TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES VIA TWO TOMINAGA'S RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space. In this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0 < m \leq A$, $B \leq M$ for some constants m and M, then for $\nu \in [0,1]$

$$A^{1-\nu} \otimes B^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq S\left(\frac{M}{m}\right) A^{1-\nu} \otimes B^{\nu}$$

and

$$\begin{aligned} 0 &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1, \end{aligned}$$

where $S(\cdot)$ is the Specht's ratio and $L(\cdot,\cdot)$ is the logarithmic mean. We also have the following inequalities for the Hadamard product

$$A^{1-\nu} \circ B^{\nu} \leq [(1-\nu) A + \nu B] \circ 1 \leq S\left(\frac{M}{m}\right) A^{1-\nu} \circ B^{\nu}$$

and

$$0 \leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \circ 1,$$

where $\nu \in [0,1]$.

1. Introduction

As is known to all, the famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

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Tominaga [9] had proved a multiplicative reverse Young inequality with the Specht's ratio [8] as follows:

$$(1.2)\qquad (1-\nu)\,a+\nu b\le S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}$$

for $a,\,b>0$ and $\nu\in[0,1]$.

He also obtained the following additive reverse

$$(1.3)\qquad (1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le L(a,b)\ln S\left(\frac{a}{b}\right)$$

for $a,\,b>0$ and $\nu\in[0,1]$, where $L(\cdot,\cdot)$ is the *logarithmic mean* defined by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ for } b\neq a,\\ \\ a \text{ if } b=a.\end{cases}$$

If $0<m\le a,\,b\le M$, then also [9]

$$(1.4)\qquad \left(a^{1-\nu}b^{\nu}\le\right)(1-\nu)\,a+\nu b\le S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}$$

and

$$(1.5)\qquad (0\le)\,(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le aL\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)$$

for $\nu\in[0,1]$.

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$(1.6)\qquad f(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k)$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [4] for functions of two variables and have the property that

$$f(A_1,...,A_k)=f_1(A_1)\otimes...\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k)=f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st)\ge(\le)\,f(s)\,f(t)\ \text{ for all } s,t\in[0,\infty)$$

and if f is continuous on $[0,\infty)$, then [6, p. 173]

$$(1.7)\qquad f(A\otimes B)\ge(\le)\,f(A)\otimes f(B)\ \text{ for all } A,\ B\ge 0.$$

This follows by observing that, if

$$A=\int_{[0,\infty)} tdE(t) \text{ and } B=\int_{[0,\infty)} sdF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)} f(st)\,dE(t)\otimes dF(s) \tag{1.8}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators A, $B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A \text{ and } (A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [10] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B)&\le\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\le\frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.9}$$

for A, $B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A\circ B=\mathcal{U}^*(A\otimes B)\mathcal{U} \tag{1.10}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [6, p. 173]

$$f(A\circ B)\ge(\le)\,f(A)\circ f(B) \text{ for all } A,\ B\ge 0. \tag{1.11}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\le\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\ge 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\ge 1 \text{ for } A>0. \tag{1.12}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq \left(A^{2}\circ B^{2}\right)^{1/2}\text{ for }A,\ B\geq 0.$$

It has been shown in [7] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0<m\leq A,\ B\leq M$ for some constants m and M, then for $\nu\in[0,1]$

$$A^{1-\nu}\otimes B^{\nu}\leq (1-\nu)\,A\otimes 1+\nu 1\otimes B\leq S\left(\frac{M}{m}\right)A^{1-\nu}\otimes B^{\nu}$$

and

$$\begin{aligned}0&\leq (1-\nu)\,A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\\&\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\otimes 1,\end{aligned}$$

where $S(\cdot)$ is the Specht's ratio and $L(\cdot,\cdot)$ is the logarithmic mean. We also have the following inequalities for the Hadamard product

$$A^{1-\nu}\circ B^{\nu}\leq [(1-\nu)\,A+\nu B]\circ 1\leq S\left(\frac{M}{m}\right)A^{1-\nu}\circ B^{\nu}$$

and

$$0\leq [(1-\nu)\,A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\circ 1,$$

where $\nu\in[0,1]$.

2. Main Results

Our first main result is as follows:

Theorem 1. *Assume that the selfadjoint operators A and B satisfy the condition $0<m\leq A,\ B\leq M$ for some constants m and M, then for $\nu\in[0,1]$,*

$$A^{1-\nu}\otimes B^{\nu}\leq (1-\nu)\,A\otimes 1+\nu 1\otimes B\leq S\left(\frac{M}{m}\right)A^{1-\nu}\otimes B^{\nu}\tag{2.1}$$

and

$$0\leq (1-\nu)\,A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\otimes 1.\tag{2.2}$$

In particular,

$$A^{1/2}\otimes B^{1/2}\leq \frac{1}{2}\left(A\otimes 1+1\otimes B\right)\leq S\left(\frac{M}{m}\right)A^{1/2}\otimes B^{1/2}\tag{2.3}$$

and

$$0\leq \frac{1}{2}\left(A\otimes 1+1\otimes B\right)-A^{1/2}\otimes B^{1/2}\leq L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)A\otimes 1.\tag{2.4}$$

Proof. From (1.4) we get

$$t^{1-\nu}s^{\nu} \le (1-\nu)\,t + \nu s \le S\left(\frac{M}{m}\right) t^{1-\nu}s^{\nu} \tag{2.5}$$

for all $t,\ s \in [m, M]$ and $\nu \in [0,1]$.

Assume that

$$A = \int_m^M t dE\,(t) \text{ and } B = \int_m^M s dF\,(s)$$

are the spectral resolutions of A and B. Now, if we take the double integral $\int_m^M \int_m^M$ over $dE\,(t) \otimes dF\,(s)$ in (2.5), then we get

$$\begin{aligned}\int_m^M \int_m^M t^{1-\nu}s^{\nu} dE\,(t) \otimes dF\,(s) &\le \int_m^M \int_m^M \left[(1-\nu)\,t + \nu s\right] dE\,(t) \otimes dF\,(s) \\ &\le S\left(\frac{M}{m}\right) \int_m^M \int_m^M t^{1-\nu}s^{\nu} dE\,(t) \otimes dF\,(s)\,.\end{aligned} \tag{2.6}$$

Since

$$\int_m^M \int_m^M t^{1-\nu}s^{\nu} dE\,(t) \otimes dF\,(s) = A^{1-\nu} \otimes B^{\nu}$$

and

$$\int_m^M \int_m^M \left[(1-\nu)\,t + \nu s\right] dE\,(t) \otimes dF\,(s) = (1-\nu)\,A \otimes 1 + \nu 1 \otimes B,$$

hence by (2.6) we get (2.1).

By (1.5) we get

$$\begin{aligned}0 &\le \int_m^M \int_m^M \left[(1-\nu)\,t + \nu s\right] dE\,(t) \otimes dF\,(s) \\ &- \int_m^M \int_m^M t^{1-\nu}s^{\nu} dE\,(t) \otimes dF\,(s) \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \int_m^M \int_m^M t dE\,(t) \otimes dF\,(s)\end{aligned} \tag{2.7}$$

and since

$$\int_m^M \int_m^M t dE\,(t) \otimes dF\,(s) = A \otimes 1,$$

hence by (2.7) we deduce (2.2). □

Remark 1. *If $0 < m \le A \le M$ for some constants m and M, then*

$$A^{1-\nu} \otimes A^{\nu} \le (1-\nu)\,A \otimes 1 + \nu 1 \otimes A \le S\left(\frac{M}{m}\right) A^{1-\nu} \otimes A^{\nu} \tag{2.8}$$

and

$$0 \le (1-\nu)\,A \otimes 1 + \nu 1 \otimes A - A^{1-\nu} \otimes A^{\nu} \le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1. \tag{2.9}$$

In particular,

$$A^{1/2} \otimes A^{1/2} \le \frac{1}{2}\,(A \otimes 1 + 1 \otimes A) \le S\left(\frac{M}{m}\right) A^{1/2} \otimes A^{1/2} \tag{2.10}$$

and

$$0 \le \frac{1}{2}(A \otimes 1 + 1 \otimes A) - A^{1/2} \otimes A^{1/2} \le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1. \tag{2.11}$$

Corollary 1. *Assume that the selfadjoint operators A_i and B_i satisfy the condition $0 < m \le A_i,\ B_i \le M$, $p_i \ge 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$, then*

$$\begin{aligned} &\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right) \\ &\le (1-\nu)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^n p_i B_i\right) \\ &\le S\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} 0 &\le (1-\nu)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^n p_i B_i\right) \\ &- \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right) \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1. \end{aligned} \tag{2.13}$$

Proof. From (2.1) we have

$$A_i^{1-\nu} \otimes B_j^{\nu} \le (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j \le S\left(\frac{M}{m}\right) A_i^{1-\nu} \otimes B_j^{\nu}$$

for $i, j \in \{1, ..., n\}$.

If we multiply by $p_i p_j \ge 0$ and sum, then we get

$$\begin{aligned} \sum_{i,j=1}^n p_i p_j A_i^{1-\nu} \otimes B_j^{\nu} &\le \sum_{i,j=1}^n p_i p_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j\right] \\ &\le S\left(\frac{M}{m}\right) \sum_{i,j=1}^n p_i p_j A_i^{1-\nu} \otimes B_j^{\nu}, \end{aligned}$$

which is equivalent to (2.12).

