SOME IMPROVEMENTS OF THE MONOTONICITY PROPERTY FOR RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

In this paper we show among others that, if $C \geq m_1 A > 0$, $B \geq m_2 A > 0$, $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then

$$\begin{aligned}
&\exp\left(-\Phi(m_1,m_2)\operatorname{tr}(PA)\left\|A^{-1/2}(B-C)A^{-1/2}\right\|\right)\\
&\leq \frac{D_P(A|B)}{D_P(A|C)}\\
&\leq \exp\left(\Phi(m_1,m_2)\operatorname{tr}(PA)\left\|A^{-1/2}(B-C)A^{-1/2}\right\|\right),
\end{aligned}$$

where

$$\Phi(m_1,m_2) := \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \text{ if } m_2 \neq m_1, \\ \\ \frac{1}{m} \text{ if } m_2 = m_1 = m. \end{cases}$$

1. Introduction

In 1952, in the paper [13], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.
Key words and phrases. Positive operators, Trace class operators, Determinants, Inequalities.

RGMIA Res. Rep. Coll. **25** (2022), Art. 71, 18 pp. Received 01/07/22

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [19], [20], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [23].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \leq \|T\| \, \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A|\, e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \ \text{and} \ \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \ \mid B \in \mathcal{B}_2(H), \ \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P\ge 0$ and $P\in\mathcal{B}_1(H)$, then for all $T\in\mathcal{B}(H)$, PT, $TP\in\mathcal{B}_1(H)$ and $\operatorname{tr}(PT)=\operatorname{tr}(TP)$. Also, since $P^{1/2}\in\mathcal{B}_2(H)$, $TP^{1/2}\in\mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2}=TP\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)=\operatorname{tr}(TP)$. Therefore, if $P\ge 0$ and $P\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(PT)=\operatorname{tr}(TP)=\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T\in\mathcal{B}(H)$.

If $T\ge 0$, then $P^{1/2}TP^{1/2}\ge 0$, which implies that $\operatorname{tr}(PT)\ge 0$ that shows that the functional $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n\to T$ for $n\to\infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty}\operatorname{tr}(PT_n)=\operatorname{tr}(PT)$, namely $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [3]-[10] and the references therein.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A)=\exp\operatorname{tr}((\ln A)P)=\exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. We observe that we have the following elementary properties [11]:

(i) *continuity*: the map $A\to\Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t)=\Delta_P(A)^t$ for all $t>0$;
(iii) *homogeneity*: $\Delta_P(tA)=t\Delta_P(A)$ and $\Delta_P(t1_H)=t$ for all $t>0$;
(iv) *monotonicity*: $0<A\le B$ implies $\Delta_P(A)\le\Delta_P(B)$.

In [11], we presented some fundamental properties of this determinant. Among others we showed that

$$1\le\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\le\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]$$

and

$$1\le\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\le\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right],$$

for $A>0$ and $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

For the entropy function $\eta(t)=-t\ln t$, $t>0$, the operator entropy has the following expression:

$$\eta(A)=-A\ln A$$

for positive A.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *entropic P-determinant* of the positive invertible operator A by [12]

$$\eta_P(A):=\exp\left[-\operatorname{tr}(PA\ln A)\right]=\exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\}=\exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right)\\
&=\exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t+\ln A\right)\right]\right\}\right)=\exp\left(-\operatorname{tr}\left\{P\left(tA\ln t+tA\ln A\right)\right\}\right)\\
&=\exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right)\\
&=\exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA)=t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_P(1_H)=1 \text{ and } \eta_P(t1_H)=t^{-t} \tag{1.14}$$

for $t>0$.

Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If A, $B>0$, then we have the Ky Fan type inequality [12]

$$\eta_P((1-t)A+tB)\geq\left[\eta_P(A)\right]^{1-t}\left[\eta_P(B)\right]^{t} \tag{1.15}$$

for all $t\in[0,1]$.

Also we have the inequalities [12]:

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1$$

and if there exists the constants $0<m<M$ such that $m1_H\leq A\leq M1_H$, then [12]

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M}&\leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)}\leq\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\\
&\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1.
\end{aligned}$$

Kamei and Fujii [17], [18] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B):=A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [27]. For various results on relative operator entropy see [14]-[28] and the references therein.

