REVERSE INEQUALITIES FOR RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

In this paper we show, among others that, if $0 < mA_j \leq B_j \leq MA_j$ with m, $M > 0$ for $j \in \{1, ..., n\}$ and $P_j \in B_1(H)$, $P_j \geq 0$ with $\operatorname{tr}(P_j) = 1$ for $j \in \{1, ..., n\}$ and $\sum_{j=1}^{n}\operatorname{tr}(P_jA_j) > 0$, then

$$1 \leq \frac{\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}}{\prod_{i=1}^{n}D_{P_i}(A_i|B_i)\Big)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}}}$$
$$\leq \exp\left[\frac{1}{mM}\quad M - \frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}\Big)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)} - m\right)\right]$$
$$\leq \exp\left[\frac{1}{4mM}(M - m)^2\right].$$

1. Introduction

In 1952, in the paper [13], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

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For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [19], [20], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [22].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*, then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT,\ TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B \in \mathcal{B}_2(H)$ then $AB,\ BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [3]-[10] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [11]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;

(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;

(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(t1_H) = t$ for all $t > 0$;

(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In [11], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by [12]

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA\ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_P(1_H)=1 \text{ and } \eta_P(t1_H)=t^{-t} \tag{1.14}$$

for $t>0$.

Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If A, $B>0$, then we have the Ky Fan type inequality [12]

$$\eta_P((1-t)A+tB)\geq\left[\eta_P(A)\right]^{1-t}\left[\eta_P(B)\right]^{t} \tag{1.15}$$

for all $t\in[0,1]$.

Also we have the inequalities [12]:

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1$$

and if there exists the constants $0<m<M$ such that $m1_H\leq A\leq M1_H$, then [12]

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M}&\leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)}\leq\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\\
&\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1.
\end{aligned}$$

Kamei and Fujii [17], [18] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [26]. For various results on relative operator entropy see [14]-[27] and the references therein.

Definition 1. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}
D_P(A|B)&:=\exp\{\operatorname{tr}\left[PS(A|B)\right]\}\\
&=\exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.
\end{aligned}$$

We observe that for $A>0$,

$$D_P(A|1_H) := \exp\{\operatorname{tr}[PS(A|1_H)]\} = \exp\{\operatorname{tr}(-PA\ln A)\} = \eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B>0$,

$$D_P(1_H|B) := \exp\{\operatorname{tr}[PS(1_H|B)]\} = \exp\{\operatorname{tr}(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

Motivated by the above results, in this paper we show, among others that, if $0 < mA_j \le B_j \le MA_j$ with $m,\ M>0$ for $j\in\{1,...,n\}$ and $P_j\in B_1(H)$, $P_j\ge 0$ with $\operatorname{tr}(P_j)=1$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^n \operatorname{tr}(P_jA_j)>0$, then

$$\begin{aligned}
1 &\le \frac{\frac{\sum_{j=1}^n \operatorname{tr}(P_jB_j)}{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}}{\left(\prod_{i=1}^n D_{P_i}(A_i|B_i)\right)^{\frac{1}{\sum_{i=1}^n \operatorname{tr}(P_iA_i)}}}\\
&\le \exp\left[\frac{1}{mM}\left(M-\frac{\sum_{j=1}^n \operatorname{tr}(P_jB_j)}{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}\right)\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jB_j)}{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}-m\right)\right]\\
&\le \exp\left[\frac{1}{4mM}(M-m)^2\right].
\end{aligned}$$

2. Some Trace Inequalities

We use the following result that was obtained in [2]:

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then*

$$\begin{aligned}
0 &\le \frac{(b-t)f(a)+(t-a)f(b)}{b-a}-f(t) \\
&\le (b-t)(t-a)\frac{f'_-(b)-f'_+(a)}{b-a} \le \frac{1}{4}(b-a)\left[f'_-(b)-f'_+(a)\right]
\end{aligned} \tag{2.1}$$

for any $t\in[a,b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have the following reverse for the Jensen's trace inequality:

Theorem 4. *Assume that f is differentiable convex on the interior $\mathring{I}$ of an interval. Let $C_j,\ Q_j\ge 0$ with $Q_j\in B_1(H)$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^n \operatorname{tr}(Q_jC_j)>0$, then for all V_j with the spectra $\operatorname{Sp}(V_j)\subseteq[m,M]\subset\mathring{I}$ for $j\in\{1,...,n\}$, we have*

$$\begin{aligned}
0 &\le \frac{\sum_{j=1}^n \operatorname{tr}\left[C_j^{1/2}Q_jC_j^{1/2}f(V_j)\right]}{\sum_{j=1}^n \operatorname{tr}(Q_jC_j)} - f\left(\frac{\sum_{j=1}^n \operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^n \operatorname{tr}(Q_jC_j)}\right)\\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}\\
&\times\left(M-\frac{\sum_{j=1}^n \operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^n \operatorname{tr}(Q_jC_j)}\right)\left(\frac{\sum_{j=1}^n \operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^n \operatorname{tr}(Q_jC_j)}-m\right)\\
&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right].
\end{aligned} \tag{2.2}$$

Proof. Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1_H$ and the convexity of f on $[m, M]$, we have

$$f(m(1_H - T) + MT) \leq f(m)(1_H - T) + f(M)T \tag{2.3}$$

in the operator order.

If we take in (2.3)

$$0 \leq T = \frac{V_j - m1_H}{M - m} \leq 1_H,\ j \in \{1, ..., n\}$$

then we get

$$\begin{aligned} & f\left(m\left(1_H - \frac{V_j - m1_H}{M - m}\right) + M\frac{V_j - m1_H}{M - m}\right) \\ & \leq f(m)\left(1_H - \frac{V_j - m1_H}{M - m}\right) + f(M)\frac{V_j - m1_H}{M - m}. \end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned} & m\left(1_H - \frac{V_j - m1_H}{M - m}\right) + M\frac{V_j - m1_H}{M - m} \\ & = \frac{m(M1_H - V_j) + M(V_j - m1_H)}{M - m} = V_j \end{aligned}$$

and

$$\begin{aligned} & f(m)\left(1_H - \frac{V_j - m1_H}{M - m}\right) + f(M)\frac{V_j - m1_H}{M - m} \\ & = \frac{f(m)(M1_H - V_j) + f(M)(V_j - m1_H)}{M - m} \end{aligned}$$

and by (2.4) we get the following inequality of interest

$$f(V_j) \leq \frac{f(m)(M1_H - V_j) + f(M)(V_j - m1_H)}{M - m} \tag{2.5}$$

for all $j \in \{1, ..., n\}$.

If we multiply (2.5) both sides with $C_j^{1/2}$ and then with $Q_j^{1/2}$ we get by summing that

$$\begin{aligned}
&\sum_{j=1}^{n} Q_j^{1/2} C_j^{1/2} f(V_j) C_j^{1/2} Q_j^{1/2} \\
&\leq \sum_{j=1}^{n} Q_j^{1/2} C_j^{1/2} \left[\frac{f(m)(M1_H - V_j) + f(M)(V_j - m1_H)}{M-m} \right] C_j^{1/2} Q_j^{1/2} \\
&= \frac{f(m) \sum_{j=1}^{n} Q_j^{1/2} C_j^{1/2} (M1_H - V_j) C_j^{1/2} Q_j^{1/2}}{M-m} \\
&+ \frac{f(M) \sum_{j=1}^{n} Q_j^{1/2} C_j^{1/2} (V_j - m1_H) C_j^{1/2} Q_j^{1/2}}{M-m} \\
&= \frac{1}{M-m} \left[f(m) \left(M \sum_{j=1}^{n} Q_j^{1/2} C_j Q_j^{1/2} - \sum_{j=1}^{n} Q_j^{1/2} C_j^{1/2} V_j C_j^{1/2} Q_j^{1/2} \right) \right. \\
&\left. + f(M) \left(\sum_{j=1}^{n} Q_j^{1/2} C_j^{1/2} V_j C_j^{1/2} Q_j^{1/2} - m \sum_{j=1}^{n} Q_j^{1/2} C_j Q_j^{1/2} \right) \right],
\end{aligned}$$

which implies, by taking the trace and using its properties, that

$$\begin{aligned}
&\sum_{j=1}^{n} \operatorname{tr} \left[C_j^{1/2} Q_j C_j^{1/2} f(V_j) \right] \\
&\leq \frac{1}{M-m} \left[f(m) \left(M \sum_{j=1}^{n} \operatorname{tr} \left(Q_j^{1/2} C_j Q_j^{1/2} \right) - \sum_{j=1}^{n} \operatorname{tr} \left(C_j^{1/2} Q_j C_j^{1/2} V_j \right) \right) \right. \\
&\left. + f(M) \left(\sum_{j=1}^{n} \operatorname{tr} \left(C_j^{1/2} Q_j C_j^{1/2} V_j \right) - m \sum_{j=1}^{n} \operatorname{tr} \left(Q_j^{1/2} C_j Q_j^{1/2} \right) \right) \right],
\end{aligned}$$

which gives that

$$\begin{aligned}
&\frac{\sum_{j=1}^{n} \operatorname{tr} \left[C_j^{1/2} Q_j C_j^{1/2} f(V_j) \right]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} \\
&\leq \frac{f(m) \left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j \right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} \right) + f(M) \left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j \right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - m \right)}{M-m},
\end{aligned}$$

namely

$$0 \leq \frac{\sum_{j=1}^{n} \operatorname{tr}\left[C_j^{1/2} Q_j C_j^{1/2} f(V_j)\right]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right) \tag{2.6}$$

$$\leq \frac{f(m)\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right)}{M-m} + \frac{f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - m\right)}{M-m} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right).$$

Here the first inequality is Jensen's inequality.

Using the inequality (2.1) for

$$t = \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} \in [m, M],$$

$a = m$ and $b = M$ we have

$$\frac{f(m)\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right)}{M-m} + \frac{f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - m\right)}{M-m} \tag{2.7}$$

$$- f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right)$$

$$\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right) \times \left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - m\right)$$

$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right].$$

By making use of (2.6) and (2.7) we derive (2.2). □

We also have [2]:

Lemma 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$*. If* f' *is* K*-Lipschitzian on* $[a,b]$*, then*

$$\begin{aligned} &\left|(1-t)f(a)+tf(b)-f((1-t)a+tb)\right| \\ &\leq \frac{1}{2}K(b-t)(t-a)\leq\frac{1}{8}K(b-a)^2 \end{aligned} \tag{2.8}$$

for all $t\in[0,1]$*.*

The constants $1/2$ *and* $1/8$ *are the best possible in (2.8).*

Remark 1. *If* $f:[a,b]\to\mathbb{R}$ *is twice differentiable and* $f''\in L_\infty[a,b]$*, then*

$$\begin{aligned} &\left|(1-t)f(a)+tf(b)-f((1-t)a+tb)\right| \\ &\leq \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a)\leq\frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2, \end{aligned} \tag{2.9}$$

where $\|f''\|_{[a,b],\infty}:=\operatorname{essup}_{t\in[a,b]}|f''(t)|<\infty$*. The constants* $1/2$ *and* $1/8$ *are the best possible in (2.9).*

Theorem 5. *Assume that* f *is twice differentiable convex on the interior* $\mathring{I}$ *of the interval* I *and the derivative* f'' *is bounded on* $\mathring{I}$*. Let* $Q_j\geq 0$ *with* $Q_j\in B_1(H)$ *for* $j\in\{1,...,n\}$ *and* $\sum_{j=1}^n\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\right)>0$*, then for all* V_j *with the spectra* $\operatorname{Sp}(V_j)\subseteq[m,M]\subset\mathring{I}$ *for* $j\in\{1,...,n\}$*, we have*

$$\begin{aligned} 0 &\leq \frac{\sum_{j=1}^n\operatorname{tr}\left[C_j^{1/2}Q_jC_j^{1/2}f(V_j)\right]}{\sum_{j=1}^n\operatorname{tr}(Q_jC_j)}-f\left(\frac{\sum_{j=1}^n\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^n\operatorname{tr}(Q_jC_j)}\right) \\ &\leq \frac{1}{2}\|f''\|_{[m,M],\infty}\left(M-\frac{\sum_{j=1}^n\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^n\operatorname{tr}(Q_jC_j)}\right) \\ &\times\left(\frac{\sum_{j=1}^n\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^n\operatorname{tr}(Q_jC_j)}-m\right) \\ &\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2. \end{aligned} \tag{2.10}$$

Proof. From (2.9) and the continuous functional calculus, we get

$$\begin{aligned} 0 &\leq \frac{f(m)(M1_H-V_j)+f(M)(V_j-m1_H)}{M-m}-f(V_j) \\ &\leq \frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-V_j)(V_j-m1_H) \\ &\leq \frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H, \end{aligned} \tag{2.11}$$

where V_j are selfadjoint operators with the spectra $\operatorname{Sp}(V_j)\subset[m,M]$, $j\in\{1,...,n\}$.

Now, by employing a similar argument to the one in the proof of Theorem 4 we derive the desired result (2.10). □

We also have the following scalar inequality of interest:

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $t\in[0,1]$, then*

$$\begin{aligned}(2.12)\qquad & 2\min\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\\ &\le (1-t)f(a)+tf(b)-f((1-t)a+tb)\\ &\le 2\max\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

The proof follows, for instance, by Corollary 1 from [1] for $n=2$, $p_1=1-t$, $p_2=t$, $t\in[0,1]$ and $x_1=a$, $x_2=b$.

Theorem 6. *Assume that f is convex on the interior $\mathring{I}$ of an interval I. Let $Q_j\ge 0$ with $Q_j\in B_1(H)$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^{n}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\right)>0$, then for all V_j with the spectra $\operatorname{Sp}(V_j)\subseteq[m,M]\subset\mathring{I}$ for $j\in\{1,...,n\}$, we have*

$$\begin{aligned}(2.13)\qquad 0 &\le \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\quad\times\left(\frac{1}{2}(M-m)-\frac{\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\left|V_j-\frac{1}{2}(m+M)1_H\right|\right)}{\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)}\right)\end{aligned}$$

$$\begin{aligned}&\le\frac{f(m)\left(M-\frac{\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)}\right)}{M-m}+\frac{f(M)\left(\frac{\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)}-m\right)}{M-m}\\ &-\frac{\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}f(V_j)\right)}{\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)}\end{aligned}$$

$$\begin{aligned}&\le\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\times\left(\frac{1}{2}(M-m)+\frac{\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\left|V_j-\frac{1}{2}(m+M)1_H\right|\right)}{\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)}\right)\\ &\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

Proof. We have from (2.12) that

$$\begin{aligned}(2.14)\qquad 0&\le 2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\le(1-t)f(m)+tf(M)-f((1-t)m+tM)\\ &\le 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],\end{aligned}$$

for all $t\in[0,1]$.

Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1_H$ we get from (2.14) that

$$0 \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \tag{2.15}$$
$$\leq (1-T)f(m)+Tf(M)-f((1-T)m+TM)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),$$

in the operator order.

If we take in (2.15)

$$0 \leq T = \frac{V_j - m1_H}{M-m} \leq 1_H,\ j \in \{1,...,n\},$$

then we get

$$0 \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{2.16}$$
$$\times\left(\frac{1}{2}(M-m)1_H-\left|V_j-\frac{1}{2}(m+M)1_H\right|\right)$$
$$\leq \frac{f(m)(M1_H-V_j)+f(M)(V_j-m1_H)}{M-m}-f(V_j)$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H+\left|V_j-\frac{1}{2}(m+M)1_H\right|\right).$$

If we multiply both sides by $C_j^{1/2}$ and then by $Q_j^{1/2}$ we derive

$$0 \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)Q_j^{1/2}C_jQ_j^{1/2}-Q_j^{1/2}C_j^{1/2}\left|V_j-\frac{1}{2}(m+M)1_H\right|C_j^{1/2}Q_j^{1/2}\right)$$

$$\leq \frac{f(m)\left(MQ_j^{1/2}C_jQ_j^{1/2}-Q_j^{1/2}C_j^{1/2}V_jC_j^{1/2}Q_j^{1/2}\right)}{M-m}$$
$$+\frac{f(M)\left(Q_j^{1/2}C_j^{1/2}V_jC_j^{1/2}Q_j^{1/2}-mQ_j^{1/2}C_jQ_j^{1/2}\right)}{M-m}$$
$$-Q_j^{1/2}C_j^{1/2}f(V_j)C_j^{1/2}Q_j^{1/2}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)Q_j^{1/2}C_jQ_j^{1/2}+Q_j^{1/2}C_j^{1/2}\left|V_j-\frac{1}{2}(m+M)1_H\right|C_j^{1/2}Q_j^{1/2}\right).$$

Now, by taking the trace and summing over j from 1 to n, we derive

$$0 \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)-\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\left|V_j-\frac{1}{2}(m+M)1_H\right|\right)\right)$$

$$\leq \frac{1}{M-m}\left[f(m)\left(M\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)-\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)\right)\right.$$
$$\left.+f(M)\left(\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)-m\sum_{j=1}^{k}\operatorname{tr}(Q_jC_j)\right)\right]$$
$$-\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}f(V_j)\right)$$

$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\right)+\sum_{j=1}^{k}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}\left|V_j-\frac{1}{2}(m+M)1_H\right|\right)\right).$$

This proves (2.13). □

We also have:

Proposition 2. *With the assumptions of Theorem 6 we have*

$$0 \leq \frac{\sum_{j=1}^{n}\operatorname{tr}\left[C_j^{1/2}Q_jC_j^{1/2}f(V_j)\right]}{\sum_{j=1}^{n}\operatorname{tr}(Q_jC_j)}-f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{n}\operatorname{tr}(Q_jC_j)}\right) \tag{2.17}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)+\left|\frac{\sum_{j=1}^{n}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{n}\operatorname{tr}(Q_jC_j)}-\frac{1}{2}(m+M)\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

Proof. From (2.6) we have

$$0 \leq \frac{\sum_{j=1}^{n} \operatorname{tr}\left[C_j^{1/2} Q_j C_j^{1/2} f(V_j)\right]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right) \tag{2.18}$$
$$\leq \frac{f(m)\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right)}{M-m} + \frac{f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - m\right)}{M-m} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right).$$

From the second part of the scalar version of (2.16) we also have the scalar inequality

$$\frac{f(m)\left(M - \frac{\sum_{j=1}^{k} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{k} \operatorname{tr}(Q_j C_j)}\right)}{M-m} + \frac{f(M)\left(\frac{\sum_{j=1}^{k} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{k} \operatorname{tr}(Q_j C_j)} - m\right)}{M-m} - f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right) \tag{2.19}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \times \left(\frac{1}{2}(M-m)1_H + \left|\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - \frac{1}{2}(m+M)\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right].$$

By utilizing (2.18) and (2.19) we obtain the desired result (2.17). □

3. Determinant Inequalities

We can provide now several inequalities for the relative entropic normalized P-determinants.

Theorem 7. *Assume that* $0 < mA_j \leq B_j \leq MA_j$ *with* $m, M > 0$ *for* $j \in \{1, ..., n\}$ *and* $P_j \in B_1(H)$, $P_j \geq 0$ *with* $\operatorname{tr}(P_j) = 1$ *for* $j \in \{1, ..., n\}$ *and* $\sum_{j=1}^{n} \operatorname{tr}(P_j A_j) > 0$.

Then

$$1 \leq \frac{\frac{\sum_{j=1}^{n} \operatorname{tr}(P_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}}{\left(\prod_{i=1}^{n} D_{P_i}(A_i|B_i)\right)^{\frac{1}{\sum_{i=1}^{n} \operatorname{tr}(P_i A_i)}}} \tag{3.1}$$
$$\leq \exp\left[\frac{1}{mM}\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}(P_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}\right)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}(P_j B_j)}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)} - m\right)\right]$$
$$\leq \exp\left[\frac{1}{4mM}(M-m)^2\right].$$

Proof. If we take the convex function $f(t) = -\ln t$, $t > 0$ in (2.2) then we get the inequality

$$0 \leq \ln\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right) - \frac{\sum_{j=1}^{n} \operatorname{tr}\left[C_j^{1/2} Q_j C_j^{1/2} \ln V_j\right]}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} \tag{3.2}$$
$$\leq \frac{1}{mM}\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)}\right)$$
$$\times \left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(C_j^{1/2} Q_j C_j^{1/2} V_j\right)}{\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j)} - m\right) \leq \frac{1}{4mM}(M-m)^2,$$

where C_j, $Q_j \geq 0$ with $Q_j \in B_1(H)$ for $j \in \{1, ..., n\}$ and $\sum_{j=1}^{n} \operatorname{tr}(Q_j C_j) > 0$ while $\operatorname{Sp}(V_j) \subseteq [m, M] \subset (0, \infty)$ for $j \in \{1, ..., n\}$.

Now if $0 < mA_j \leq B_j \leq M_j A_j$ for $j \in \{1, ..., n\}$, then $0 < m1_H \leq A_j^{-1/2} B_j A_j^{-1/2} \leq M_j 1_H$ and by taking $V_j = A_j^{-1/2} B_j A_j^{-1/2}$, $C_j = A_j$ and $Q_j = P_j$ for $j \in \{1, ..., n\}$ in (3.2), then we get

$$0 \leq \ln\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(A_j^{1/2} P_j A_j^{1/2} A_j^{-1/2} B_j A_j^{-1/2}\right)}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}\right)$$
$$- \frac{\sum_{j=1}^{n} \operatorname{tr}\left[A_j^{1/2} P_j A_j^{1/2} \ln\left(A_j^{-1/2} B_j A_j^{-1/2}\right)\right]}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}$$
$$\leq \frac{1}{mM}\left(M - \frac{\sum_{j=1}^{n} \operatorname{tr}\left(A_j^{1/2} P_j A_j^{1/2} A_j^{-1/2} B_j A_j^{-1/2}\right)}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}\right)$$
$$\times \left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(A_j^{1/2} P_j A_j^{1/2} A_j^{-1/2} B_j A_j^{-1/2}\right)}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)} - m\right)$$
$$\leq \frac{1}{4mM}(M-m)^2,$$

namely

$$\begin{aligned}0&\le\ln\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}A_{j}\right)}\right)-\frac{\sum_{j=1}^{n}\operatorname{tr}\left[A_{j}^{1/2}P_{j}A_{j}^{1/2}\ln\left(A_{j}^{-1/2}B_{j}A_{j}^{-1/2}\right)\right]}{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}A_{j}\right)}\\&\le\frac{1}{mM}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}A_{j}\right)}\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}A_{j}\right)}-m\right)\\&\le\frac{1}{4mM}\left(M-m\right)^{2}.\end{aligned}$$

Now, if we take the exponential, then we get

$$\begin{aligned}1&\le\frac{\frac{\sum_{j=1}^{n}\operatorname{tr}(P_{j}B_{j})}{\sum_{j=1}^{n}\operatorname{tr}(P_{j}A_{j})}}{\exp\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left[A_{j}^{1/2}P_{j}A_{j}^{1/2}\ln\left(A_{j}^{-1/2}B_{j}A_{j}^{-1/2}\right)\right]}{\sum_{j=1}^{n}\operatorname{tr}(P_{j}A_{j})}\right)}\\&\le\exp\left[\frac{1}{mM}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}A_{j}\right)}\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}A_{j}\right)}-m\right)\right]\\&\le\frac{1}{4mM}\left(M-m\right)^{2}.\end{aligned}\tag{3.3}$$

Since

$$\begin{aligned}&\exp\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(A_{i}^{1/2}P_{i}A_{i}^{1/2}\ln\left(A_{i}^{-1/2}B_{i}A_{i}^{-1/2}\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(P_{i}A_{i}\right)}\right)\\&=\left(\exp\left[\sum_{i=1}^{n}\operatorname{tr}\left(A_{i}^{1/2}P_{i}A_{i}^{1/2}\ln\left(A_{i}^{-1/2}B_{i}A_{i}^{-1/2}\right)\right)\right]\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_{i}A_{i}\right)}}\\&=\left(\prod_{i=1}^{n}\exp\left[\operatorname{tr}\left(A_{i}^{1/2}P_{i}A_{i}^{1/2}\ln\left(A_{i}^{-1/2}B_{i}A_{i}^{-1/2}\right)\right)\right]\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_{i}A_{i}\right)}}\\&=\left(\prod_{i=1}^{n}D_{P_{i}}\left(A_{i}|B_{i}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}\left(P_{i}A_{i}\right)}},\end{aligned}$$

hence by (3.3) we derive (3.1). □

Corollary 1. *Assume that* $0<m1_{H}\le B_{j}\le M1_{H}$ *with* $m,M>0$ *for* $j\in\{1,...,n\}$ *and* $P_{j}\in B_{1}(H)$, $P_{j}\ge0$ *with* $\operatorname{tr}(P_{j})=1$ *for* $j\in\{1,...,n\}$. *Then*

$$\begin{aligned}1&\le\frac{\frac{\sum_{j=1}^{n}\operatorname{tr}(P_{j}B_{j})}{n}}{\left(\prod_{i=1}^{n}\Delta_{P_{i}}(B_{j})\right)^{\frac{1}{n}}}\\&\le\exp\left[\frac{1}{mM}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{n}\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_{j}B_{j}\right)}{n}-m\right)\right]\\&\le\exp\left[\frac{1}{4mM}\left(M-m\right)^{2}\right].\end{aligned}\tag{3.4}$$

The proof follows by (3.1) for $A_j = 1_H$ with $j \in \{1, ..., n\}$.

Corollary 2. *Assume that $0 < k1_H \le A_j \le K1_H$ with $k,\ K > 0$ for $j \in \{1, ..., n\}$ and $P_j \in B_1(H)$, $P_j \ge 0$ with $\operatorname{tr}(P_j) = 1$ for $j \in \{1, ..., n\}$. Then*

$$
\begin{aligned}
1 &\le \frac{\frac{n}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}}{\left(\prod_{i=1}^{n} \eta_{P_i}(A_i)\right)^{\frac{1}{\sum_{i=1}^{n} \operatorname{tr}(P_i A_i)}}} \\
&\le \exp\left[kK\left(k^{-1} - \frac{n}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}\right)\left(\frac{n}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)} - K^{-1}\right)\right] \\
&\le \exp\left[\frac{1}{4kK}(K-k)^2\right].
\end{aligned} \tag{3.5}
$$

Proof. Since $0 < k1_H \le A_j \le K1_H$, hence $\frac{1}{K} A_j \le 1_H \le \frac{1}{k} A_j$ and by taking $B_j = 1_H$ with $j \in \{1, ..., n\}$ in (3.1), we derive (3.5). □

Remark 2. *The case of a pair of operators is as follows. If $0 < mA \le B \le MA$ with $m,\ M > 0$ and $P \in B_1(H)$, $P \ge 0$ with $\operatorname{tr}(P) = 1$ and $\operatorname{tr}(PA) > 0$. Then*

$$
\begin{aligned}
1 &\le \frac{\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}}{\left(D_P(A|B)\right)^{\frac{1}{\operatorname{tr}(PA)}}} \\
&\le \exp\left[\frac{1}{mM}\left(M - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)} - m\right)\right] \\
&\le \exp\left[\frac{1}{4mM}(M-m)^2\right].
\end{aligned} \tag{3.6}
$$

If $0 < m1_H \le B \le M1_H$, then

$$
\begin{aligned}
1 &\le \frac{\operatorname{tr}(PB)}{\Delta_P(B)} \le \exp\left[\frac{1}{mM}(M - \operatorname{tr}(PB))(\operatorname{tr}(PB) - m)\right] \\
&\le \exp\left[\frac{1}{4mM}(M-m)^2\right].
\end{aligned} \tag{3.7}
$$

Assume that $0 < k1_H \le A_j \le K1_H$, then

$$
\begin{aligned}
1 &\le \frac{\frac{1}{\operatorname{tr}(PA)}}{\left[\eta_P(A)\right]^{\frac{1}{\operatorname{tr}(PA)}}} \\
&\le \exp\left[kK\left(k^{-1} - \frac{1}{\operatorname{tr}(PA)}\right)\left(\frac{1}{\operatorname{tr}(PA)} - K^{-1}\right)\right] \\
&\le \exp\left[\frac{1}{4kK}(K-k)^2\right].
\end{aligned} \tag{3.8}
$$

Theorem 8. *Assume that $0 < mA_j \le B_j \le MA_j$ with $m,\ M > 0$ for $j \in \{1, ..., n\}$ and $P_j \in B_1(H)$, $P_j \ge 0$ with $\operatorname{tr}(P_j) = 1$ for $j \in \{1, ..., n\}$ and $\sum_{j=1}^{n} \operatorname{tr}(P_j A_j) > 0$.*

Then

$$1\leq\frac{\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}}{\left(\prod_{i=1}^{n}D_{P_i}\left(A_i|B_i\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}}}\tag{3.9}$$
$$\leq\exp\left[\frac{1}{2m^2}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_jA_j\right)}\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_jA_j\right)}-m\right)\right]$$
$$\leq\exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right].$$

Proof. If we take the convex function $f(t)=-\ln t$, $t>0$ in (2.10) then we get the operator inequality

$$0\leq\ln\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jC_j\right)}\right)-\frac{\sum_{j=1}^{n}\operatorname{tr}\left[C_j^{1/2}Q_jC_j^{1/2}\ln V_j\right]}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jC_j\right)}$$
$$\leq\frac{1}{2m^2}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jC_j\right)}\right)$$
$$\times\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(C_j^{1/2}Q_jC_j^{1/2}V_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jC_j\right)}-m\right)\leq\frac{1}{8}\left(\frac{M}{m}-1\right)^2,$$

where $C_j,\ Q_j\geq0$ with $Q_j\in B_1(H)$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^{n}\operatorname{tr}(Q_jC_j)>0$ while $\operatorname{Sp}(V_j)\subseteq[m,M]\subset(0,\infty)$ for $j\in\{1,...,n\}$.

By utilizing a similar argument as in the proof of Theorem 7 we deduce the desired result (3.9). □

Corollary 3. *we the assumptions of Corollary 1 we have*

$$1\leq\frac{\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{n}}{\left(\prod_{i=1}^{n}\Delta_{P_i}(B_j)\right)^{\frac{1}{n}}}\tag{3.10}$$
$$\leq\exp\left[\frac{1}{2m^2}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_jB_j\right)}{n}\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(P_jB_j\right)}{n}-m\right)\right]$$
$$\leq\exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right].$$

The proof follows by (3.9) for $A_j=1_H$ with $j\in\{1,...,n\}$.

Corollary 4. *With the assumptions of Corollary 2 we have*

$$1 \leq \frac{\frac{n}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}}{\left(\prod_{i=1}^{n} \eta_{P_i}(A_i)\right)^{\frac{1}{\sum_{i=1}^{n} \operatorname{tr}(P_i A_i)}}} \tag{3.11}$$
$$\leq \exp\left[\frac{K^2}{2}\left(k^{-1} - \frac{n}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)}\right)\left(\frac{n}{\sum_{j=1}^{n} \operatorname{tr}(P_j A_j)} - K^{-1}\right)\right]$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{K}{k} - 1\right)^2\right].$$

It follows by taking $B_j = 1_H$ with $j \in \{1, ..., n\}$ in (3.9).

Remark 3. *The case of a pair of operators is as follows. If* $0 < mA \leq B \leq MA$ *with* m, $M > 0$ *and* $P \in B_1(H)$, $P \geq 0$ *with* $\operatorname{tr}(P) = 1$ *and* $\operatorname{tr}(PA) > 0$, *then*

$$1 \leq \frac{\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}}{[D_P(A|B)]^{\frac{1}{\operatorname{tr}(PA)}}} \tag{3.12}$$
$$\leq \exp\left[\frac{1}{2m^2}\left(M - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)}\right)\left(\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PA)} - m\right)\right]$$
$$\leq \exp\left[\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]\right].$$

If $0 < m1_H \leq B \leq M1_H$, *then*

$$1 \leq \frac{\operatorname{tr}(PB)}{\Delta_P(B)} \leq \exp\left[\frac{1}{2m^2}(M - \operatorname{tr}(PB))(\operatorname{tr}(PB) - m)\right] \tag{3.13}$$
$$\leq \exp\left[\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]\right].$$

Assume that $0 < k1_H \leq A_j \leq K1_H$, *then*

$$1 \leq \frac{\frac{1}{\operatorname{tr}(PA)}}{[\eta_P(A)]^{\frac{1}{\operatorname{tr}(PA)}}} \leq \exp\left[\frac{K^2}{2}\left(k^{-1} - \frac{1}{\operatorname{tr}(PA)}\right)\left(\frac{1}{\operatorname{tr}(PA)} - K^{-1}\right)\right]$$
$$\leq \exp\left[\left[\frac{1}{8}\left(\frac{K}{k} - 1\right)^2\right]\right].$$

4. Some Related Results

We also have the complementary inequalities:

Theorem 9. *Assume that* $0 < mA_j \leq B_j \leq MA_j$ *with* m, $M > 0$ *for* $j \in \{1, ..., n\}$ *and* $P_j \in B_1(H)$, $P_j \geq 0$ *with* $\operatorname{tr}(P_j) = 1$ *for* $j \in \{1, ..., n\}$ *and* $\sum_{j=1}^{n} \operatorname{tr}(P_j A_j) > 0$.

Then

$$1 \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)-\mathcal{V}_{m,M}(\mathbf{A},\mathbf{B},\mathbf{P})\right)} \tag{4.1}$$
$$\le \frac{\left(\prod_{i=1}^{n} D_{P_i}\left(A_i|B_i\right)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}}}{M^{\frac{1}{M-m}\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}-m\right)} m^{\frac{1}{M-m}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}\right)}}$$
$$\le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\mathcal{V}_{m,M}(\mathbf{A},\mathbf{B},\mathbf{P})\right)} \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{2}$$

with

$$\mathcal{V}_{m,M}\left(\mathbf{A},\mathbf{B},\mathbf{P}\right) := \frac{\sum_{j=1}^{n}\operatorname{tr}\left(A_j^{1/2}P_jA_j^{1/2}\left|A_j^{-1/2}B_jA_j^{-1/2}-\frac{1}{2}\left(m+M\right)1_H\right|\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(P_jA_j\right)}.$$

Proof. If we take the convex function $f(t) = -\ln t$, $t > 0$ in (2.13) then we get the inequality

$$0 \le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}}\left(\frac{1}{2}\left(M-m\right)-\mathcal{V}_{m,M}\left(\mathbf{A},\mathbf{B},\mathbf{P}\right)\right)$$
$$\le \frac{\sum_{j=1}^{n}\operatorname{tr}\left[A_j^{1/2}P_jA_j^{1/2}\ln\left(A_j^{-1/2}B_jA_j^{-1/2}\right)\right]}{\sum_{j=1}^{n}\operatorname{tr}\left(P_jA_j\right)}$$
$$-\frac{\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}-m\right)\ln M+\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}\right)\ln m}{M-m}$$
$$\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}}\left(\frac{1}{2}\left(M-m\right)+\mathcal{V}_{m,M}\left(\mathbf{A},\mathbf{B},\mathbf{P}\right)\right) \le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{2}.$$

By taking the exponential and performing the required calculations, we derive (4.1). □

Corollary 5. *Assume that* $0 < m1_H \le B_j \le M1_H$ *with* $m, M > 0$ *for* $j \in \{1, ..., n\}$ *and* $P_j \in B_1(H)$, $P_j \ge 0$ *with* $\operatorname{tr}(P_j) = 1$ *for* $j \in \{1, ..., n\}$. *Then*

Corollary 6.

$$1 \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)-\mathcal{V}_{m,M}(\mathbf{B},\mathbf{P})\right)} \tag{4.2}$$
$$\le \frac{\left(\prod_{i=1}^{n}\Delta_{P_i}\left(B_i\right)\right)^{\frac{1}{n}}}{M^{\frac{1}{M-m}\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{n}-m\right)} m^{\frac{1}{M-m}\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jB_j)}{n}\right)}}$$
$$\le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\mathcal{V}_{m,M}(\mathbf{B},\mathbf{P})\right)} \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{2}$$

with

$$\mathcal{V}_{m,M}(\mathbf{B},\mathbf{P}) := \frac{\sum_{j=1}^{n} \operatorname{tr}\left(P_j \left|B_j - \frac{1}{2}(m+M)1_H\right|\right)}{n}.$$

Corollary 7. *Assume that* $0 < k1_H \le A_j \le K1_H$ *with* $k,\ K > 0$ *for* $j \in \{1,...,n\}$ *and* $P_j \in B_1(H)$, $P_j \ge 0$ *with* $\operatorname{tr}(P_j) = 1$ *for* $j \in \{1,...,n\}$. *Then*

$$1 \le \left(\frac{k+K}{2\sqrt{kK}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)-\mathcal{V}_{k,K}(\mathbf{A},\mathbf{P})\right)} \tag{4.3}$$

$$\le \frac{\left(\prod_{i=1}^{n} \eta_{P_i}(A_i)\right)^{\frac{1}{\sum_{i=1}^{n}\operatorname{tr}(P_iA_i)}}}{k^{\frac{-1}{k^{-1}-K^{-1}}\left(\frac{n}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}-K^{-1}\right)} K^{\frac{-1}{k^{-1}-K^{-1}}\left(k^{-1}-\frac{n}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}\right)}}$$

$$\le \left(\frac{k+K}{2\sqrt{kK}}\right)^{\frac{2}{M-m}\left(\frac{1}{2}(M-m)+\mathcal{V}_{k,K}(\mathbf{A},\mathbf{P})\right)} \le \left(\frac{k+K}{2\sqrt{kK}}\right)^{2}$$

with

$$\mathcal{V}_{k,K}(\mathbf{A},\mathbf{P}) := \frac{\sum_{j=1}^{n} \operatorname{tr}\left(A_j^{1/2}P_jA_j^{1/2}\left|A_j^{-1} - \frac{1}{2}\left(K^{-1}+k^{-1}\right)1_H\right|\right)}{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}.$$

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