SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR S-CONVEX FUNCTIONS

LOREDANA CIURDARIU

Abstract. In this paper an identity is presented in order to establish several Hermite-Hadamard type inequalities for functions whose powers of absolute values of third derivatives are s-convex. Some consequences are also presented.

1. Introduction

The convex analysis has an important role in mathematics and in many other fields such as numerical analysis, convex programming, statistics and approximation theory. The classical inequality of Hermite-Hadamard was extended and generalized in many directions in recent years by many authors, like for example, [9, 8, 1, 12, 11, 15, 10, 5, 2, 3, 13, 4, 16, 17] and the references therein.

We begin by recalling below the classical definition for the convex functions and then for s-convex functions([5], [8],[6],[7]).

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be s-convex if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

for each $x, y \in \mathbb{R}$ and $t \in (0,1)$, $s \in (0,1]$.

The classical Hermite-Hadamard's inequality for convex functions, see [14] is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

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The aim of this paper is to give several Hermite-Hadamard type inequalities for functions whose powers of absolute values of third derivatives are s-convex. For this goal an identity is presented as a main tool in the demonstrations of these results.

2. Several Hermite-Hadamard type inequalities for convex functions

Starting from a result from [2], the aim of this section is to present some Hermite-Hadamard type inequalities for functions whose powers of absolute values of third derivatives are s-convex .

Lemma 1. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, then for all $x \in I^0$ the following inequality takes place:*

$$\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du =$$

$$= \frac{1}{6(b-a)} \int_0^1 t(1-t)^2[(x-a)^4 f^{'''}(tx+(1-t)a) - (x-b)^4 f^{'''}(tx+(1-t)b)]dt$$

Proof. It will be denoted $I_1 = \int_0^1 t(1-t)^2(x-a)^4 f^{'''}(tx+(1-t)a)dt$ and $I_2 = \int_0^1 t(1-t)^2(x-b)^4 f^{'''}(tx+(1-t)b)$. By integrating by parts three times I_1 and I_2 we get,

$$I_1 = -(x-a)^3 \int_0^1 (1-4t+3t^2) f^{''}(tx+(1-t)a)dt =$$

$$= (x-a)^2 f^{'}(a) + 2f(x)(x-a) + 4f(a)(x-a) - 6(x-a)\int_0^1 f(tx+(1-t)a)dt$$

and here by using $u = tx + (1-t)a$, it is obtained

$$I_1 = (x-a)^2 f^{'}(a) + 2f(x)(x-a) + 4f(a)(x-a) - 6\int_a^x f(u)du,$$

and

$$I_2 = (x-b)^2 f^{'}(b) + 2f(x)(x-b) + 4f(b)(x-b) + 6\int_x^b f(v)dv,$$

where $v = tx + (1-t)b$. Now substracting I_2 from I_1, we have,

$$I_1 - I_2 = (x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2(b-a)f(x) + 4[(x-a)f(a) - (x-b)f(b)] - 6\int_a^b f(u)du,$$

and dividing by $6(b-a)$ last equality the desired inequality is obtained.

□

Theorem 1. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, and $|f^{'''}|$ is convex on $[a,b]$, then for all $x \in I^0$ the following inequality is satisfied:*

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{1}{60(b-a)}[\frac{(x-a)^4 |f^{'''}(a) + (x-b)^4 |f^{'''}(b)|}{2} + \frac{(x-a)^4 + (x-b)^4}{3} |f^{'''}(x)|].$$

Proof. It will be used Lemma 1, the definition of convex functions for $|f^{'''}|$ and the properties of the Gamma and Beta functions. We will have then

$$|\frac{(x-a)^2 f^{'}(a)-(x-b)^2 f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du| \le$$

$$\le \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)a)|dt+\frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)b)|dt \le$$

$$\le \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2[t|f^{'''}(x)|+(1-t)|f^{'''}(a)|]dt+$$

$$+\frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2[t|f^{'''}(x)|+(1-t)|f^{'''}(b)|]dt \le$$

$$\le \frac{(x-a)^4}{6(b-a)}[|f^{'''}(x)|\int_0^1 t^2(1-t)^2dt+|f^{'''}(a)|\int_0^1 t(1-t)^3dt]+$$

$$+\frac{(x-b)^4}{6(b-a)}[|f^{'''}(x)|\int_0^1 t^2(1-t)^2dt+|f^{'''}(b)|\int_0^1 t(1-t)^3dt]=$$