Definition 1. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}
D_P(A|B)&:=\exp\{\operatorname{tr}\left[PS\left(A|B\right)\right]\}\\
&=\exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.
\end{aligned}$$

We observe that for $A>0$,

$$D_P(A|1_H):=\exp\{\operatorname{tr}[PS(A|1_H)]\}=\exp\{\operatorname{tr}(-PA\ln A)\}=\eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B>0$,

$$D_P(1_H|B):=\exp\{\operatorname{tr}[PS(1_H|B)]\}=\exp\{\operatorname{tr}(P\ln B)\}=\Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

Motivated by the above results, in this paper we show among others that, if $C\geq m_1A>0$, $B\geq m_2A>0$, $A>0$ and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then

$$\begin{aligned}&\exp\left(-\Phi(m_1,m_2)\operatorname{tr}(PA)\left\|A^{-1/2}(B-C)A^{-1/2}\right\|\right)\\&\leq\frac{D_P(A|B)}{D_P(A|C)}\\&\leq\exp\left(\Phi(m_1,m_2)\operatorname{tr}(PA)\left\|A^{-1/2}(B-C)A^{-1/2}\right\|\right),\end{aligned}$$

where

$$\Phi(m_1,m_2):=\begin{cases}\frac{\ln m_2-\ln m_1}{m_2-m_1} \text{ if } m_2\neq m_1,\\ \\ \frac{1}{m} \text{ if } m_2=m_1=m.\end{cases}$$

2. Main Results

We can state the following representation result that is of interest in itself. In order to simplify the notations, instead of $\lambda 1_H$ with scalar λ, we write just λ.

Lemma 1. *For all T, $V>0$ we have*

$$\begin{aligned}&\ln V-\ln T\\&=\int_0^\infty\left[(\lambda+T)^{-1}-(\lambda+V)^{-1}\right]d\lambda\\&=\int_0^\infty\left(\int_0^1(\lambda+(1-t)T+tV)^{-1}(V-T)(\lambda+(1-t)T+tV)^{-1}dt\right)d\lambda.\end{aligned}\tag{2.1}$$

Proof. Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)}\tag{2.2}$$

for all $t>0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T=\int_0^\infty\frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}d\lambda\tag{2.3}$$

for all operators $T>0$.

We have from (2.3) for $T,\ V>0$ that

$$\ln V-\ln T=\int_0^\infty \frac{1}{\lambda+1}\left[(V-1)(\lambda+V)^{-1}-(T-1)(\lambda+T)^{-1}\right]d\lambda. \tag{2.4}$$

Since

$$\begin{aligned}&(V-1)(\lambda+V)^{-1}-(T-1)(\lambda+T)^{-1}\\&=V(\lambda+V)^{-1}-T(\lambda+T)^{-1}-\left((\lambda+V)^{-1}-(\lambda+T)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&V(\lambda+V)^{-1}-T(\lambda+T)^{-1}\\&=(V+\lambda-\lambda)(\lambda+V)^{-1}-(T+\lambda-\lambda)(\lambda+T)^{-1}\\&=1-\lambda(\lambda+V)^{-1}-1+\lambda(\lambda+T)^{-1}=\lambda(\lambda+T)^{-1}-\lambda(\lambda+V)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(V-1)(\lambda+V)^{-1}-(T-1)(\lambda+T)^{-1}\\&=\lambda(\lambda+T)^{-1}-\lambda(\lambda+V)^{-1}-\left((\lambda+V)^{-1}-(\lambda+T)^{-1}\right)\\&=(\lambda+1)\left[(\lambda+T)^{-1}-(\lambda+V)^{-1}\right]\end{aligned}$$

and by (2.4) we get

$$\ln V-\ln T=\int_0^\infty\left[(\lambda+T)^{-1}-(\lambda+V)^{-1}\right]d\lambda, \tag{2.5}$$

we proves the first equality in (2.1).

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt.$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt. \tag{2.6}$$

Now, if we take in (2.6) $C=\lambda+T$, $D=\lambda+V$, then

$$\begin{aligned}&(\lambda+T)^{-1}-(\lambda+V)^{-1}\\&=\int_0^1((1-t)(\lambda+T)+t(\lambda+V))^{-1}(V-T)\\&\times((1-t)(\lambda+T)+t(\lambda+V))^{-1}dt\\&=\int_0^1(\lambda+(1-t)T+tV)^{-1}(V-T)(\lambda+(1-t)T+tV)^{-1}dt.\end{aligned} \tag{2.7}$$

By employing (2.7) and (2.5) we derive the desired result (2.1). □

Theorem 4. *Assume that* A, B, $C>0$ *and* $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$, *then*

$$\begin{aligned}(2.8)\quad \frac{D_P(A|B)}{D_P(A|C)} &= \exp\left(\int_0^\infty \left\{\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+A^{-1/2}CA^{-1/2}\right)^{-1}\right]\right.\right.\\ &\qquad \left.\left.-\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+A^{-1/2}BA^{-1/2}\right)^{-1}\right]\right\}d\lambda\right)\\ &= \exp\int_0^\infty\int_0^1 \operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+A^{-1/2}BA^{-1/2}\right)^{-1}\right.\\ &\qquad\times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right)\\ &\qquad\left.\times\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right]dtd\lambda\Bigg).\end{aligned}$$

Proof. If we take $V=A^{-1/2}BA^{-1/2}$ and $T=A^{-1/2}CA^{-1/2}$, then we get the identity

$$\begin{aligned}(2.9)\quad &\ln\left(A^{-1/2}BA^{-1/2}\right)-\ln\left(A^{-1/2}CA^{-1/2}\right)\\ &=\int_0^\infty\left[\left(\lambda+A^{-1/2}CA^{-1/2}\right)^{-1}-\left(\lambda+A^{-1/2}BA^{-1/2}\right)^{-1}\right]d\lambda\\ &=\int_0^\infty\int_0^1\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+A^{-1/2}BA^{-1/2}\right)^{-1}\\ &\times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right)\\ &\times\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}dtd\lambda.\end{aligned}$$

We multiply both sides by $A^{1/2}$, then we get

$$\begin{aligned}&A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}-A^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)A^{1/2}\\ &=\int_0^\infty\left[A^{1/2}\left(\lambda+A^{-1/2}CA^{-1/2}\right)^{-1}A^{1/2}-A^{1/2}\left(\lambda+A^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2}\right]d\lambda\\ &=\int_0^\infty\int_0^1 A^{1/2}\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+A^{-1/2}BA^{-1/2}\right)^{-1}\\ &\times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right)\\ &\times\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2}dtd\lambda.\end{aligned}$$

If we multiply both sides of this equality by $P^{1/2}$, take the trace and use its properties, then we get

$$\begin{aligned}
(2.10)\quad & \operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]-\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)\right]\\
&=\int_0^\infty\left\{\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+A^{-1/2}CA^{-1/2}\right)^{-1}\right]\right.\\
&\left.-\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+A^{-1/2}BA^{-1/2}\right)^{-1}\right]\right\}d\lambda\\
&=\int_0^\infty\int_0^1\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+(1-t)\,A^{-1/2}CA^{-1/2}+A^{-1/2}BA^{-1/2}\right)^{-1}\right.\\
&\times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right)\\
&\left.\times\left(\lambda+(1-t)\,A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right]dtd\lambda.
\end{aligned}$$

If we take the exponential, then we get the desired identity (2.11). □

Corollary 1. *Assume that B, $C>0$ and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then*

$$\begin{aligned}
(2.11)\quad \frac{\Delta_P(B)}{\Delta_P(C)}&=\exp\int_0^\infty\left(\operatorname{tr}\left[P\,(\lambda+C)^{-1}\right]-\operatorname{tr}\left[P\,(\lambda+B)^{-1}\right]\right)d\lambda\\
&=\exp\int_0^\infty\int_0^1\operatorname{tr}\left[P\,(\lambda+(1-t)\,C+B)^{-1}\,(B-C)\right.\\
&\left.\times(\lambda+(1-t)\,C+tB)^{-1}\right]dtd\lambda.
\end{aligned}$$

It follows by (2.8) for $A=1_H$.

Theorem 5. *Assume that $C\geq m_1A>0$, $B\geq m_2A>0$, $A>0$ and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then*

$$\begin{aligned}
(2.12)\quad &\exp\left(-\Phi(m_1,m_2)\operatorname{tr}(PA)\left\|A^{-1/2}(B-C)\,A^{-1/2}\right\|\right)\\
&\leq\frac{D_P(A|B)}{D_P(A|C)}\\
&\leq\exp\left(\Phi(m_1,m_2)\operatorname{tr}(PA)\left\|A^{-1/2}(B-C)\,A^{-1/2}\right\|\right),
\end{aligned}$$

where

$$\Phi(m_1,m_2):=\begin{cases}\frac{\ln m_2-\ln m_1}{m_2-m_1} & \text{if } m_2\neq m_1,\\[2mm] \frac{1}{m} & \text{if } m_2=m_1=m.\end{cases}$$

Proof. If we take the modulus in (2.10), then we get

$$\begin{aligned}
(2.13)\quad & \left|\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]-\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)\right]\right| \\
& \leq \int_0^\infty \int_0^1 \left|\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+A^{-1/2}BA^{-1/2}\right)^{-1}\right.\right. \\
& \times \left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right) \\
& \left.\left.\times \left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right]\right| dtd\lambda \\
& \leq \left\|A^{1/2}PA^{1/2}\right\|_1 \left\|A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right\| \\
& \times \int_0^\infty \int_0^1 \left\|\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right\|^2 dtd\lambda \\
& = \operatorname{tr}(PA)\left\|A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right\| \\
& \times \int_0^\infty \int_0^1 \left\|\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right\|^2 dtd\lambda
\end{aligned}$$

Since $C \geq m_1 A > 0$, $B \geq m_2 A > 0$, then $A^{-1/2}CA^{-1/2} \geq m_1$ and $A^{-1/2}BA^{-1/2} \geq m_2$.

Assume that $m_2 > m_1$. Then

$$(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}+\lambda \geq (1-t)m_1+tm_2+\lambda,$$

which implies that

$$\begin{aligned}
& \left((1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}+\lambda\right)^{-1} \\
& \leq ((1-t)m_1+tm_2+\lambda)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
& \left\|\left((1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}+\lambda\right)^{-1}\right\|^2 \\
& \leq ((1-t)m_1+tm_2+\lambda)^{-2}
\end{aligned}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore

$$\begin{aligned}
(2.14)\quad & \left|\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]-\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)\right]\right| \\
& \leq \operatorname{tr}(PA)\left\|A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right\| \\
& \times \int_0^\infty \left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-2} dt\right) d\lambda \\
& = \frac{\operatorname{tr}(PA)\left\|A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right\|}{m_2-m_1} \\
& \times \int_0^\infty \left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-1}\right. \\
& \left.\times (m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1} dt\right) d\lambda,
\end{aligned}$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

If we use the identity (2.1) for $T = m_1$, $V = m_2$ we get the scalar identity

$$\ln m_2 - \ln m_1 = \int_0^\infty \left(\int_0^1 ((1-t) m_1 + t m_2 + \lambda)^{-1} (m_2 - m_1) \times ((1-t) m_1 + t m_2 + \lambda)^{-1} dt \right) d\lambda$$

and by (2.14) we obtain

$$\left| \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} B A^{-1/2} \right) \right] - \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} C A^{-1/2} \right) \right] \right| \leq \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \operatorname{tr}(PA) \left\| A^{-1/2} B A^{-1/2} - A^{-1/2} C A^{-1/2} \right\|$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

The case $m_2 < m_1$ goes in a similar way.

Now, assume that $C \geq mA > 0$, $B \geq mA > 0$. Let $\epsilon > 0$, then $A^{-1/2} B A^{-1/2} + \epsilon \geq m + \epsilon$. Put $m_2 = m + \epsilon > m = m_1$. If we write the inequality (2.13) for $A^{-1/2} B A^{-1/2} + \epsilon$ and A, we get

$$\left| \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} B A^{-1/2} \right) \right] - \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} C A^{-1/2} \right) \right] \right| \leq \frac{\ln(m+\epsilon) - \ln m}{\epsilon} \operatorname{tr}(PA) \left\| A^{-1/2} B A^{-1/2} + \epsilon - A^{-1/2} C A^{-1/2} \right\|$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

If we take the limit over $\epsilon \to 0+$ and observe that

$$\lim_{\epsilon \to 0+} \frac{\ln(m+\epsilon) - \ln m}{\epsilon} = \frac{1}{m},$$

then we get

$$\left| \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} B A^{-1/2} \right) \right] - \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} C A^{-1/2} \right) \right] \right| \leq \frac{1}{m} \operatorname{tr}(PA) \left\| A^{-1/2} B A^{-1/2} + 1 - A^{-1/2} C A^{-1/2} \right\|$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

Therefore

$$\begin{aligned} &- \Phi(m_1, m_2) \operatorname{tr}(PA) \left\| A^{-1/2} B A^{-1/2} - A^{-1/2} C A^{-1/2} \right\| \\ &\leq \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} B A^{-1/2} \right) \right] - \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln \left(A^{-1/2} C A^{-1/2} \right) \right] \\ &\leq \Phi(m_1, m_2) \operatorname{tr}(PA) \left\| A^{-1/2} B A^{-1/2} - A^{-1/2} C A^{-1/2} \right\| \end{aligned}$$

for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, which, by taking the exponential, gives the desired result (2.12). □

Corollary 2. *Assume that $C \geq m_1 > 0$, $B \geq m_2 > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then*

$$\exp(-\Phi(m_1, m_2) \|B - C\|) \leq \frac{\Delta_P(B)}{\Delta_P(C)} \leq \exp(\Phi(m_1, m_2) \|B - C\|). \tag{2.15}$$

The proof follows by Theorem 5 for $A = 1_H$.

Theorem 6. *Assume that for $A>0$, $0<mA\leq B-C\leq MA$ and $0<\gamma A\leq C\leq \Gamma A$ for some constants m, M, γ and Γ, then*

$$1<\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}\operatorname{tr}(PA)}\leq \frac{D_P\left(A|B\right)}{D_P\left(A|C\right)}\leq \left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}\operatorname{tr}(PA)} \tag{2.16}$$

for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

Proof. Since $0<mA\leq B-C\leq MA$, hence by multiplying both sides by $A^{-1/2}$ then

$$0<m1_H\leq A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\leq M1_H$$

by multiplying both sides by $\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}>0$ we derive

$$\begin{aligned} &m\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-2} \\ &\leq \left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1} \\ &\times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right) \\ &\times\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1} \\ &\leq M\left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-2} \end{aligned} \tag{2.17}$$

for all $t\in[0,1]$ and $\lambda>0$.

Observe that

$$\begin{aligned} &(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2} \\ &=A^{-1/2}CA^{-1/2}+t\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right), \end{aligned}$$

and since $\gamma\leq A^{-1/2}CA^{-1/2}\leq\Gamma$, hence

$$\begin{aligned} \lambda+\gamma+tm &\leq \lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2} \\ &\leq \lambda+\Gamma+tM, \end{aligned}$$

namely,

$$\begin{aligned} (\lambda+\Gamma+tM)^{-1} &\leq \left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1} \\ &\leq (\lambda+\gamma+tm)^{-1}, \end{aligned}$$

which gives that

$$\begin{aligned} (\lambda+\Gamma+tM)^{-2} &\leq \left(\lambda+(1-t)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-2} \\ &\leq (\lambda+\gamma+tm)^{-2} \end{aligned} \tag{2.18}$$

for all $t\in[0,1]$ and $\lambda>0$.

By utilizing (2.17) and (2.18), we derive

$$\begin{aligned} & m\left(\lambda+\Gamma+tM\right)^{-2} \\ & \le \left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1} \\ & \left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right) \\ & \left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1} \\ & \le M\left(\lambda+\gamma+tm\right)^{-2} \end{aligned} \tag{2.19}$$

for all $t\in[0,1]$ and $\lambda>0$.

If we multiply both sides by $A^{1/2}$and then by $P^{1/2}$ we get

$$\begin{aligned} & mP^{1/2}A^{1/2}\left(\lambda+\Gamma+tM\right)^{-2}A^{1/2}P^{1/2} \\ & \le P^{1/2}A^{1/2}\left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1} \\ & \times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right) \\ & \times\left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2}P^{1/2} \\ & \le MP^{1/2}A^{1/2}\left(\lambda+\gamma+tm\right)^{-2}A^{1/2}P^{1/2} \end{aligned}$$

and by taking the trace, we derive

$$\begin{aligned} & m\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+\Gamma+tM\right)^{-2}\right] \\ & \le \operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right. \\ & \times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right) \\ & \left.\times\left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right] \\ & \le M\operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+\gamma+tm\right)^{-2}\right] \end{aligned} \tag{2.20}$$

for all $t\in[0,1]$ and $\lambda>0$.

This is equivalent to

$$\begin{aligned} & m\left(\lambda+\Gamma+tM\right)^{-2}\operatorname{tr}\left(A^{1/2}PA^{1/2}\right) \\ & \le \operatorname{tr}\left[A^{1/2}PA^{1/2}\left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right. \\ & \times\left(A^{-1/2}BA^{-1/2}-A^{-1/2}CA^{-1/2}\right) \\ & \left.\times\left(\lambda+\left(1-t\right)A^{-1/2}CA^{-1/2}+tA^{-1/2}BA^{-1/2}\right)^{-1}\right] \\ & \le M\left(\lambda+\gamma+tm\right)^{-2}\operatorname{tr}\left(A^{1/2}PA^{1/2}\right) \end{aligned} \tag{2.21}$$

for all $t\in[0,1]$ and $\lambda>0$.

If we take the integrals in (2.21), then we get

$$
\begin{aligned}
& m \operatorname{tr}\left(A^{1/2} P A^{1/2}\right) \int_0^\infty \int_0^1 (\lambda + \Gamma + tM)^{-2} \, dt d\lambda \\
& \leq \int_0^\infty \int_0^1 \operatorname{tr}\left[A^{1/2} P A^{1/2} \left(\lambda + (1-t) A^{-1/2} C A^{-1/2} + t A^{-1/2} B A^{-1/2}\right)^{-1}\right. \\
& \times \left(A^{-1/2} B A^{-1/2} - A^{-1/2} C A^{-1/2}\right) \\
& \times \left.\left(\lambda + (1-t) A^{-1/2} C A^{-1/2} + t A^{-1/2} B A^{-1/2}\right)^{-1}\right] dt d\lambda \\
& \leq M \operatorname{tr}\left(A^{1/2} P A^{1/2}\right) \int_0^\infty \int_0^1 (\lambda + \gamma + tm)^{-2} \, dt d\lambda
\end{aligned}
$$

namely, by (2.10)

$$
\begin{aligned}
& m \operatorname{tr}\left(A^{1/2} P A^{1/2}\right) \int_0^\infty \left(\int_0^1 (\lambda + \Gamma + tM)^{-2} \, dt\right) d\lambda \\
& \leq \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln\left(A^{-1/2} B A^{-1/2}\right)\right] \\
& - \operatorname{tr}\left[A^{1/2} P A^{1/2} \ln\left(A^{-1/2} C A^{-1/2}\right)\right] \\
& \leq M \operatorname{tr}\left(A^{1/2} P A^{1/2}\right) \int_0^\infty \left(\int_0^1 (\lambda + \gamma + tm)^{-2} \, dt\right) d\lambda.
\end{aligned} \tag{2.22}
$$

Observe that

$$
\begin{aligned}
\int_0^1 (\lambda + \gamma + tm)^{-2} \, dt & = -\frac{1}{m} (\lambda + \gamma + m)^{-1} + \frac{1}{m} (\lambda + \gamma)^{-1} \\
& = \frac{1}{m} \left((\lambda + \gamma)^{-1} - (\lambda + \gamma + m)^{-1}\right),
\end{aligned}
$$

which gives

$$
M \int_0^\infty \left(\int_0^1 (\lambda + \gamma + tm)^{-2} \, dt\right) d\lambda = \frac{M}{m} \int_0^\infty \left((\lambda + \gamma)^{-1} - (\lambda + \gamma + m)^{-1}\right) d\lambda.
$$

By the first identity in (2.1) in the scalar case, we have

$$
\ln(\gamma + m) - \ln \gamma = \int_0^\infty \left[(\lambda + \gamma)^{-1} - (\lambda + \gamma + m)^{-1}\right] d\lambda
$$

and then

$$
\begin{aligned}
M \int_0^\infty \left(\int_0^1 (\lambda + \gamma + tm)^{-2} \, dt\right) d\lambda & = M \frac{\ln(\gamma + m) - \ln \gamma}{m} \\
& = \ln\left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
m \int_0^\infty \left(\int_0^1 (\lambda + \Gamma + tM)^{-2} \, dt\right) d\lambda & = m \frac{\ln(\Gamma + M) - \ln \Gamma}{M} \\
& = \ln\left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}}
\end{aligned}
$$

and by (2.22) we get

$$\begin{aligned}
&\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\operatorname{tr}(PA)\\
&\le \operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]-\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)\right]\\
&\le \ln\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}\operatorname{tr}(PA).
\end{aligned}$$

By taking the exponential, we derive

$$\begin{aligned}
1<\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}\operatorname{tr}(PA)} &\le \frac{\exp\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right]}{\exp\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)\right]}\\
&\le \left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}\operatorname{tr}(PA)}
\end{aligned}$$

for all $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$ and the inequality (2.14) is obtained. □

Corollary 3. *Assume that* $0<m\le B-C\le M$ *and* $0<\gamma\le C\le\Gamma$ *for some constants* m, M, γ *and* Γ, *then*

$$1<\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\frac{\Delta_P(B)}{\Delta_P(C)}\le\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}} \tag{2.23}$$

for all $P\ge 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$.

3. Related Results

Let C and B be strictly positive operators on a Hilbert space H such that $B-C\ge m>0$. In 2015, [21], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$f(B)-f(C)\ge f(\|C\|+m)-f(\|C\|)\ge f(\|B\|)-f(\|B\|-m)>0. \tag{3.1}$$

If $B>C>0$, then

$$\begin{aligned}
f(B)-f(C) &\ge f\left(\|C\|+\frac{1}{\left\|(B-C)^{-1}\right\|}\right)-f(\|C\|)\\
&\ge f(\|B\|)-f\left(\|B\|-\frac{1}{\left\|(B-C)^{-1}\right\|}\right)>0.
\end{aligned} \tag{3.2}$$

The inequality between the first and third term in (3.2) was obtained earlier by H. Zuo and G. Duan in [31].

If we write the inequality (3.1) for $f(t)=\ln t$, then we get for $B-C\ge m>0$

$$\ln B-\ln C\ge\ln\left(\frac{\|C\|+m}{\|C\|}\right)\ge\ln\left(\frac{\|B\|}{\|B\|-m}\right)>0. \tag{3.3}$$

If $B > C > 0$, then by (3.2) we get

$$\ln B - \ln C \geq \ln\left(\|C\| + \frac{1}{\left\|(B-C)^{-1}\right\|}\right) - \ln(\|C\|) \geq \ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-C)^{-1}\right\|}\right) > 0. \tag{3.4}$$

Proposition 2. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that $B - C \geq mA > 0$ with $m > 0$, then*

$$\frac{D_P(A|B)}{D_P(A|C)} \geq \left(\frac{\|A^{-1/2}CA^{-1/2}\| + m}{\|A^{-1/2}CA^{-1/2}\|}\right)^{\operatorname{tr}(PA)} \geq \left(\frac{\|A^{-1/2}BA^{-1/2}\|}{\|A^{-1/2}BA^{-1/2}\| - m}\right)^{\operatorname{tr}(PA)} \geq 1 \tag{3.5}$$

If $B > C > 0$, then

$$\frac{D_P(A|B)}{D_P(A|C)} \geq \left(\frac{\|A^{-1/2}CA^{-1/2}\| \left\|A^{1/2}(B-C)^{-1}A^{1/2}\right\| + 1}{\left\|A^{1/2}(B-C)^{-1}A^{1/2}\right\| \|A^{-1/2}CA^{-1/2}\|}\right)^{\operatorname{tr}(PA)} \geq \left(\frac{\|A^{-1/2}BA^{-1/2}\| \left\|A^{1/2}(B-C)^{-1}A^{1/2}\right\|}{\|A^{-1/2}BA^{-1/2}\| \left\|A^{1/2}(B-C)^{-1}A^{1/2}\right\| - 1}\right) \geq 1. \tag{3.6}$$

Proof. Since $B - C \geq mA > 0$, hence by multiplying both sides by $A^{-1/2} > 0$, we get $A^{-1/2}BA^{-1/2} - A^{-1/2}CA^{-1/2} \geq m1_H > 0$ and by (3.3) we get

$$\ln\left(A^{-1/2}BA^{-1/2}\right) - \ln\left(A^{-1/2}CA^{-1/2}\right) \geq \ln\left(\frac{\|A^{-1/2}CA^{-1/2}\| + m}{\|A^{-1/2}CA^{-1/2}\|}\right) \geq \ln\left(\frac{\|A^{-1/2}BA^{-1/2}\|}{\|A^{-1/2}BA^{-1/2}\| - m}\right) > 0.$$

If we multiply both sides by $A^{1/2} > 0$, then by $P^{1/2} \geq 0$ and take the trace, we get

$$\begin{aligned}
&\operatorname{tr}\left(P^{1/2}A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}P^{1/2}\right) \\
&- \operatorname{tr}\left(P^{1/2}A^{1/2}\ln\left(A^{-1/2}CA^{-1/2}\right)A^{1/2}P^{1/2}\right) \\
&\geq \ln\left(\frac{\|A^{-1/2}CA^{-1/2}\| + m}{\|A^{-1/2}CA^{-1/2}\|}\right)\operatorname{tr}\left(P^{1/2}AP^{1/2}\right) \\
&\geq \ln\left(\frac{\|A^{-1/2}BA^{-1/2}\|}{\|A^{-1/2}BA^{-1/2}\| - m}\right)\operatorname{tr}\left(P^{1/2}AP^{1/2}\right) > 0.
\end{aligned}$$

By taking the exponential, we get (3.5).

If $A^{-1/2}BA^{-1/2} > A^{-1/2}CA^{-1/2} > 0$, then by (3.4) we get

$$\begin{aligned}
&\ln A^{-1/2}BA^{-1/2} - \ln A^{-1/2}CA^{-1/2} \\
&\geq \ln\left(\left\|A^{-1/2}CA^{-1/2}\right\| + \frac{1}{\left\|A^{1/2}(B-C)^{-1}A^{1/2}\right\|}\right) \\
&- \ln\left(\left\|A^{-1/2}CA^{-1/2}\right\|\right) \\
&\geq \ln\left(\left\|A^{-1/2}BA^{-1/2}\right\|\right) \\
&- \ln\left(\left\|A^{-1/2}BA^{-1/2}\right\| - \frac{1}{\left\|A^{1/2}(B-C)^{-1}A^{1/2}\right\|}\right) \\
&> 0.
\end{aligned}$$

If we multiply both sides by $A^{1/2} > 0$, then by $P^{1/2} \geq 0$ and take the trace, we get (3.6). □

Corollary 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that $B - C \geq m > 0$ with $m > 0$, then*

$$\frac{\Delta_P(B)}{\Delta_P(C)} \geq \frac{\|C\| + m}{\|C\|} \geq \frac{\|B\|}{\|B\| - m} \geq 1 \tag{3.7}$$

If $B > C > 0$, then

$$\frac{\Delta_P(B)}{\Delta_P(C)} \geq \frac{\|C\|\left\|(B-C)^{-1}\right\| + 1}{\left\|(B-C)^{-1}\right\|\|C\|} \geq \frac{\|B\|\left\|(B-C)^{-1}\right\|}{\|B\|\left\|(B-C)^{-1}\right\| - 1} \geq 1. \tag{3.8}$$

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