 1 &\leq \exp\left[\frac{1}{2M^2}\operatorname{tr}\left[P(MA - B)A^{-1}(B - mA)\right]\right] \\
&\leq \frac{D_P(A|B)}{m^{\frac{M\operatorname{tr}(PA) - \operatorname{tr}(PB)}{M-m}} M^{\frac{\operatorname{tr}(PB) - m\operatorname{tr}(PA)}{M-m}}} \\
&\leq \exp\left[\frac{1}{2m^2}\operatorname{tr}\left[P(MA - B)A^{-1}(B - mA)\right]\right] \\
&\leq \exp\left[\frac{\operatorname{tr}(PA)}{2m^2}\left(M - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)} - m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2 \operatorname{tr}(PA)\right].
\end{aligned}
$$

Proof. From (3.1) we have

$$
\begin{aligned}
& \exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{m}{M}\right)^2\right] \\
&\leq \frac{(1-\nu)m + \nu M}{m^{1-\nu}M^{\nu}} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m} - 1\right)^2\right],
\end{aligned}
$$

for $\nu \in [0,1]$.

By taking the logarithm, we obtain

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2 \\
&\leq \ln((1-\nu)m+\nu M)-(1-\nu)\ln m-\nu\ln M \\
&\leq \frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2,
\end{aligned} \tag{3.3}$$

for $\nu \in [0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$\begin{aligned}
\frac{(M-t)(t-m)}{2M^2} &\leq \ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M \\
&\leq \frac{(M-t)(t-m)}{2m^2}
\end{aligned}$$

for $t\in[m,M]$.

If we use the continuous functional calculus for the operator $0<m1_H\leq T\leq M1_H$, then we get

$$\begin{aligned}
&\frac{1}{2M^2}(M1_H-T)(T-m1_H) \\
&\leq \ln T-\ln m\frac{M1_H-T}{M-m}-\ln M\frac{T-m1_H}{M-m} \\
&\leq \frac{1}{2m^2}(M1_H-t)(t-m1_H).
\end{aligned} \tag{3.4}$$

Now, if we take $T=A^{-1/2}BA^{-1/2}$ in (3.4), then we get

$$\begin{aligned}
&\frac{1}{2M^2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right) \\
&\leq \ln\left(A^{-1/2}BA^{-1/2}\right) \\
&-\ln m\frac{M1_H-A^{-1/2}BA^{-1/2}}{M-m}-\ln M\frac{A^{-1/2}BA^{-1/2}-m1_H}{M-m} \\
&\leq \frac{1}{2m^2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right).
\end{aligned}$$

If we multiply both sides by $A^{1/2}>0$, then we get

$$\begin{aligned}
&\frac{1}{2M^2}A^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)A^{1/2} \\
&\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\
&-\ln m\frac{MA-B}{M-m}-\ln M\frac{B-mA}{M-m} \\
&\leq \frac{1}{2m^2}A^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)A^{1/2}.
\end{aligned}$$

Now, if we multiply both sides by $P^{1/2} \geq 0$ and take the trace, then we get

$$\begin{aligned} &\frac{1}{2M^2}\operatorname{tr}\left[PA^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)A^{1/2}\right] \\ &\leq \operatorname{tr}\left(PA^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\right) \\ &- \frac{M\operatorname{tr}(PA) - \operatorname{tr}(PB)}{M-m}\ln m - \frac{\operatorname{tr}(PB) - m\operatorname{tr}(PA)}{M-m}\ln M \\ &\leq \frac{1}{2m^2}\operatorname{tr}\left[PA^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - m1_H\right)A^{1/2}\right]. \end{aligned}$$

Then rest follows from the proof of Theorem 5 and the details are omitted. □

Corollary 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If B is an operator satisfying the condition $0 < m1_H \leq B \leq M1_H$ then*

$$\begin{aligned} 1 &\leq \exp\left[\frac{1}{2M^2}\operatorname{tr}\left[P(M1_H - B)(B - m1_H)\right]\right] \\ &\leq \frac{\Delta_P(B)}{m^{\frac{M-\operatorname{tr}(PB)}{M-m}}M^{\frac{\operatorname{tr}(PB)-m}{M-m}}} \\ &\leq \exp\left[\frac{1}{2m^2}\operatorname{tr}\left[P(M1_H - B)(B - m1_H)\right]\right] \\ &\leq \exp\left[\frac{1}{2m^2}(M - \operatorname{tr}(PB))(\operatorname{tr}(PB) - m)\right] \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]. \end{aligned} \tag{3.5}$$

It follows by (3.2) on taking $A = 1_H$.

Corollary 6. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A is an operator satisfying the condition $0 < n1_H \leq A \leq N1_H$, then*

$$\begin{aligned} 1 &\leq \exp\left\{\frac{\operatorname{tr}(PA)}{2}n^2\left(n^{-1} - [\operatorname{tr}(PA)]^{-1}\right)\left([\operatorname{tr}(PA)]^{-1} - N^{-1}\right)\right\} \\ &\leq \frac{\eta_P(A)}{M^{\frac{-N[\operatorname{tr}(PA)-n]}{N-n}}n^{\frac{-n[N-\operatorname{tr}(PA)]}{N-n}}} \\ &\leq \exp\left[\frac{1}{2}N^2\operatorname{tr}\left[P\left(n^{-1}A - 1_H\right)A^{-1}\left(1_H - N^{-1}A\right)\right]\right] \\ &\leq \exp\left[\frac{\operatorname{tr}(PA)}{2}N^2\left(n^{-1} - [\operatorname{tr}(PA)]^{-1}\right)\left([\operatorname{tr}(PA)]^{-1} - N^{-1}\right)\right] \\ &\leq \exp\left[\frac{1}{8}\left(\frac{N}{n} - 1\right)^2\operatorname{tr}(PA)\right]. \end{aligned} \tag{3.6}$$

It follows by (3.2) on taking $B = 1_H$.

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