ON THE SUB-MULTIPLICATIVE PROPERTY FOR THE RELATIVE ENTROPIC NORMALIZED P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by

$$D_P(A|B) = \exp\left\{\operatorname{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.$$

Assume that A, B, $C > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. In this paper we show among others that, if $BA^{-1}C + CA^{-1}B \geq 0$, then

$$D_P(A|B+C+A) \leq D_P(A|B+A)\, D_P(A|C+A).$$

If $B + C \leq \Theta A$, with Θ a positive constant, then

$$\frac{D_P(A|B+A)\, D_P(A|C+A)}{D_P(A|B+C+A)} \leq \exp\left[\Theta^2 \operatorname{tr}\left(A^{1/2}PA^{1/2}(B+C+A)^{-1}\right)\right].$$

1. Introduction

In 1952, in the paper [13], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

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Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and 1_H stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [19], [20], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [24].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \text{ and } \ |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For recent results on trace inequalities see [3]-[10] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties [11]:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(t1_H) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In [11], we presented some fundamental properties of this determinant. Among others we showed that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *entropic P-determinant* of the positive invertible operator A by [12]

$$\eta_P(A) := \exp\left[-\operatorname{tr}(PA \ln A)\right] = \exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\} = \exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A) P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA \ln(tA)\right]\right\}\right) \\
&= \exp\left(-\operatorname{tr}\left\{P\left[tA(\ln t + \ln A)\right]\right\}\right) = \exp\left(-\operatorname{tr}\left\{P\left(tA \ln t + tA \ln A\right)\right\}\right) \\
&= \exp\left(-t \ln t \operatorname{tr}(PA)\right) \exp\left(-t \operatorname{tr}(PA \ln A)\right) \\
&= \exp \ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}(PA \ln A)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\,\mathrm{tr}(PA)} [\eta_P(A)]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(1_H) = 1 \text{ and } \eta_P(t1_H) = t^{-t} \tag{1.14}$$

for $t > 0$.

Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\mathrm{tr}(P) = 1$. If A, $B > 0$, then we have the Ky Fan type inequality [12]

$$\eta_P((1-t)A + tB) \geq [\eta_P(A)]^{1-t} [\eta_P(B)]^t \tag{1.15}$$

for all $t \in [0,1]$.

Also we have the inequalities [12]:

$$\left[\frac{\mathrm{tr}\left(PA^2\right)}{\mathrm{tr}^2(PA)}\right]^{-\mathrm{tr}(PA)} \leq \frac{\eta_P(A)}{[\mathrm{tr}(PA)]^{-\mathrm{tr}(PA)}} \leq 1$$

and if there exists the constants $0 < m < M$ such that $m1_H \leq A \leq M1_H$, then [12]

$$\begin{aligned}\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\,\mathrm{tr}(PA)} \leq \left[\frac{\mathrm{tr}\left(PA^2\right)}{\mathrm{tr}^2(PA)}\right]^{-\mathrm{tr}(PA)} \\ &\leq \frac{\eta_P(A)}{[\mathrm{tr}(PA)]^{-\mathrm{tr}(PA)}} \leq 1.\end{aligned}$$

Kamei and Fujii [17], [18] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}} \left(\ln\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)\right) A^{\frac{1}{2}}, \tag{1.16}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [28]. For various results on relative operator entropy see [14]-[29] and the references therein.

Definition 1. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\mathrm{tr}(P) = 1$. For positive invertible operators A, B we define the relative entropic normalized P-determinant by*

$$\begin{aligned}D_P(A|B) &:= \exp\{\mathrm{tr}[PS(A|B)]\} \\ &= \exp\left\{\mathrm{tr}\left[PA^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}\right]\right\}.\end{aligned}$$