$$=\frac{(x-a)^4+(x-b)^4}{6(b-a)}|f^{'''}(x)|B(3,3)+\frac{(x-a)^4|f^{'''}(a)|+(x-b)^4|f^{'''}(b)|}{6(b-a)}B(2,4)=$$

$$=\frac{1}{60(b-a)}[\frac{(x-a)^4+(x-b)^4}{3}|f^{'''}(x)|+\frac{(x-a)^4|f^{'''}(a)|+(x-b)^4|f^{'''}(b)|}{2}]$$

where $B(x,y)$ is the Beta function, $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x>0,\ y>0$ and the Gamma function is $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ x>0$. □

Corollary 1. *Let $f: I^0\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^0, a,b\in I^0$ with $a<b$. If $f\in C^3(\mathbb{R})$, $f^{'''}\in L[a,b]$, and $|f^{'''}|$ is convex on $[a,b]$, then the following inequality is holds:*

$$|\frac{b-a}{24}[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du| \le$$

$$\le \frac{(b-a)^3}{960}\{\frac{1}{2}[|f^{'''}(a)|+|f^{'''}(b)|]+\frac{2}{3}|f^{'''}(\frac{a+b}{2})|].$$

Proof. We put in previous theorem $x=\frac{a+b}{2}$. □

Corollary 2. *As in Corollary 1, by using the convexity of $|f^{'''}|$, we can obtain further,*

$$|\frac{b-a}{24}[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du| \le$$

$$\le \frac{(b-a)^3}{1152}\{|f^{'''}(a)|+|f^{'''}(b)|\}.$$

Theorem 2. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, with $a, b \in I^0$ and $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, and $|f^{'''}|$ is s-convex on $[a,b]$, then for all $x \in I^0$ the following inequality takes place:*

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{1}{6(b-a)}\{(x-a)^4[2\frac{|f^{'''}(x)|}{(s+4)(s+3)(s+2)} + \frac{|f^{''}(a)|}{(s+4)(s+3)}] +$$
$$+(x-b)^4[2\frac{|f^{'''}(x)|}{(s+4)(s+3)(s+2)} + \frac{|f^{'''}(b)|}{(s+4)(s+3)}]\}.$$

Proof. The method of demonstration is analogue to Theorem 1 but it is used the definition of s-convexity instead of convexity for function $|f^{'''}|^q$. □

Corollary 3. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, and $|f^{'''}|$ is s-convex on $[a,b]$, then we get the following:*

$$|\frac{b-a}{24}[f^{'}(a) - f^{'}(b)] + \frac{1}{3}[f(a) + f(b) + f(\frac{a+b}{2})] - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{(b-a)^3}{6}\{[2\frac{|f^{'''}(\frac{a+b}{2})|}{(s+4)(s+3)(s+2)} + \frac{|f^{''}(a)|}{(s+4)(s+3)}] +$$
$$+[2\frac{|f^{'''}(\frac{a+b}{2})|}{(s+4)(s+3)(s+2)} + \frac{|f^{'''}(b)|}{(s+4)(s+3)}]\}.$$

Theorem 3. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, and $|f^{'''}|^q$ is s-convex on $[a,b]$, $q > 1$ then for all $x \in I^0$ the following inequality holds:*

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{1}{6(b-a)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(x-a)^4[2\frac{|f^{'''}(x)|^q}{(s+4)(s+3)(s+2)} + \frac{|f^{''}(a)|^q}{(s+4)(s+3)}]^{\frac{1}{q}} +$$
$$+(x-b)^4[2\frac{|f^{'''}(x)|^q}{(s+4)(s+3)(s+2)} + \frac{|f^{'''}(b)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}.$$

Proof. Using Lemma 1, Holder's inequality and that $|f^{'''}|^q$ is s-convex function we have succesively next inequalities,

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)a)|dt + \frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)b)|dt =$$

$$= \frac{(x-a)^4}{6(b-a)}\int_0^1 [t(1-t)^2]^{1-\frac{1}{q}}[t(1-t)^2]^{\frac{1}{q}}|f^{'''}(tx+(1-t)a)|dt +$$
$$+\frac{(x-b)^4}{6(b-a)}\int_0^1 [t(1-t)^2]^{1-\frac{1}{q}}[t(1-t)^2]^{\frac{1}{q}}|f^{'''}(tx+(1-t)b)|dt \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-b)^4}{6(b-a)}\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}[B(2,3)]^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2[t^s|f^{'''}(x)|^q+(1-t)^s|f^{'''}(a)|^q]dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-b)^4}{6(b-a)}[B(2,3)]^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2[t^s|f^{'''}(x)|^q+(1-t)^s|f^{'''}(b)|^q]dt\right)^{\frac{1}{q}}=$$

$$=\frac{\left(\frac{1}{12}\right)^{1-\frac{1}{q}}}{6(b-a)}\{(x-a)^4[|f^{''}(x)|^qB(s+2,3)+|f^{'''}(a)|^qB(2,s+3)]^{\frac{1}{q}}+$$

$$+(x-b)^4[|f^{''}(x)|^qB(s+2,3)+|f^{'''}(b)|^qB(2,s+3)]^{\frac{1}{q}}\}.$$

Now by taking into account that $B(s+2,3)=\frac{2}{(s+4)(s+3)(s+2)}$ and $B(2,s+3)=\frac{1}{(s+4)(s+3)}$ we find the inequality from conclusion.

□

Corollary 4. *Let $f: I^0\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^0, a, b\in I^0$ with $a<b$. If $f\in C^3(\mathbb{R})$, $f^{'''}\in L[a,b]$, and $|f^{'''}|^q$ is s-convex on $[a,b]$, $q>1$ then the next inequality holds:*

$$|\frac{1}{24}(b-a)[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du|\leq$$

$$\leq\frac{(b-a)^3}{96}\frac{1}{12^{1-\frac{1}{q}}}\{[2\frac{|f^{'''}(\frac{a+b}{2})|^q}{(s+4)(s+3)(s+2)}+\frac{|f^{'''}(a)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}+$$

$$+[2\frac{|f^{'''}(\frac{a+b}{2})|^q}{(s+4)(s+3)(s+2)}+\frac{|f^{'''}(b)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}\}.$$

Proof. We put $x=\frac{a+b}{2}$ in Theorem 2. □

Theorem 4. *Suppose that $f: I^0\subset\mathbb{R}\to\mathbb{R}$ is a differentiable mapping on $I^0, a, b\in I^0$ with $a<b$. If $f\in C^3(\mathbb{R})$, $f^{'''}\in L[a,b]$, and $|f^{'''}|^q$ is s-convex on $[a,b]$, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ then we have the following inequality:*

$$|\frac{(x-a)^2f^{'}(a)-(x-b)^2f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq\frac{p^{\frac{1}{p}}\sqrt{\pi}^{\frac{1}{p}}}{b-a}\frac{2}{81}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{p}}\frac{\Gamma(p)\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}\frac{1}{(s+1)^{\frac{1}{q}}}\{(x-a)^4[|f^{'''}(x)|^q+|f^{'''}(a)|^q]^{\frac{1}{q}}+$$

$$+(x-b)^4[|f^{'''}(x)|^q+|f^{'''}(b)|^q]^{\frac{1}{q}}\}.$$

Proof. Like in the proof of Theorem 1 we have,

$$|\frac{(x-a)^2 f^{'}(a)-(x-b)^2 f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du|\leq$$

$$\leq\frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)a)|dt+\frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)b)|dt.$$

In view of Holder's inequality and s-convexity for $|f^{'''}|^q$ we get,

$$|\frac{(x-a)^2 f^{'}(a)-(x-b)^2 f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du|\leq$$

$$\leq\frac{(x-a)^4}{6(b-a)}\left(\int_0^1[t(1-t)^2]^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f^{'''}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-b)^4}{6(b-a)}\left(\int_0^1[t(1-t)^2]^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f^{'''}(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\leq$$

$$\leq\frac{B^{\frac{1}{p}}(p+1,2p+1)}{6(b-a)}\{(x-a)^4[|f^{'''}(x)|^q\int_0^1 t^s dt+|f^{'''}(a)|^q\int_0^1(1-t)^s dt]^{\frac{1}{q}}+$$

$$+(x-b)^4[|f^{'''}(x)|^q\int_0^1 t^s dt+|f^{'''}(b)|^q\int_0^1(1-t)^s dt]^{\frac{1}{q}}\}=$$

$$=\frac{B^{\frac{1}{p}}(p+1,2p+1)}{6(b-a)}\{(x-a)^4[\frac{|f^{'''}(x)|^q+|f^{'''}(a)|^q}{s+1}]^{\frac{1}{q}}+$$

$$+(x-b)^4[\frac{|f^{'''}(x)|^q+|f^{'''}(b)|^q}{s+1}]^{\frac{1}{q}}\}.$$

On the other hand, by Gauss multiplication formula,

$$\Gamma(z)\prod_{k=1}^{n-1}\Gamma(z+\frac{k}{n})=n^{\frac{1}{2}-nz}(2\pi)^{(n-1)/2}\Gamma(nz)$$

we get

$$\Gamma(3p)=\frac{\Gamma(p)\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}{2\pi 3^{\frac{1}{2}-3p}}$$

for $n=3$ and by Legendre's duplication formula for the Gamma function,

$$\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\Gamma(2z),\ Rez>0$$

, we have,

$$B(p+1,2p+1)=\frac{2p\Gamma(p)\Gamma(2p)}{3(3p+1)\Gamma(3p)}=p\sqrt{\pi}\frac{\Gamma(p)\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}\frac{2^{2p+1}}{3^{\frac{1}{2}+3p}}.$$

Last expression will be replaced in last inequality and the proof will be finished. □

Corollary 5. *Under conditions of previous theorem, if we take* $x = \frac{a+b}{2}$ *the following inequality holds:*

$$|\frac{1}{24}(b-a)[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq p^{\frac{1}{p}}\sqrt{\pi}^{\frac{1}{p}}\frac{2}{81}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{p}}\frac{\Gamma(p)\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}\frac{(b-a)^3}{(s+1)^{\frac{1}{q}}}\{[|f^{'''}(\frac{a+b}{2})|^q+|f^{'''}(a)|^q]^{\frac{1}{q}}+$$

$$+[|f^{'''}(\frac{a+b}{2})|^q+|f^{'''}(b)|^q]^{\frac{1}{q}}\}.$$

Theorem 5. *Suppose that* $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping on* $I^0, a, b \in I^0$ *with* $a < b$. *If* $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, *and* $|f^{'''}|^q$ *is s-convex on* $[a,b]$, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$ *then we have the following inequality:*

$$|\frac{(x-a)^2f^{'}(a)-(x-b)^2f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{1}{6(b-a)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\{(x-a)^4[|f^{'''}(x)|^qB(2q+1,s+1)+\frac{|f^{'''}(a)|^q}{2q+s+1}]^{\frac{1}{q}}+$$

$$+(x-b)^4[|f^{'''}(x)|^qB(2q+1,s+1)+\frac{|f^{'''}(b)|^q}{2q+s+1}]^{\frac{1}{q}}\}.$$

Proof. In view of s-convexity of $|f^{'''}|^q$ and Holder's inequality, the first inequality from the proof of Theorem becomes,

$$|\frac{(x-a)^2f^{'}(a)-(x-b)^2f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{1}{6(b-a)}\{(x-a)^4\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{2q}|f^{'''}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+(x-b)^4\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{2q}|f^{'''}(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\} \leq$$

$$\leq \frac{1}{6(b-a)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\{(x-a)^4[\int_0^1 (1-t)^{2q}\left(t^s|f^{'''}(x)|^q+(1-t)^s|f^{'''}(a)|^q\right)dt]^{\frac{1}{q}}+$$

$$+(x-b)^4[\int_0^1 (1-t)^{2q}\left(t^s|f^{'''}(x)|^q+(1-t)^s|f^{'''}(b)|^q\right)dt]^{\frac{1}{q}}\}$$

which leads to desired inequality, by taking into account the definition of Beta function $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt,\ p,q > 0$.

□

Corollary 6. *Under conditions of previous theorem, if we take* $x = \frac{a+b}{2}$ *we have,*

$$|\frac{1}{24}(b-a)[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{(b-a)^3}{6}\{[|f^{'''}(\frac{a+b}{2})|^qB(2q+1,s+1)+\frac{|f^{'''}(a)|^q}{2q+s+1}]^{\frac{1}{q}}+$$

$$+[|f^{'''}(\frac{a+b}{2})|^q B(2q+1,s+1)+\frac{|f^{'''}(b)|^q}{2q+s+1}]^{\frac{1}{q}}\}.$$

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Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara