SOME BOUNDS FOR RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA OSTROWSKI'S INEQUALITY

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

In this paper we show, among others, that, if A satisfies the condition $0 < mA \leq B \leq MA$, where m, M are positive numbers, then

$$\begin{aligned}\left(\frac{m}{M}\right)^{\operatorname{tr}(PA)} &\leq \left(\frac{m}{M}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|\right)} \\ &\leq \frac{D_P(A|B)}{[I(m,M)]^{\operatorname{tr}(PA)}} \\ &\leq \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|\right)} \\ &\leq \left(\frac{M}{m}\right)^{\operatorname{tr}(PA)}.\end{aligned}$$

1. Introduction

In 1952, in the paper [16], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive operators, Trace class operators, Determinants, Inequalities.

RGMIA Res. Rep. Coll. **25** (2022), Art. 68, 16 pp. Received 01/07/22

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [23], [24], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [26].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT,\ TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \ \textit{and}\ \ |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}(P^{1/2}TP^{1/2}) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [4]-[11] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [12]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(t1_H) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In [12], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by [13]

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA\ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln(tA)\right]\right\}\right)\\
&= \exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t + \ln A\right)\right]\right\}\right) = \exp\left(-\operatorname{tr}\left\{P\left(tA\ln t + tA\ln A\right)\right\}\right)\\
&= \exp\left(-t\ln t\operatorname{tr}(PA)\right)\exp\left(-t\operatorname{tr}(PA\ln A)\right)\\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}(PA\ln A)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(1_H) = 1 \text{ and } \eta_P(t1_H) = t^{-t} \tag{1.14}$$

for $t > 0$.

Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, $B > 0$, then we have the Ky Fan type inequality [13]

$$\eta_P((1-t)A + tB) \geq \left[\eta_P(A)\right]^{1-t}\left[\eta_P(B)\right]^{t} \tag{1.15}$$

for all $t \in [0, 1]$.

Also we have the inequalities [13]:

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)} \leq \frac{\eta_P(A)}{\left[\operatorname{tr}(PA)\right]^{-\operatorname{tr}(PA)}} \leq 1$$

and if there exists the constants $0 < m < M$ such that $m1_H \leq A \leq M1_H$, then [13]

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)} \leq \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)}\\
&\leq \frac{\eta_P(A)}{\left[\operatorname{tr}(PA)\right]^{-\operatorname{tr}(PA)}} \leq 1.
\end{aligned}$$

Kamei and Fujii [21], [22] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [30]. For various results on relative operator entropy see [18]-[31] and the references therein.

Definition 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}
D_P(A|B) &:= \exp\{\operatorname{tr}\left[PS(A|B)\right]\}\\
&= \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.
\end{aligned}$$

We observe that for $A>0$,

$$D_P(A|1_H) := \exp\{\operatorname{tr}[PS(A|1_H)]\} = \exp\{\operatorname{tr}(-PA\ln A)\} = \eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic* P*-determinant* and for $B>0$,

$$D_P(1_H|B) := \exp\{\operatorname{tr}[PS(1_H|B)]\} = \exp\{\operatorname{tr}(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the P*-determinant.*

Motivated by the above results, in this paper we show among others that, if A satisfies the condition $0 < mA \le B \le MA$, where m, M are positive numbers, then

$$\begin{aligned}\left(\frac{m}{M}\right)^{\operatorname{tr}(PA)} &\le \left(\frac{m}{M}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|\right)}\\ &\le \frac{D_P(A|B)}{[I(m,M)]^{\operatorname{tr}(PA)}}\\ &\le \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|\right)} \le \left(\frac{M}{m}\right)^{\operatorname{tr}(PA)}.\end{aligned}$$

2. Main Results

Recall the *identric mean*

$$I(a,b) := \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b \ne a \\ a & \text{if} \quad b = a \end{cases} ;\ a,b>0.$$

It is easy to observe the connection between the integral mean of the logarithmic function and the logarithm of the identric mean,

$$\frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b)$$

for $a \ne b$ positive numbers.

Theorem 4. *Let $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that the operator A satisfies the condition $0 < mA \le B \le MA$, where m, M are positive numbers,*

then

$$\begin{aligned}
(2.1)\qquad & \exp\left\{-\frac{1}{2}\left(\frac{M}{m}-1\right)\operatorname{tr}(PA)\right\} \\
& \leq \exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\right. \\
& \left.\left.+\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\right]\right\} \\
& \leq \frac{D_P(A|B)}{[I(m,M)]^{\operatorname{tr}(PA)}} \\
& \leq \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\right. \\
& \left.\left.+\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\right]\right\} \\
& \leq \exp\left\{\frac{1}{2}\left(\frac{M}{m}-1\right)\operatorname{tr}(PA)\right\}.
\end{aligned}$$

Proof. We use Ostrowski's inequality [32]:

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{s\in(a,b)}|f'(s)|<\infty$, then

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{2.2}$$

for all $t\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (2.2) and observe that

$$\|f'\|_\infty = \sup_{t\in[a,b]} t^{-1} = \frac{1}{a},$$

then we get

$$|\ln t-\ln I(a,b)| \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right),$$

for all $t\in[a,b]$.

This inequality is equivalent to

$$\begin{aligned}
(2.3)\qquad & -\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right) \\
& \leq \ln t-\ln I(a,b) \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right),
\end{aligned}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.3) that

$$-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}1_H+\frac{1}{(M-m)^2}\left(T-\frac{m+M}{2}1_H\right)^2\right] \tag{2.4}$$
$$\leq \ln T-\ln I(m,M)\,1_H$$
$$\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}1_H+\frac{1}{(M-m)^2}\left(T-\frac{m+M}{2}1_H\right)^2\right].$$

Since $0<mA\leq B\leq MA$, then by multiplying both sides by $A^{-1/2}$ we get $0<m1_H\leq A^{-1/2}BA^{-1/2}\leq M1_H$ and by taking $T=A^{-1/2}BA^{-1/2}$ in (2.4), then we get

$$-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}1_H+\frac{1}{(M-m)^2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]$$
$$\leq \ln\left(A^{-1/2}BA^{-1/2}\right)-\ln I(m,M)\,1_H$$
$$\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}1_H+\frac{1}{(M-m)^2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right].$$

If we multiply both sides by $A^{1/2}$ we derive

$$-\left(\frac{M}{m}-1\right)$$
$$\times\left[\frac{1}{4}A+\frac{1}{(M-m)^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2A^{1/2}\right]$$
$$\leq \ln\left(A^{-1/2}BA^{-1/2}\right)-\ln I(m,M)\,A$$
$$\leq \left(\frac{M}{m}-1\right)$$
$$\times\left[\frac{1}{4}A+\frac{1}{(M-m)^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2A^{1/2}\right].$$

If we multiply both sides with $P^{1/2}$, then we get

$$-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}P^{1/2}AP^{1/2}\right.$$
$$\left.+\frac{1}{(M-m)^2}P^{1/2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2A^{1/2}P^{1/2}\right]$$
$$\leq P^{1/2}A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}P^{1/2}-\ln I(m,M)\,P^{1/2}AP^{1/2}$$
$$\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}P^{1/2}AP^{1/2}\right.$$
$$\left.+\frac{1}{(M-m)^2}P^{1/2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2A^{1/2}P^{1/2}\right].$$

Now, if we take the trace and use its properties, then we obtain

$$
\begin{aligned}
(2.5)\qquad & -\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\\
&\left.+\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\right]\\
&\leq \operatorname{tr}\left[A^{1/2}P^{1/2}A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]-\ln I(m,M)\operatorname{tr}(PA)\\
&\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\\
&\left.+\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\right].
\end{aligned}
$$

By taking the exponential in (2.5) we derive

$$
\begin{aligned}
&\exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\right.\\
&\left.\left.+\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\right]\right\}\\
&\leq \frac{\exp\operatorname{tr}\left[A^{1/2}P^{1/2}A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]}{[I(m,M)]^{\operatorname{tr}(PA)}}\\
&\leq \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\right.\\
&\left.\left.+\frac{1}{(M-m)^2}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\right]\right\}.
\end{aligned}
$$

Since

$$
\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\leq\frac{1}{4}(M-m)^2\operatorname{tr}(PA),
$$

hence

$$
\begin{aligned}
&\frac{1}{4}\operatorname{tr}(PA)+\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right]\\
&\leq\frac{1}{2}\operatorname{tr}(PA)
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\operatorname{tr}(PA)\leq&-\frac{1}{4}\operatorname{tr}(PA)\\
&-\frac{1}{(M-m)^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right)^2\right].
\end{aligned}
$$

These prove the desired result (2.1). □

Corollary 1. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. Assume that the operator B satisfies the condition $0<m1_H\leq B\leq M1_H$, where m,M are positive numbers,*

then

$$
\begin{aligned}
(2.6)\qquad & \exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right] \\
& \leq \exp\left\{-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(BA-\frac{m+M}{2}1_H\right)^2\right]\right]\right\} \\
& \leq \frac{\Delta_P(B)}{[I(m,M)]^{\operatorname{tr}(PA)}} \\
& \leq \exp\left\{\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(BA-\frac{m+M}{2}1_H\right)^2\right]\right]\right\} \\
& \leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right]
\end{aligned}
$$

The proof follows by (2.1) for $A = 1_H$.

Corollary 2. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that the operator A satisfies the condition $0 < n1_H \leq A \leq N1_H$, where n, N are positive numbers, then*

$$
\begin{aligned}
(2.7)\qquad & \exp\left[-\frac{1}{2}\left(\frac{N}{n}-1\right)\operatorname{tr}(PA)\right] \\
& \leq \exp\left\{-\left(\frac{N}{n}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\right. \\
& \left.\left.+\frac{1}{(n^{-1}-N^{-1})^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1}-\frac{n^{-1}+N^{-1}}{2}1_H\right)^2\right]\right]\right\} \\
& \leq \frac{\eta_P(A)}{[I(n^{-1},N^{-1})]^{\operatorname{tr}(PA)}} \\
& \leq \exp\left\{\left(\frac{N}{n}-1\right)\left[\frac{1}{4}\operatorname{tr}(PA)\right.\right. \\
& \left.\left.+\frac{1}{(n^{-1}-N^{-1})^2}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1}-\frac{n^{-1}+N^{-1}}{2}1_H\right)^2\right]\right]\right\}. \\
& \leq \exp\left[\frac{1}{2}\left(\frac{N}{n}-1\right)\operatorname{tr}(PA)\right].
\end{aligned}
$$

Since $0 < n1_H \leq A \leq N1_H$, hence $\frac{1}{N}A \leq 1_H \leq \frac{1}{n}A$ and by taking $B = 1_H$ in (2.1) we derive (2.7).

Theorem 5. *With the assumptions of Theorem 4, we have the inequalities*

$$\left(\frac{m}{M}\right)^{\operatorname{tr}(PA)} \leq \left(\frac{m}{M}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|\right)} \leq \frac{D_P(A|B)}{[I(m,M)]^{\operatorname{tr}(PA)}} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}1_H\right|\right)} \leq \left(\frac{M}{m}\right)^{\operatorname{tr}(PA)}. \tag{2.8}$$

Proof. In 1997, Dragomir and Wang proved the following Ostrowski type inequality [14]:

Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$, then

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \leq \left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}, \tag{2.9}$$

for all $t\in[a,b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],1} := \int_a^b |g(t)|\,dt.$$

The constant $\frac{1}{2}$ is best possible.

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (2.9) and observe that

$$\|f'\|_{[a,b],1} = \ln b - \ln a,$$

then by (2.9) we get

$$|\ln t - \ln I(a,b)| \leq \left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b - \ln a),$$

for all $t\in[a,b]$.

This inequality is equivalent to

$$-\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a) \leq \ln t - \ln I(a,b) \leq \left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a), \tag{2.10}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.3) that

$$-(\ln M-\ln m)\left[\frac{1}{2}1_H+\frac{1}{M-m}\left|T-\frac{m+M}{2}1_H\right|\right] \leq \ln T - \ln I(m,M)1_H \leq (\ln M-\ln m)\left[\frac{1}{2}1_H+\frac{1}{M-m}\left|T-\frac{m+M}{2}1_H\right|\right], \tag{2.11}$$

By taking $T = A^{-1/2}BA^{-1/2}$ in (2.11) we get

$$\begin{aligned}
&-(\ln M - \ln m)\left[\frac{1}{2}1_H + \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right] \\
&\leq \ln\left(A^{-1/2}BA^{-1/2}\right) - \ln I(m,M)\,1_H \\
&\leq (\ln M - \ln m)\left[\frac{1}{2}1_H + \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right].
\end{aligned}$$

Now, if we multiply both sides by $A^{1/2}$ and then by $P^{1/2}$ we get

$$\begin{aligned}
&-(\ln M - \ln m) \\
&\times\left[\frac{1}{2}P^{1/2}AP^{1/2} + \frac{1}{M-m}P^{1/2}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|A^{1/2}P^{1/2}\right] \\
&\leq P^{1/2}A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}P^{1/2} - \ln I(m,M)\,P^{1/2}AP^{1/2} \\
&\leq (\ln M - \ln m) \\
&\times\left[\frac{1}{2}AP^{1/2}AP^{1/2} + \frac{1}{M-m}P^{1/2}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|A^{1/2}P^{1/2}\right].
\end{aligned}$$

This implies the trace inequalities

$$\begin{aligned}
&-(\ln M - \ln m) \\
&\times\left[\frac{1}{2}\operatorname{tr}(PA) + \frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right)\right] \\
&\leq \operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right] - \ln I(m,M)\operatorname{tr}(PA) \\
&\leq (\ln M - \ln m) \\
&\times\left[\frac{1}{2}\operatorname{tr}(PA) + \frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right)\right].
\end{aligned}$$

By taking the exponential, we derive

$$\begin{aligned}
&\left(\frac{m}{M}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA) + \frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right)\right]} \\
&\leq \frac{\exp\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]}{[I(m,M)]^{\operatorname{tr}(PA)}} \\
&\leq \left(\frac{M}{m}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA) + \frac{1}{M-m}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right)\right]}.
\end{aligned}$$

Also, we notice that

$$\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}1_H\right|\right) \leq \frac{1}{2}(M-m)\operatorname{tr}(PA)$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. These prove the desired result (2.8). $\square$

Corollary 3. *With the assumptions of Corollary 1, we have*

$$\frac{m}{M} \le \left(\frac{m}{M}\right)^{\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|B-\frac{m+M}{2}1_H\right|\right)} \le \frac{\Delta_P(B)}{I(m,M)} \le \left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|B-\frac{m+M}{2}1_H\right|\right)} \le \frac{M}{m}. \tag{2.12}$$

Also we have:

Corollary 4. *With the assumptions of Corollary 2, we have*

$$\begin{aligned} &\left(\frac{n}{N}\right)^{\operatorname{tr}(PA)} \\ &\le \left(\frac{n}{N}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{\left(n^{-1}-N^{-1}\right)^2}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1}-\frac{n^{-1}+N^{-1}}{2}1_H\right|\right)\right]} \\ &\le \frac{\eta_P(A)}{\left[I\left(n^{-1},N^{-1}\right)\right]^{\operatorname{tr}(PA)}} \\ &\le \left(\frac{N}{n}\right)^{\left[\frac{1}{2}\operatorname{tr}(PA)+\frac{1}{\left(n^{-1}-N^{-1}\right)^2}\operatorname{tr}\left(A^{1/2}PA^{1/2}\left|A^{-1}-\frac{n^{-1}+N^{-1}}{2}1_H\right|\right)\right]} \\ &\le \left(\frac{N}{n}\right)^{\operatorname{tr}(PA)}. \end{aligned} \tag{2.13}$$

3. Related Results

The following results of Ostrowski type holds, see [1]:

Lemma 1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $t\in[a,b]$ one has the inequality*

$$\begin{aligned} &\frac{1}{2}\left[(b-t)^2 f'_+(t)-(t-a)^2 f'_-(t)\right] \\ &\le \int_a^b f(s)\,ds-(b-a)f(t) \\ &\le \frac{1}{2}\left[(b-t)^2 f'_-(b)-(t-a)^2 f'_+(a)\right]. \end{aligned} \tag{3.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $t=a$ or $t=b$.

If the function is differentiable in $t\in(a,b)$ then the first inequality in (3.1) becomes

$$\left(\frac{a+b}{2}-t\right)f'(t)\le\frac{1}{b-a}\int_a^b f(s)\,ds-f(t). \tag{3.2}$$

We also have:

Theorem 6. *Let $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. Assume that the operator A satisfies the condition $0<mA\le B\le MA$, where m,M are positive numbers,*

then

$$\exp\left[\operatorname{tr}(PA)-\frac{m+M}{2}\operatorname{tr}\left(APAB^{-1}\right)\right] \tag{3.3}$$
$$\leq \frac{D_P(A|B)}{[I(m,M)]^{\operatorname{tr}(PA)}}$$
$$\leq \exp\left\{\frac{1}{m}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-m1_H\right)^2\right]\right.$$
$$\left.-\frac{1}{M}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)^2\right]\right\}.$$

Proof. Writing (3.1) and (3.2) for the convex function $f(t)=-\ln t$, then we get

$$1-\frac{a+b}{2}t^{-1}\leq \ln t-\ln I(a,b)\leq \frac{(t-a)^2}{a}-\frac{(b-t)^2}{b},$$

for all $t\in[a,b]\subset(0,\infty)$.

If we use the functional calculus, we derive the operator inequality

$$1-\frac{m+M}{2}T^{-1}\leq \ln T-\ln I(m,M)\leq \frac{(T-m1_H)^2}{m}-\frac{(M1_H-T)^2}{M},$$

provided that $0<m1_H\leq T\leq M1_H$.

This gives for $T=A^{-1/2}BA^{-1/2}$ that

$$1-\frac{m+M}{2}A^{1/2}B^{-1}A^{1/2}$$
$$\leq \ln\left(A^{-1/2}BA^{-1/2}\right)-\ln I(m,M)$$
$$\leq \frac{\left(A^{-1/2}BA^{-1/2}-m1_H\right)^2}{m}-\frac{\left(M1_H-A^{-1/2}BA^{-1/2}\right)^2}{M}.$$

If we multiply both sides by $A^{1/2}$ and then by $P^{1/2}$ then we get

$$P^{1/2}AP^{1/2}-\frac{m+M}{2}P^{1/2}AB^{-1}AP^{1/2}$$
$$\leq P^{1/2}A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}P^{1/2}-\ln I(m,M)P^{1/2}AP^{1/2}$$
$$\leq \frac{1}{m}P^{1/2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-m1_H\right)^2A^{1/2}P^{1/2}$$
$$-\frac{1}{M}P^{1/2}A^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)^2A^{1/2}P^{1/2}.$$

If we take the trace, then we get

$$\operatorname{tr}(PA)-\frac{m+M}{2}\operatorname{tr}\left(APAB^{-1}\right)$$
$$\leq \operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]-\ln I(m,M)\operatorname{tr}(PA)$$
$$\leq \frac{1}{m}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1/2}BA^{-1/2}-m1_H\right)^2\right]$$
$$-\frac{1}{M}\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(M1_H-A^{-1/2}BA^{-1/2}\right)^2\right].$$

If we take the exponential, we then get the desired result (3.3). □

Corollary 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that the operator B satisfies the condition $0 < m1_H \leq B \leq M1_H$, where m, M are positive numbers, then*

$$\exp\left[1 - \frac{m+M}{2}\operatorname{tr}\left(PB^{-1}\right)\right] \leq \frac{\Delta_P(B)}{I(m,M)} \leq \exp\left\{\frac{1}{m}\operatorname{tr}\left[P(B - m1_H)^2\right] - \frac{1}{M}\operatorname{tr}\left[P(M1_H - B)^2\right]\right\}. \tag{3.4}$$

Finally, we can also state

Corollary 6. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that the operator A satisfies the condition $0 < n1_H \leq A \leq N1_H$, where n, N are positive numbers, then*

$$\exp\left[\operatorname{tr}(PA) - \frac{n^{-1} + N^{-1}}{2}\operatorname{tr}\left(PA^2\right)\right] \leq \frac{\eta_P(A)}{\left[I\left(n^{-1}, N^{-1}\right)\right]^{\operatorname{tr}(PA)}} \leq \frac{\exp\left(N\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(A^{-1} - N^{-1}1_H\right)^2\right]\right)}{\exp\left(n\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(n^{-1}1_H - A^{-1}\right)^2\right]\right)}. \tag{3.5}$$

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