We observe that for $A > 0$,

$$D_P(A|1_H) := \exp\{\mathrm{tr}[PS(A|1_H)]\} = \exp\{\mathrm{tr}(-PA\ln A)\} = \eta_P(A),$$

where $\eta_P(\cdot)$ is the *entropic P-determinant* and for $B > 0$,

$$D_P(1_H|B) := \exp\{\mathrm{tr}[PS(1_H|B)]\} = \exp\{\mathrm{tr}(P\ln B)\} = \Delta_P(B),$$

where $\Delta_P(\cdot)$ is the *P-determinant.*

Assume that A, B, $C > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\mathrm{tr}(P) = 1$. In this paper we show among others that, if $BA^{-1}C + CA^{-1}B \geq 0$, then

$$D_P(A|B+C+A) \leq D_P(A|B+A)\, D_P(A|C+A).$$

If $B+C\leq\Theta A$, with Θ a positive constant, then

$$\frac{D_P\left(A|B+A\right)D_P\left(A|C+A\right)}{D_P\left(A|B+C+A\right)}\leq\exp\left[\Theta^2\operatorname{tr}\left(A^{1/2}PA^{1/2}\left(B+C+A\right)^{-1}\right)\right].$$

2. Some Preliminary Facts

To simplify the notations, instead of $a1_H$ we write a when this is a scalar. The following representation result holds:

Lemma 1. *For all U, $B\geq 0$ and $a>0$ we have*

$$\begin{aligned}&\ln\left(U+a\right)+\ln\left(V+a\right)-\ln\left(U+V+a\right)-\ln a\\&=\int_0^\infty\left(a+\lambda\right)^{-1}S\left(\lambda,a,U,V\right)d\lambda+\int_0^\infty\left(a+\lambda\right)^{-1}Q\left(\lambda,a,U,V\right)d\lambda,\end{aligned}\tag{2.1}$$

where

$$S\left(\lambda,a,U,V\right):=\left(U+V+a+\lambda\right)^{-1}\left(UV+VU\right)\left(U+V+a+\lambda\right)^{-1}$$

and

$$\begin{aligned}Q\left(\lambda,a,U,V\right):=&\left(U+V+a+\lambda\right)^{-1}\\&\times\left[V\left(U+a+\lambda\right)^{-1}UV+U\left(V+a+\lambda\right)^{-1}VU\right]\\&\times\left(U+V+a+\lambda\right)^{-1}\end{aligned}$$

for $\lambda>0$.

Proof. Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{\left(\lambda+t\right)\left(\lambda+1\right)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{\left(\lambda+t\right)\left(\lambda+1\right)},$$

which gives the representation for the logarithm

$$\ln t=\left(t-1\right)\int_0^\infty\frac{d\lambda}{\left(\lambda+1\right)\left(\lambda+t\right)}\tag{2.2}$$

for all $t>0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T=\int_0^\infty\frac{1}{\lambda+1}\left(T-1\right)\left(\lambda+T\right)^{-1}d\lambda\tag{2.3}$$

for all operators $T>0$.

Observe that

$$\begin{aligned}\int_0^\infty\frac{1}{\lambda+1}\left(T-1\right)\left(\lambda+T\right)^{-1}d\lambda&=\int_0^\infty\frac{1}{\lambda+1}\left(T+\lambda-\lambda-1\right)\left(\lambda+T\right)^{-1}d\lambda\\&=\int_0^\infty\left[\left(\lambda+1\right)^{-1}-\left(\lambda+T\right)^{-1}\right]d\lambda\end{aligned}$$

and then

$$\ln T = \int_0^\infty \left[(\lambda + 1)^{-1} - (\lambda + T)^{-1} \right] d\lambda.$$

Therefore

$$\ln(U + a) + \ln(V + a) - \ln(U + V + a) - \ln a = \int_0^\infty K_\lambda d\lambda \tag{2.4}$$

where

$$K_\lambda := (U + V + a + \lambda)^{-1} + (a + \lambda)^{-1} - (U + a + \lambda)^{-1} - (V + a + \lambda)^{-1}.$$

To simplify calculations, consider $\delta := a + \lambda$ and set

$$L_\delta := (U + V + \delta)^{-1} + \delta^{-1} - (U + \delta)^{-1} - (V + \delta)^{-1}.$$

If we multiply both sides by $U + V + \delta$ we get

$$\begin{aligned} W_\delta &:= (U + V + \delta) L_\delta (U + V + \delta) \\ &= (U + V + \delta) + \delta^{-1} (U + V + \delta)^2 \\ &- (U + V + \delta)(U + \delta)^{-1} (U + V + \delta) \\ &- (U + V + \delta)(V + \delta)^{-1} (U + V + \delta) \\ &= (U + V + \delta) + \delta^{-1} (U + V + \delta)^2 \\ &- (U + V + \delta) - V (U + \delta)^{-1} (U + V + \delta) \\ &- U (V + \delta)^{-1} (U + V + \delta) - (U + V + \delta) \\ &= \delta^{-1} (U + V + \delta)^2 - V (U + \delta)^{-1} V - V \\ &- U (V + \delta)^{-1} U - U - (U + V + \delta) \end{aligned}$$

$$\begin{aligned} &= \delta^{-1} \left(U^2 + UV + \delta U + VU + V^2 + \delta V + \delta U + \delta V + \delta^2 \right) \\ &- V (U + \delta)^{-1} V - 2V - U (V + \delta)^{-1} U - 2U - \delta \\ &= \delta^{-1} \left(U^2 + UV + VU + V^2 \right) + 2V + 2U + \delta \\ &- V (U + \delta)^{-1} V - U (V + \delta)^{-1} U - 2U - 2V - \delta \\ &= \delta^{-1} \left(U^2 + UV + VU + V^2 \right) - V (U + \delta)^{-1} V - U (V + \delta)^{-1} U \\ &= \delta^{-1} \left[U^2 + UV + VU + V^2 - \delta V (U + \delta)^{-1} V - \delta U (V + \delta)^{-1} U \right] \\ &= \delta^{-1} \left[U^2 + UV + VU + V^2 - V \left(\delta^{-1} U + 1 \right)^{-1} V - U \left(\delta^{-1} V + 1 \right)^{-1} U \right]. \end{aligned}$$

Observe that

$$\begin{aligned} &V^2 - V \left(\delta^{-1} U + 1 \right)^{-1} V \\ &= V \left(\delta^{-1} U + 1 \right)^{-1} \left(\delta^{-1} U + 1 \right) V - V \left(\delta^{-1} U + 1 \right)^{-1} V \\ &= V \left(\delta^{-1} U + 1 \right)^{-1} \left(\delta^{-1} U + 1 - 1 \right) V \\ &= \delta^{-1} V \left(\delta^{-1} U + 1 \right)^{-1} UV = V (U + \delta)^{-1} UV \end{aligned}$$

and

$$\begin{aligned}&U^2-U\left(\delta^{-1}V+1\right)^{-1}U\\&=U\left(\delta^{-1}V+1\right)^{-1}\left(\delta^{-1}V+1\right)U-U\left(\delta^{-1}V+1\right)^{-1}U\\&=U\left(\delta^{-1}V+1\right)^{-1}\left(\delta^{-1}V+1-1\right)U\\&=\delta^{-1}U\left(\delta^{-1}V+1\right)^{-1}VU=U\left(V+\delta\right)^{-1}VU.\end{aligned}$$

Therefore

$$W_\delta=\delta^{-1}\left[UV+VU+V\left(U+\delta\right)^{-1}UV+U\left(V+\delta\right)^{-1}VU\right]$$

which gives that

$$L_\delta:=\left(U+V+\delta\right)^{-1}W_\delta\left(U+V+\delta\right)^{-1}.$$

We obtain then the following representation

$$\begin{aligned}K_\lambda&=(a+\lambda)^{-1}\left(U+V+a+\lambda\right)^{-1}\left(UV+VU\right)\left(U+V+a+\lambda\right)^{-1}\\&+(a+\lambda)^{-1}\left(U+V+a+\lambda\right)^{-1}\\&\times\left[V\left(U+a+\lambda\right)^{-1}UV+U\left(V+a+\lambda\right)^{-1}VU\right]\left(U+V+a+\lambda\right)^{-1}\\&=(a+\lambda)^{-1}S\left(\lambda,a,U,V\right)+(a+\lambda)^{-1}P\left(\lambda,a,U,V\right)\end{aligned}\tag{2.5}$$

for $a,\ \lambda>0$.

By utilizing (2.4) and (2.5) we derive the representation (2.1). □

Corollary 1. *For all* $U,\ V\geq 0$ *we have*

$$\begin{aligned}&\ln\left(U+1\right)+\ln\left(V+1\right)-\ln\left(U+V+1\right)\\&=\int_0^\infty(1+\lambda)^{-1}S\left(\lambda,U,V\right)d\lambda+\int_0^\infty(1+\lambda)^{-1}Q\left(\lambda,U,V\right)d\lambda,\end{aligned}\tag{2.6}$$

where

$$S\left(\lambda,U,V\right):=\left(U+V+1+\lambda\right)^{-1}\left(UV+VU\right)\left(U+V+1+\lambda\right)^{-1}$$

and

$$\begin{aligned}Q\left(\lambda,a,U,V\right):=&\left(U+V+1+\lambda\right)^{-1}\\&\times\left[V\left(U+1+\lambda\right)^{-1}UV+U\left(V+1+\lambda\right)^{-1}VU\right]\\&\times\left(U+V+1+\lambda\right)^{-1}\end{aligned}$$

for $\lambda>0$.

We have the following operator inequalities:

Theorem 4. *For all* $U,\ V>0$ *and* $a>0$ *we have*

$$\begin{aligned}&\int_0^\infty(a+\lambda)^{-1}S\left(\lambda,a,U,V\right)d\lambda\\&\leq\ln\left(U+a\right)+\ln\left(V+a\right)-\ln\left(U+V+a\right)-\ln a\\&\leq\int_0^\infty(a+\lambda)^{-1}R\left(\lambda,a,U,V\right)d\lambda,\end{aligned}\tag{2.7}$$

where

$$R\left(\lambda,a,U,V\right)=\left(U+V+a+\lambda\right)^{-1}\left(U+V\right)^2\left(U+V+a+\lambda\right)^{-1}$$

for $\lambda \geq 0$.

In particular,

$$\int_0^{\infty} (1+\lambda)^{-1} S(\lambda, U, V)\, d\lambda \tag{2.8}$$
$$\leq \ln(U+1) + \ln(V+1) - \ln(U+V+1)$$
$$\leq \int_0^{\infty} (1+\lambda)^{-1} R(\lambda, U, V)\, d\lambda,$$

where

$$R(\lambda, U, V) = (U+V+1+\lambda)^{-1} (U+V)^2 (U+V+1+\lambda)^{-1}$$

for $\lambda \geq 0$.

Proof. Assume that U, $V \geq 0$. Observe that for a, $\lambda > 0$

$$(U+a+\lambda)^{-1} U = (U+a+\lambda)^{-1} (U+a+\lambda-a-\lambda)$$
$$= 1 - (a+\lambda)(U+a+\lambda)^{-1},$$

which shows that

$$0 \leq (U+a+\lambda)^{-1} U \leq 1.$$

If we multiply this inequality both sides by V, then we get

$$0 \leq V(U+a+\lambda)^{-1} UV \leq V^2.$$

Similarly,

$$0 \leq U(V+a+\lambda)^{-1} VU \leq U^2.$$

Therefore

$$0 \leq V(U+a+\lambda)^{-1} UV + U(V+a+\lambda)^{-1} VU \leq U^2 + V^2$$

and by multiplying both sides by $(U+V+1+\lambda)^{-1}$ we deduce

$$0 \leq Q(\lambda, a, U, V) \leq (U+V+a+\lambda)^{-1} \left(U^2+V^2\right) (U+V+a+\lambda)^{-1}$$

for a, $\lambda > 0$.

Now, if to this inequality we add $S(\lambda, a, U, V)$, then we obtain

$$S(\lambda, a, U, V) \leq Q(\lambda, a, U, V) + S(\lambda, a, U, V) \tag{2.9}$$
$$\leq (U+V+a+\lambda)^{-1} \left(U^2+V^2\right) (U+V+a+\lambda)^{-1}$$
$$+ (U+V+a+\lambda)^{-1} (UV+VU) (U+V+a+\lambda)^{-1}$$
$$= (U+V+a+\lambda)^{-1} (U+V)^2 (U+V+a+\lambda)^{-1}$$
$$= R(\lambda, a, U, V)$$

for a, $\lambda > 0$.

If we multiply (2.9) by $(1+\lambda)^{-1} > 0$, integrate over λ on $[0, \infty)$ and use representation (2.1) we derive (2.7). □

Corollary 2. *Let* U, $V > 0$ *and* $a > 0$.

(i) *If* $UV + VU \geq 0$, *then*

$$\ln(U+V+a) + \ln a \leq \ln(U+a) + \ln(V+a). \tag{2.10}$$

In particular,

$$\ln(U+V+1) \leq \ln(U+1) + \ln(V+1). \tag{2.11}$$

(ii) *If* $U+V\leq\Omega$, *with* Ω *a positive constant, then*

$$\ln(U+a)+\ln(V+a)-\ln(U+V+a)-\ln a\leq\frac{\Omega^2}{a}(U+V+a)^{-1}. \tag{2.12}$$

In particular,

$$\ln(U+1)+\ln(V+1)-\ln(U+V+1)\leq\Omega^2(U+V+1)^{-1}. \tag{2.13}$$

Proof. (i) If $UV+VU\geq 0$, then by multiplying both sides by $(U+V+a+\lambda)^{-1}$ we get

$$0\leq(U+V+a+\lambda)^{-1}(UV+VU)(U+V+a+\lambda)^{-1}$$

for a, $\lambda>0$, which implies that

$$\begin{aligned}0&\leq\int_0^\infty(a+\lambda)^{-1}(U+V+a+\lambda)^{-1}\\&\quad\times(UV+VU)(U+V+a+\lambda)^{-1}d\lambda\\&=\int_0^\infty(a+\lambda)^{-1}S(\lambda,a,U,V)\,d\lambda\end{aligned}$$

and by (2.7) we get (2.10).

(ii) If $U+V\leq\Omega$, then

$$(U+V+a+\lambda)^{-1}(U+V)^2(U+V+a+\lambda)^{-1}\leq\Omega^2(U+V+a+\lambda)^{-2}$$

for a, $\lambda>0$. This implies that

$$\begin{aligned}&\int_0^\infty(a+\lambda)^{-1}(U+V+a+\lambda)^{-1}(U+V)^2(U+V+a+\lambda)^{-1}d\lambda \tag{2.14}\\&\leq\Omega^2\int_0^\infty(a+\lambda)^{-1}(U+V+a+\lambda)^{-2}d\lambda\\&\leq\frac{\Omega^2}{a}\int_0^\infty(U+V+a+\lambda)^{-2}d\lambda.\end{aligned}$$

Now, if we take the derivative over t in (2.2), then we get

$$\begin{aligned}t^{-1}&=\int_0^\infty(\lambda+1)^{-1}\left(\frac{t-1}{\lambda+t}\right)'d\lambda\\&=\int_0^\infty(\lambda+1)^{-1}\frac{\lambda+1}{(\lambda+t)^2}d\lambda=\int_0^\infty(\lambda+t)^{-2}d\lambda.\end{aligned}$$

This gives that

$$\int_0^\infty(U+V+a+\lambda)^{-2}d\lambda=(U+V+a)^{-1}$$

and by (2.14) and (2.7) we obtain (2.12). □

3. Main Results

We also have the following representation result:

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, B, $C > 0$ and $a > 0$ we have the representation*

$$\begin{aligned} &\frac{D_P(A|B+aA)\, D_P(A|C+aA)}{a^{\operatorname{tr}(PA)} D_P(A|B+C+aA)} \\ &= \int_0^\infty (a+\lambda)^{-1} \operatorname{tr}\left[A^{1/2}PA^{1/2}Y(A,B,C,a,\lambda)\right] d\lambda \\ &+ \int_0^\infty (a+\lambda)^{-1} \operatorname{tr}\left[A^{1/2}PA^{1/2}Z(A,B,C,a,\lambda)\right] d\lambda, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &Y(A,B,C,a,\lambda) \\ &:= A^{1/2}(B+C+(a+\lambda)A)^{-1}\left(BA^{-1}C+CA^{-1}B\right) \\ &\times (B+C+(a+\lambda)A)^{-1}A^{1/2} \end{aligned}$$

and

$$\begin{aligned} &Z(A,B,C,a,\lambda) \\ &:= A^{1/2}(B+C+(a+\lambda)A)^{-1} \\ &\times \left[C(B+(a+\lambda)A)^{-1}BA^{-1}C + B(C+(a+\lambda)A)^{-1}CA^{-1}B\right] \\ &\times (B+C+(a+\lambda)A)^{-1}A^{1/2}. \end{aligned}$$

Proof. From (2.1) we have for $U = A^{-1/2}BA^{-1/2}$ and $V = A^{-1/2}CA^{-1/2}$ that

$$\begin{aligned} &\ln\left(A^{-1/2}(B+aA)A^{-1/2}\right) + \ln\left(A^{-1/2}(C+aA)A^{-1/2}\right) \\ &- \ln\left(A^{-1/2}(B+C+aA)A^{-1/2}\right) - \ln a \\ &= \int_0^\infty (a+\lambda)^{-1} S\left(\lambda, a, A^{-1/2}BA^{-1/2}, A^{-1/2}CA^{-1/2}\right) d\lambda \\ &+ \int_0^\infty (a+\lambda)^{-1} Q\left(\lambda, a, A^{-1/2}BA^{-1/2}, A^{-1/2}CA^{-1/2}\right) d\lambda, \end{aligned}$$

where

$$\begin{aligned} &S\left(\lambda, a, A^{-1/2}BA^{-1/2}, A^{-1/2}CA^{-1/2}\right) \\ &= \left(A^{-1/2}BA^{-1/2} + A^{-1/2}CA^{-1/2} + a + \lambda\right)^{-1} \\ &\times \left(A^{-1/2}BA^{-1/2}A^{-1/2}CA^{-1/2} + A^{-1/2}CA^{-1/2}A^{-1/2}BA^{-1/2}\right) \\ &\times \left(A^{-1/2}BA^{-1/2} + A^{-1/2}CA^{-1/2} + a + \lambda\right)^{-1} \\ &= A^{1/2}(B+C+(a+\lambda)A)^{-1}A^{1/2}A^{-1/2}\left(BA^{-1}C+CA^{-1}B\right)A^{-1/2} \\ &\times A^{1/2}(B+C+(a+\lambda)A)^{-1}A^{1/2} \\ &= A^{1/2}(B+C+(a+\lambda)A)^{-1}\left(BA^{-1}C+CA^{-1}B\right) \\ &\times (B+C+(a+\lambda)A)^{-1}A^{1/2} \\ &= Y(A,B,C,a,\lambda) \end{aligned}$$

and

$$
\begin{aligned}
&Q\left(\lambda, a, A^{-1/2}BA^{-1/2}, A^{-1/2}CA^{-1/2}\right) \\
&= \left(A^{-1/2}BA^{-1/2} + A^{-1/2}CA^{-1/2} + a + \lambda\right)^{-1} \\
&\times \left[A^{-1/2}CA^{-1/2}\left(A^{-1/2}BA^{-1/2} + a + \lambda\right)^{-1} A^{-1/2}BA^{-1/2}A^{-1/2}CA^{-1/2}\right. \\
&\left. + A^{-1/2}BA^{-1/2}\left(A^{-1/2}CA^{-1/2} + a + \lambda\right)^{-1} A^{-1/2}CA^{-1/2}A^{-1/2}BA^{-1/2}\right] \\
&\times \left(A^{-1/2}BA^{-1/2} + A^{-1/2}CA^{-1/2} + a + \lambda\right)^{-1} \\
&= A^{1/2}\left(B + C + (a+\lambda)A\right)^{-1} A^{1/2} \\
&\times \left[A^{-1/2}CA^{-1/2}A^{1/2}\left(B + (a+\lambda)A\right)^{-1} A^{1/2}A^{-1/2}BA^{-1/2}A^{-1/2}CA^{-1/2}\right. \\
&\left. + A^{-1/2}BA^{-1/2}A^{1/2}\left(C + (a+\lambda)A\right)^{-1} A^{1/2}A^{-1/2}CA^{-1/2}A^{-1/2}BA^{-1/2}\right] \\
&\times A^{1/2}\left(B + C + (a+\lambda)A\right)^{-1} A^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= A^{1/2}\left(B + C + (a+\lambda)A\right)^{-1} \\
&\times \left[C\left(B + (a+\lambda)A\right)^{-1} BA^{-1}C + B\left(C + (a+\lambda)A\right)^{-1} CA^{-1}B\right] \\
&\times \left(B + C + (a+\lambda)A\right)^{-1} A^{1/2} \\
&= Z(A, B, C, a, \lambda).
\end{aligned}
$$

If we multiply both sides of (2.1) by $A^{1/2}$ and then by $P^{1/2}$, then we get

$$
\begin{aligned}
&P^{1/2}A^{1/2}\ln\left(A^{-1/2}(B + aA)A^{-1/2}\right)A^{1/2}P^{1/2} \\
&+ P^{1/2}A^{1/2}\ln\left(A^{-1/2}(C + aA)A^{-1/2}\right)A^{1/2}P^{1/2} \\
&- P^{1/2}A^{1/2}\ln\left(A^{-1/2}(B + C + aA)A^{-1/2}\right)A^{1/2}P^{1/2} - \ln a P^{1/2}AP^{1/2} \\
&= \int_0^\infty (a+\lambda)^{-1} P^{1/2}A^{1/2}Y(A, B, C, a, \lambda)A^{1/2}P^{1/2}d\lambda \\
&+ \int_0^\infty (a+\lambda)^{-1} P^{1/2}A^{1/2}Z(A, B, C, a, \lambda)A^{1/2}P^{1/2}d\lambda.
\end{aligned}
$$

If we take the trace and use its properties, then we get

$$
\begin{aligned}
&\operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}(B + aA)A^{-1/2}\right)\right] \\
&+ \operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}(C + aA)A^{-1/2}\right)\right] \\
&- \operatorname{tr}\left[A^{1/2}PA^{1/2}\ln\left(A^{-1/2}(B + C + aA)A^{-1/2}\right)\right] - \ln a \operatorname{tr}(PA) \\
&= \int_0^\infty (a+\lambda)^{-1} \operatorname{tr}\left[A^{1/2}PA^{1/2}Y(A, B, C, a, \lambda)\right]d\lambda \\
&+ \int_0^\infty (a+\lambda)^{-1} \operatorname{tr}\left[A^{1/2}PA^{1/2}Z(A, B, C, a, \lambda)\right]d\lambda.
\end{aligned} \tag{3.2}
$$

Further, if we take the exponential in (3.2), then we get the desired result (3.1). □

Corollary 3. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, B, and $C > 0$ we have the representation*

$$\frac{D_P(A|B+A)\,D_P(A|C+A)}{D_P(A|B+C+A)} = \int_0^\infty (1+\lambda)^{-1} \operatorname{tr}\left[A^{1/2}PA^{1/2}Y(A,B,C,\lambda)\right] d\lambda + \int_0^\infty (1+\lambda)^{-1} \operatorname{tr}\left[A^{1/2}PA^{1/2}Z(A,B,C,\lambda)\right] d\lambda, \tag{3.3}$$

where

$$Y(A,B,C,\lambda) := A^{1/2}(B+C+(1+\lambda)A)^{-1}\left(BA^{-1}C+CA^{-1}B\right) \times (B+C+(1+\lambda)A)^{-1}A^{1/2}$$

and

$$Z(A,B,C,\lambda) := A^{1/2}(B+C+(1+\lambda)A)^{-1} \times \left[C(B+(1+\lambda)A)^{-1}BA^{-1}C + B(C+(1+\lambda)A)^{-1}CA^{-1}B\right] \times (B+C+(1+\lambda)A)^{-1}A^{1/2}.$$

Remark 1. *If we take $A = 1_H$ in Theorem 5, then we get*

$$\frac{\Delta_P(B+a)\,\Delta_P(C+a)}{a\Delta_P(B+C+a)} = \int_0^\infty (a+\lambda)^{-1} \operatorname{tr}[PY(B,C,a,\lambda)]\,d\lambda + \int_0^\infty (a+\lambda)^{-1} \operatorname{tr}[PZ(AB,C,a,\lambda)]\,d\lambda, \tag{3.4}$$

where

$$Y(B,C,a,\lambda) := (B+C+a+\lambda)^{-1}(BC+CB)(B+C+a+\lambda)^{-1}$$

and

$$Z(B,C,a,\lambda) := (B+C+a+\lambda)^{-1} \times \left[C(B+a+\lambda)^{-1}BC + B(C+a+\lambda)^{-1}CB\right] \times (B+C+a+\lambda)^{-1}.$$

In particular, we have

$$\frac{\Delta_P(B+1)\,\Delta_P(C+1)}{\Delta_P(B+C+1)} = \int_0^\infty (1+\lambda)^{-1} \operatorname{tr}[PY(B,C,\lambda)]\,d\lambda + \int_0^\infty (1+\lambda)^{-1} \operatorname{tr}[PZ(AB,C,\lambda)]\,d\lambda, \tag{3.5}$$

where

$$Y(B,C,\lambda) := (B+C+1+\lambda)^{-1}(BC+CB)(B+C+1+\lambda)^{-1}$$

and

$$Z(B,C,\lambda) := (B+C+1+\lambda)^{-1} \times \left[C(B+1+\lambda)^{-1}BC + B(C+1+\lambda)^{-1}CB\right] \times (B+C+1+\lambda)^{-1}.$$

We have the following super-multiplicative properties:

Theorem 6. *Assume that A, B, $C>0$ and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.*

(i) *If $BA^{-1}C+CA^{-1}B\geq 0$, then*

$$D_P(A|B+C+A)\leq D_P(A|B+A)\,D_P(A|C+A). \tag{3.6}$$

(ii) *If $B+C\leq\Theta A$, with Θ a positive constant, then*

$$\frac{D_P(A|B+A)\,D_P(A|C+A)}{D_P(A|B+C+A)} \leq \exp\left[\Theta^2\operatorname{tr}\left(A^{1/2}PA^{1/2}(B+C+A)^{-1}\right)\right]. \tag{3.7}$$

The proof follows by Corollary 2 and Theorem 5 and the details are omitted.

Corollary 4. *Assume that B, $C>0$ and and $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.*

(i) *If $BC+CB\geq 0$, then*

$$\Delta_P(B+C+1)\leq\Delta_P(B+1)\,\Delta_P(C+1). \tag{3.8}$$

(ii) *If $B+C\leq\Theta$, with Θ a positive constant, then*

$$\frac{\Delta_P(B+1)\,\Delta_P(C+1)}{\Delta_P(B+C+1)}\leq\exp\left[\Theta^2\operatorname{tr}\left(P(B+C+1)^{-1}\right)\right]. \tag{3.9}$$

The symmetrized product of two operators $C, B\in B(H)$ is defined by $S(C,B)=CB+BC$. In general, the symmetrized product of two operators C, B is not positive. Also Gustafson [23] showed that if $0\leq m\leq C\leq M$ and $0\leq n\leq B\leq N$, then we have the lower bound

$$S(A,B)\geq 2mn-\frac{1}{4}(M-m)(N-n), \tag{3.10}$$

which can take positive or negative values depending on the parameters m, M, n, N.

So, if $0\leq m\leq C\leq M$ and $0\leq n\leq B\leq N$ with

$$8mn\geq(M-m)(N-n),$$

then by (3.8) we get that

$$\Delta_P(B+C+1)\leq\Delta_P(B+1)\,\Delta_P(C+1), \tag{3.11}$$

for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

If $0\leq m\leq C\leq M$ and $0\leq n\leq B\leq N$, then $C+B\leq M+N$ and $(B+C+1)^{-1}\leq(m+n+1)^{-1}$ and by (3.9) we also obtain that

$$\frac{\Delta_P(B+1)\,\Delta_P(C+1)}{\Delta_P(B+C+1)}\leq\exp\left(\frac{(M+N)^2}{(m+n+1)}\right), \tag{3.12}$$

for all $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa