SEVERAL PRODUCT INEQUALITIES FOR RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

In this paper we show among others that, if A_i, $B_i > 0$ and $P_i \geq 0$ with $P_i \in B_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, then

$$\frac{\sum_{i=1}^n \operatorname{tr}(P_iA_i)}{\sum_{i=1}^n \operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)} \leq \left(\prod_{i=1}^n D_{P_i}(A_i|B_i)\right)^{\frac{1}{\sum_{i=1}^n \operatorname{tr}(P_iA_i)}} \leq \frac{\sum_{i=1}^n \operatorname{tr}(P_iB_i)}{\sum_{i=1}^n \operatorname{tr}(P_iA_i)}.$$

1. Introduction

In 1952, in the paper [11], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to

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be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [17], [18], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [20].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A\in\mathcal{B}_2(H)$ *and, if* $A\in\mathcal{B}_2(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_2(H)$ *with*

$$\|AT\|_2,\ \|TA\|_2\le\|T\|\,\|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|\,e_i,e_i\rangle<\infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A\in\mathcal{B}(H)$, *then the following are equivalent:*

(i) $A\in\mathcal{B}_1(H)$;

(ii) $|A|^{1/2}\in\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1=\|A^*\|_1\ \text{ and }\ \|A\|_2\le\|A\|_1 \tag{1.8}$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1=\sup\{\langle A,B\rangle_2\ \mid\ B\in\mathcal{B}_2(H),\ \|B\|_2\le1\};$$

(v) $(\mathcal{B}_1(H),\|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ *then* AB, $BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [1]-[8] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A)P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [9]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(t1_H) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In [9], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}(PA^{-1})\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by [10]

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA\ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln(tA)\right]\right\}\right)\\
&= \exp\left(-\operatorname{tr}\left\{P\left[tA(\ln t + \ln A)\right]\right\}\right) = \exp\left(-\operatorname{tr}\left\{P\left(tA\ln t + tA\ln A\right)\right\}\right)\\
&= \exp\left(-t\ln t\operatorname{tr}(PA)\right)\exp\left(-t\operatorname{tr}(PA\ln A)\right)\\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}(PA\ln A)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\,\mathrm{tr}(PA)}\,[\eta_P(A)]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(1_H) = 1 \text{ and } \eta_P(t1_H) = t^{-t} \tag{1.14}$$

for $t > 0$.

Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\mathrm{tr}\,(P) = 1$. If $A,\ B > 0$, then we have the Ky Fan type inequality [10]

$$\eta_P((1-t)\,A + tB) \geq [\eta_P\,(A)]^{1-t}\,[\eta_P\,(B)]^{t} \tag{1.15}$$

for all $t \in [0,1]$.

Also we have the inequalities [10]:

$$\left[\frac{\mathrm{tr}\,(PA^2)}{\mathrm{tr}^2\,(PA)}\right]^{-\mathrm{tr}(PA)} \leq \frac{\eta_P(A)}{[\mathrm{tr}\,(PA)]^{-\mathrm{tr}(PA)}} \leq 1$$

and if there exists the constants $0 < m < M$ such that $m1_H \leq A \leq M1_H$, then [10]

$$\begin{aligned}\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\,\mathrm{tr}(PA)} \leq \left[\frac{\mathrm{tr}\,(PA^2)}{\mathrm{tr}^2\,(PA)}\right]^{-\mathrm{tr}(PA)} \\ &\leq \frac{\eta_P(A)}{[\mathrm{tr}\,(PA)]^{-\mathrm{tr}(PA)}} \leq 1.\end{aligned}$$

Kamei and Fujii [15], [16] defined the *relative operator entropy* $S\,(A|B)$, for positive invertible operators A and B, by

$$S\,(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [24]. For various results on relative operator entropy see [12]-[25] and the references therein.

Definition 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\mathrm{tr}\,(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}D_P\,(A|B) &:= \exp\{\mathrm{tr}\,[PS\,(A|B)]\} \\ &= \exp\left\{\mathrm{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.\end{aligned}$$

We observe that for $A > 0$,

$$D_P\,(A|1_H) := \exp\{\mathrm{tr}\,[PS\,(A|1_H)]\} = \exp\{\mathrm{tr}\,(-PA\ln A)\} = \eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B > 0$,

$$D_P\,(1_H|B) := \exp\{\mathrm{tr}\,[PS\,(1_H|B)]\} = \exp\{\mathrm{tr}\,(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

Motivated by the above results, in this paper we show among others that, if A_i, $B_i > 0$ and $P_i \geq 0$ with $P_i \in B_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, then

$$\frac{\sum_{i=1}^{n} \operatorname{tr}(P_i A_i)}{\sum_{i=1}^{n} \operatorname{tr}\left(P_i A_i B_i^{-1} A_i\right)} \leq \left(\prod_{i=1}^{n} D_{P_i}(A_i|B_i)\right)^{\frac{1}{\sum_{i=1}^{n} \operatorname{tr}(P_i A_i)}} \leq \frac{\sum_{i=1}^{n} \operatorname{tr}(P_i B_i)}{\sum_{i=1}^{n} \operatorname{tr}(P_i A_i)}.$$

2. Preliminary Results

We start to the following double trace inequality that is of interest in itself as well:

Lemma 1. *Assume that f is differentiable convex on the interior $\mathring{I}$ of the interval I and the derivative f' is continuous on $\mathring{I}$. Let Q_i, $C_i \geq 0$ with $Q_i \in B_1(H)$ for $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} \operatorname{tr}(Q_i C_i) > 0$, then for all T_i with the spectra $\operatorname{Sp}(T_i) \subset \mathring{I}$ for $i \in \{1, ..., n\}$ and $a \in \mathring{I}$ we have the double inequality*

$$\begin{aligned} &\frac{\sum_{i=1}^{n} \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i) T_i\right)}{\sum_{i=1}^{n} \operatorname{tr}(Q_i C_i)} - a \frac{\sum_{i=1}^{n} \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i)\right)}{\sum_{i=1}^{n} \operatorname{tr}(Q_i C_i)} + f(a) \\ &\geq \frac{\sum_{i=1}^{n} \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f(T_i)\right)}{\sum_{i=1}^{n} \operatorname{tr}(Q_i C_i)} \\ &\geq f'(a) \left(\frac{\sum_{i=1}^{n} \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} T_i\right)}{\sum_{i=1}^{n} \operatorname{tr}(Q_i C_i)} - a\right) + f(a). \end{aligned} \tag{2.1}$$

Proof. We use the gradient inequality

$$f'(t)(t-a) + f(a) \geq f(t) \geq f'(a)(t-a) + f(a)$$

that holds for all t, $a \in \mathring{I}$.

Using the continuous functional calculus for the selfadjoint operators with spectra in $\mathring{I}$, we get

$$f'(T_i)(T_i - aI) + f(a) I \geq f(T_i) \geq f'(a)(T_i - a) + f(a)$$

for all $i \in \{1, ..., n\}$.

Now, if we multiply both sides by $C_i^{1/2} \geq 0$, then we get

$$\begin{aligned} &C_i^{1/2} f'(T_i) T_i C_i^{1/2} - a C_i^{1/2} f'(T_i) C_i^{1/2} + f(a) C_i \\ &\geq C_i^{1/2} f(T_i) C_i^{1/2} \\ &\geq f'(a)\left(C_i^{1/2} T_i C_i^{1/2} - a C_i\right) + f(a) C_i \end{aligned}$$

If we multiply this inequality both sides with $Q_i^{1/2}$, then we get

$$\begin{aligned} &Q_i^{1/2} C_i^{1/2} f'(T_i) T_i C_i^{1/2} Q_i^{1/2} - a Q_i^{1/2} C_i^{1/2} f'(T_i) C_i^{1/2} Q_i^{1/2} \\ &+ f(a) Q_i^{1/2} C_i Q_i^{1/2} \\ &\geq Q_i^{1/2} C_i^{1/2} f(T_i) C_i^{1/2} Q_i^{1/2} \\ &\geq f'(a) Q_i^{1/2} C_i^{1/2} T_i C_i^{1/2} Q_i^{1/2} - a f'(a) Q_i^{1/2} C_i Q_i^{1/2} \\ &+ f(a) Q_i^{1/2} C_i Q_i^{1/2} \end{aligned} \tag{2.2}$$

for all $i \in \{1, ..., n\}$.

If we take the trace in (2.2), then we get

$$\begin{aligned}
&\operatorname{tr}\left(Q_i^{1/2}C_i^{1/2}f'(T_i)T_iC_i^{1/2}Q_i^{1/2}\right) - a\operatorname{tr}\left(Q_i^{1/2}C_i^{1/2}f'(T_i)C_i^{1/2}Q_i^{1/2}\right)\\
&+f(a)\operatorname{tr}\left(Q_i^{1/2}C_iQ_i^{1/2}\right)\\
&\geq \operatorname{tr}\left(Q_i^{1/2}C_i^{1/2}f(T_i)C_i^{1/2}Q_i^{1/2}\right)\\
&\geq f'(a)\operatorname{tr}\left(Q_i^{1/2}C_i^{1/2}T_iC_i^{1/2}Q_i^{1/2}\right) - af'(a)\operatorname{tr}\left(Q_i^{1/2}C_iQ_i^{1/2}\right)\\
&+f(a)\operatorname{tr}\left(Q_i^{1/2}C_iQ_i^{1/2}\right),
\end{aligned}$$

namely, by the properties of the trace

$$\begin{aligned}
&\operatorname{tr}\left(Q_iC_i^{1/2}f'(T_i)T_iC_i^{1/2}\right) - a\operatorname{tr}\left(Q_iC_i^{1/2}f'(T_i)C_i^{1/2}\right) + f(a)\operatorname{tr}(Q_iC_i)\\
&\geq \operatorname{tr}\left(Q_iC_i^{1/2}f(T_i)C_i^{1/2}\right)\\
&\geq f'(a)\operatorname{tr}\left(Q_iC_i^{1/2}T_iC_i^{1/2}\right) - af'(a)\operatorname{tr}(Q_iC_i) + f(a)\operatorname{tr}(Q_iC_i),
\end{aligned}$$

for all $i \in \{1, ..., n\}$.

If we sum over i from 1 to n, then we get

$$\begin{aligned}
&\sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i^{1/2}f'(T_i)T_iC_i^{1/2}\right) - a\sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i^{1/2}f'(T_i)C_i^{1/2}\right)\\
&+f(a)\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)\\
&\geq \sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i^{1/2}f(T_i)C_i^{1/2}\right)\\
&\geq f'(a)\sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i^{1/2}T_iC_i^{1/2}\right) - af'(a)\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)\\
&+f(a)\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)
\end{aligned}$$

and by dividing with $\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i) > 0$, we get (2.1). □

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'(T_i)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)}\\
&-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)}\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'(T_i)\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)}\\
&\geq \frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f(T_i)\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} - f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)}\right) \geq 0.
\end{aligned} \tag{2.3}$$

Proof. Since $\operatorname{Sp}(T_i) \subset \mathring{I}$ for $i \in \{1, ..., n\}$, then there exists $m < M$ such that $\operatorname{Sp}(T_i) \subseteq [m, M] \subset \mathring{I}$ for $i \in \{1, ..., n\}$. Therefore

$$m 1_H \leq T_i \leq M 1_H, \text{ for } i \in \{1, ..., n\},$$

which implies that

$$m C_i \leq C_i^{1/2} T_i C_i^{1/2} \leq M C_i$$

If we multiply this inequality both sides with $Q_i^{1/2}$, then we get

$$m Q_i^{1/2} C_i Q_i^{1/2} \leq Q_i^{1/2} C_i^{1/2} T_i C_i^{1/2} Q_i^{1/2} \leq M Q_i^{1/2} C_i Q_i^{1/2}, \text{ for } i \in \{1, ..., n\}.$$

By taking the trace and summing, we get

$$m \leq \frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} T_i\right)}{\sum_{i=1}^n \operatorname{tr}(Q_i C_i)} \leq M.$$

Then by taking

$$a = \frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} T_i\right)}{\sum_{i=1}^n \operatorname{tr}(Q_i C_i)}$$

in (2.1) we get (2.3). □

Remark 1. *The case of one operator is as follows:*

$$\frac{\operatorname{tr}\left(C^{1/2} Q C^{1/2} f'(T) T\right)}{\operatorname{tr}(QC)} - \frac{\operatorname{tr}\left(C^{1/2} Q C^{1/2} T\right)}{\operatorname{tr}(QC)} \frac{\operatorname{tr}\left(C^{1/2} Q C^{1/2} f'(T)\right)}{\operatorname{tr}(QC)} \geq \frac{\operatorname{tr}\left(C^{1/2} Q C^{1/2} f(T)\right)}{\operatorname{tr}(QC)} - f\left(\frac{\operatorname{tr}\left(C^{1/2} Q C^{1/2} T\right)}{\operatorname{tr}(QC)}\right) \geq 0, \tag{2.4}$$

where Q, $C \geq 0$ with $Q \in B_1(H)$ and $\operatorname{tr}(QC) > 0$ and T is selfadjoint with $\operatorname{Sp}(T) \subset \mathring{I}$. This result was obtained for $C = 1_H$ in a different way in [5].

Corollary 2. *With the assumptions of Lemma 1 and if $\sum_{i=1}^n \operatorname{tr}(Q_i f'(T_i)) \neq 0$ with*

$$\frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i) T_i\right)}{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i)\right)} \in \mathring{I} \text{ for } i \in \{1, ..., n\},$$

then

$$\begin{aligned} 0 &\leq f\left(\frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i) T_i\right)}{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i)\right)}\right) - \frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f(T_i)\right)}{\sum_{i=1}^n \operatorname{tr}(Q_i C_i)} \\ &\leq f'\left(\frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i) T_i\right)}{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i)\right)}\right) \\ &\quad \times \left(\frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i) T_i\right)}{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} f'(T_i)\right)} - \frac{\sum_{i=1}^n \operatorname{tr}\left(C_i^{1/2} Q_i C_i^{1/2} T_i\right)}{\sum_{i=1}^n \operatorname{tr}(Q_i C_i)}\right). \end{aligned} \tag{2.5}$$

Proof. From (2.1) we derive for

$$a=\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}$$

that

$$\begin{aligned}
&f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right)\\
&\geq\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f\left(T_i\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i\right)}\\
&\geq f'\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right)\\
&\times\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right)\\
&+f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right),
\end{aligned}$$

namely

$$\begin{aligned}
0&\geq\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f\left(T_i\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i\right)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right)\\
&\geq f'\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right)\\
&\times\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iC_i\right)}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}f'\left(T_i\right)\right)}\right),
\end{aligned}$$

which is equivalent to (2.5). □

Remark 2. *The case of one operator is as follows: if* $\operatorname{tr}\left(C^{1/2}QC^{1/2}f'\left(T\right)\right)\neq 0$ *with* $\frac{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'(T)T\right)}{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'(T)\right)}\in\mathring{I}$*, then*

$$\begin{aligned}
(2.6)\qquad 0&\leq f\left(\frac{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'\left(T\right)T\right)}{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'\left(T\right)\right)}\right)-\frac{\operatorname{tr}\left(C^{1/2}QC^{1/2}f\left(T\right)\right)}{\operatorname{tr}\left(QC\right)}\\
&\leq f'\left(\frac{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'\left(T\right)T\right)}{\operatorname{tr}\left(C^{1/2}QC^{1/2}C^{1/2}f'\left(T\right)\right)}\right)\\
&\times\left(\frac{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'\left(T\right)T\right)}{\operatorname{tr}\left(C^{1/2}QC^{1/2}f'\left(T\right)\right)}-\frac{\operatorname{tr}\left(C^{1/2}QC^{1/2}T\right)}{\operatorname{tr}\left(QC\right)}\right),
\end{aligned}$$

which was obtained in a different way for $C = 1_H$ in [6].

3. Main Results

We have the following upper and lower bounds concerning the relative entropic normalized P-determinant.

Theorem 4. *Let A_i, $B_i > 0$ and $P_i \geq 0$ with $P_i \in B_1(H)$ with $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$. Then for all $a > 0$, we have*

$$\begin{aligned}(3.1)\qquad & ea\exp\left[-a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}\right] \\ & \leq \left(\prod_{i=1}^{n} D_{P_i}(A_i|B_i)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}} \leq \frac{a}{e}\exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{a\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}\right].\end{aligned}$$

Proof. If we take $f(t) = -\ln t$, $t > 0$ in (2.1), then we get

$$\begin{aligned}& -\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} + a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} - \ln a \\ & \geq -\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}\ln T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} \\ & \geq -a^{-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} - a\right) - \ln a,\end{aligned}$$

namely

$$\begin{aligned}(3.2)\qquad & \frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} - a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} + \ln a \\ & \leq \frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}\ln T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} \\ & \leq a^{-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(C_i^{1/2}Q_iC_i^{1/2}T_i\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_iC_i)} - a\right) + \ln a,\end{aligned}$$

for all $T_i > 0$, Q_i, $C_i \geq 0$ with $Q_i \in B_1(H)$ for $i \in \{1, ..., n\}$ and $a > 0$.

Now, if we take in (3.2) $T_i = A_i^{-1/2}B_iA_i^{-1/2}$, $C_i = A_i$ and $Q_i = P_i$ for $i \in \{1, ..., n\}$, then we get

$$\begin{aligned}& \frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)} - a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}A_i^{1/2}B_i^{-1}A_i^{1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)} + \ln a \\ & \leq \frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)} \\ & \leq a^{-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}A_i^{-1/2}B_iA_i^{-1/2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)} - a\right) + \ln a,\end{aligned}$$

namely

$$1-a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}+\ln a \le \frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)} \le a^{-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}-a\right)+\ln a, \tag{3.3}$$

for $a>0$.

Now, if we take the exponential, then we get

$$\exp\left(1-a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}+\ln a\right) \le \exp\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\right) \le \exp\left[a^{-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}-a\right)+\ln a\right],$$

for $a>0$.

Observe that

$$\exp\left(1-a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}+\ln a\right) = ea\exp\left[-a\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\right],$$

$$\exp\left[a^{-1}\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}-a\right)+\ln a\right] = \frac{a}{e}\exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{a\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\right]$$

and

$$\exp\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\right) = \left(\exp\left[\sum_{i=1}^{n}\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)\right]\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}} = \left(\prod_{i=1}^{n}\exp\left[\operatorname{tr}\left(A_i^{1/2}P_iA_i^{1/2}\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)\right]\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}} = \left(\prod_{i=1}^{n}D_{P_i}\left(A_i|B_i\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}}.$$

Now, by making use of (3.3), we deduce (3.1). □

Remark 3. *For $a=1$ we get the bounds*

$$\exp\left[1-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\right] \leq \left(\prod_{i=1}^{n}D_{P_i}\left(A_i|B_i\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}} \leq \exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}-1\right]. \tag{3.4}$$

Corollary 3. *The best inequality in the left side of (3.1) is*

$$\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)} \leq \left(\prod_{i=1}^{n}D_{P_i}\left(A_i|B_i\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}}. \tag{3.5}$$

Proof. We consider the function

$$f(t):=et\exp\left[-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}t\right],\ t>0.$$

If we take the derivative, then we get

$$\begin{aligned} f'(t) &= e\exp\left[-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}t\right] \\ &\quad - et\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\exp\left[-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}t\right] \\ &= e\exp\left[-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}t\right]\left(1-t\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}\right). \end{aligned}$$

We observe that the function f is increasing on $\left(0,\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}\right)$ and decreasing on $\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)},\infty\right)$, which shows that

$$\sup_{t\in(0,\infty)}f(t)=f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}\right)=\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}$$

and by taking

$$a=\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_iB_i^{-1}A_i\right)}$$

in (3.1) we get (3.5). □

Corollary 4. *The best inequality in the right side of (3.1) is*

$$\left(\prod_{i=1}^{n}D_{P_i}\left(A_i|B_i\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}} \leq \frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}. \tag{3.6}$$

Proof. Consider the function

$$g(t)=\frac{t}{e}\exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iA_i\right)}t^{-1}\right],\ t>0.$$

If we take the derivative, then we get

$$\begin{aligned} g'(t) &= \frac{1}{e}\exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}t^{-1}\right] \\ &\quad -\frac{t}{e}\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}t^{-2}\exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}t^{-1}\right] \\ &= \frac{1}{et}\exp\left[\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}t^{-1}\right]\left(t-\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}\right). \end{aligned}$$

We observe that the function g is decreasing on $\left(0,\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}\right)$ and increasing on $\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)},\infty\right)$, which shows that

$$\inf_{t\in(0,\infty)} g(t) = g\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}\right) = \frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}$$

and by taking $a=\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}$ in in (3.1) we get (3.6). □

4. Some Particular Cases

Let $P_i\geq 0$ with $P_i\in B_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$ .If we take in (3.5) and (3.6) $A_i=1_H$, $i\in\{1,...,n\}$, then we get

$$\frac{n}{\sum_{i=1}^{n}\operatorname{tr}\left(P_iB_i^{-1}\right)} \leq \left(\prod_{i=1}^{n}\Delta_{P_i}(B_i)\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n}\operatorname{tr}(P_iB_i)}{n} \tag{4.1}$$

for $B_i>0$, $i\in\{1,...,n\}$.

Also, if we take $B_i=1_H$, $i\in\{1,...,n\}$ in (3.5) and (3.6), then we get

$$\frac{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i^2)} \leq \left(\prod_{i=1}^{n}\eta_{P_i}(A_i)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}} \leq \frac{n}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}. \tag{4.2}$$

The case of one operator is as follows

$$\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PAB^{-1}A)} \leq D_P(A|B)^{\frac{1}{\operatorname{tr}(PA)}} \leq \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)},$$

provided that A, $B>0$ and $P\geq 0$ with $P\in B_1(H)$ and $\operatorname{tr}(P)=1$.

In particular, we have

$$\frac{1}{\operatorname{tr}(PB^{-1})} \leq \Delta_P(B) \leq \operatorname{tr}(PB) \tag{4.3}$$

and

$$\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)} \leq [\eta_P(A)]^{\frac{1}{\operatorname{tr}(PA)}} \leq \frac{1}{\operatorname{tr}(PA)} \tag{4.4}$$

provided that A, $B>0$ and $P\geq 0$ with $P\in B_1(H)$ and $\operatorname{tr}(P)=1$.

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