The inequality (2.13) follows in a similar way from (2.2). □

Remark 2. *If we take $B_i = A_i$, $i \in \{1, ..., n\}$ in Corollary 1, then we get*

$$\begin{aligned} &\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i A_i^{\nu}\right) \\ &\le (1-\nu)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^n p_i A_i\right) \\ &\le S\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i A_i^{\nu}\right) \end{aligned} \tag{2.14}$$

and

$$0 \leq (1-\nu)\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right) - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1. \tag{2.15}$$

Corollary 2. *With the assumptions of Theorem 1,*

$$A^{1-\nu} \circ B^{\nu} \leq [(1-\nu) A + \nu B] \circ 1 \leq S\left(\frac{M}{m}\right) A^{1-\nu} \circ B^{\nu} \tag{2.16}$$

and

$$0 \leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \circ 1. \tag{2.17}$$

In particular,

$$A^{1/2} \circ B^{1/2} \leq \frac{A+B}{2} \circ 1 \leq S\left(\frac{M}{m}\right) A^{1/2} \circ B^{1/2} \tag{2.18}$$

and

$$0 \leq \frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2} \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \circ 1. \tag{2.19}$$

Proof. If we use the identity (1.10) and apply $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right of inequality (2.1), we get

$$\mathcal{U}^*\left(A^{1-\nu} \otimes B^{\nu}\right)\mathcal{U} \leq \mathcal{U}^*\left[(1-\nu) A \otimes 1 + \nu 1 \otimes B\right]\mathcal{U} \leq S\left(\frac{M}{m}\right)\mathcal{U}^*\left(A^{1-\nu} \otimes B^{\nu}\right)\mathcal{U}, \tag{2.20}$$

which is equivalent to (2.16). □

Remark 3. *Assume that the selfadjoint operators A_i and B_i satisfy the condition $0 < m \leq A_i, B_i \leq M$, $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by Remark 2 we get*

$$\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \leq \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 \leq S\left(\frac{M}{m}\right) \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \tag{2.21}$$

and

$$0 \leq \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \leq L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1. \tag{2.22}$$

Theorem 2. *Assume that f, g are continuous and nonnegative on the interval I and there exists $0 \leq \gamma < \Gamma$ such that*

$$\gamma \leq \frac{f(t)}{g(t)} \leq \Gamma \text{ for all } t \in I,$$

then for the selfadjoint operators A and B with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$,

$$\begin{aligned} &\left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B)\, g^{2(1-\nu)}(B)\right) \\ &\leq (1-\nu) f^2(A) \otimes g^2(B) + \nu g^2(A) \otimes f^2(B) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B)\, g^{2(1-\nu)}(B)\right) \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} 0 &\leq (1-\nu) f^2(A) \otimes g^2(B) + \nu g^2(A) \otimes f^2(B) \\ &- \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B)\, g^{2(1-\nu)}(B)\right) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \otimes g^2(B). \end{aligned} \tag{2.24}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned} (f(A)\, g(A)) \otimes (f(B)\, g(B)) &\leq \frac{1}{2}\left[f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B)\right] \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) (f(A)\, g(A)) \otimes (f(B)\, g(B)) \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B)\right] - (f(A)\, g(A)) \otimes (f(B)\, g(B)) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \otimes g^2(B). \end{aligned} \tag{2.26}$$

Proof. For any t, $s \in I$ we have

$$\gamma^2 \leq \frac{f^2(t)}{g^2(t)}, \frac{f^2(s)}{g^2(s)} \leq \Gamma^2.$$

If we use the inequality (1.4) for

$$a = \frac{f^2(t)}{g^2(t)},\ b = \frac{f^2(s)}{g^2(s)},$$

then we get

$$\begin{aligned} \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu} \left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} &\leq (1-\nu)\frac{f^2(t)}{g^2(t)} + \nu\frac{f^2(s)}{g^2(s)} \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu} \left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \end{aligned} \tag{2.27}$$

for any t, $s \in I$.

Now, if we multiply (2.27) by $g^2(t)\,g^2(s) > 0$, then we get

$$\begin{aligned} &f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s) \\ &\leq (1-\nu)\,f^2(t)\,g^2(s) + \nu g^2(t)\,f^2(s) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s) \end{aligned} \tag{2.28}$$

for any $t,\ s \in I$.

Assume that

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B.

Further on, if we take the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (2.28), then we get

$$\begin{aligned} &\int_I \int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dE(t) \otimes dF(s) \\ &\leq (1-\nu)\int_I \int_I f^2(t)\,g^2(s)\,dE(t) \otimes dF(s) \\ &+ \nu \int_I \int_I g^2(t)\,f^2(s)\,dE(t) \otimes dF(s) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \int_I \int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dE(t) \otimes dF(s). \end{aligned} \tag{2.29}$$

Since

$$\begin{aligned} &\int_I \int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dE(t) \otimes dF(s) \\ &= \int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,dE(t) \otimes \int_I f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dF(s) \\ &= \left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right), \end{aligned}$$

$$\begin{aligned} \int_I \int_I f^2(t)\,g^2(s)\,dE(t) \otimes dF(s) &= \int_I f^2(t)\,dE(t) \otimes \int_I g^2(s)\,dF(s) \\ &= f^2(A) \otimes g^2(B), \end{aligned}$$

and

$$\begin{aligned} \int_I \int_I g^2(t)\,f^2(s)\,dE(t) \otimes dF(s) &= \int_I g^2(t)\,dE(t) \otimes \int_I f^2(s)\,dF(s) \\ &= g^2(A) \otimes f^2(B), \end{aligned}$$

hence by (2.29) we get (2.23).

From (1.5) we obtain

$$\begin{aligned} &(0 \leq)\,(1-\nu)\frac{f^2(t)}{g^2(t)} + \nu\frac{f^2(s)}{g^2(s)} - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \\ &\leq \frac{f^2(t)}{g^2(t)} L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \end{aligned}$$

for any $t,\ s \in I$.

If we multiply by $g^2(t)g^2(s)>0$ then we get

$$(1-\nu)f^2(t)g^2(s)+\nu g^2(t)f^2(s)-f^{2(1-\nu)}(t)g^{2\nu}(t)f^{2\nu}(s)g^{2(1-\nu)}(s) \le f^2(t)g^2(s)L\left(1,\left(\frac{\Gamma}{\gamma}\right)^2\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \tag{2.30}$$

for any t, $s\in I$.

If we take the double integral $\int_I\int_I$ over $dE(t)\otimes dF(s)$ in (2.30) and use a similar argument as above, we deduce the desired result (2.24). □

Corollary 3. *With the assumption of Theorem 2, we have the following inequalities for the Hadamard product*

$$\begin{aligned}&\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)\circ\left(f^{2\nu}(B)g^{2(1-\nu)}(B)\right)\\&\le(1-\nu)f^2(A)\circ g^2(B)+\nu g^2(A)\circ f^2(B)\\&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)\circ\left(f^{2\nu}(B)g^{2(1-\nu)}(B)\right)\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}0&\le(1-\nu)f^2(A)\circ g^2(B)+\nu g^2(A)\circ f^2(B)\\&\quad-\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)\circ\left(f^{2\nu}(B)g^{2(1-\nu)}(B)\right)\\&\le L\left(1,\left(\frac{\Gamma}{\gamma}\right)^2\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)f^2(A)\circ g^2(B).\end{aligned} \tag{2.32}$$

In particular, for $\nu=1/2$,

$$\begin{aligned}(f(A)g(A))\circ(f(B)g(B))&\le\frac{1}{2}\left[f^2(A)\circ g^2(B)+g^2(A)\circ f^2(B)\right]\\&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)(f(A)g(A))\circ(f(B)g(B))\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}0&\le\frac{1}{2}\left[f^2(A)\circ g^2(B)+g^2(A)\circ f^2(B)\right]-(f(A)g(A))\circ(f(B)g(B))\\&\le L\left(1,\left(\frac{\Gamma}{\gamma}\right)^2\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)f^2(A)\circ g^2(B).\end{aligned} \tag{2.34}$$

Remark 4. *If we take* $B=A$ *in Corollary 3, then we get the simpler inequalities*

$$\begin{aligned}&\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)\circ\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)\\&\le f^2(A)\circ g^2(A)\\&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)\circ\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)\end{aligned} \tag{2.35}$$

and

$$0 \le f^2(A) \circ g^2(A) - \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \circ \left(f^{2\nu}(A)\, g^{2(1-\nu)}(A)\right) \le L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \circ g^2(A). \tag{2.36}$$

In particular, for $\nu = 1/2$,

$$(f(A)\, g(A)) \circ (f(A)\, g(A)) \le f^2(A) \circ g^2(A) \le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) (f(A)\, g(A)) \circ (f(A)\, g(A)) \tag{2.37}$$

and

$$0 \le f^2(A) \circ g^2(A) - (f(A)\, g(A)) \circ (f(A)\, g(A)) \le L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \circ g^2(A). \tag{2.38}$$

Corollary 4. *With the assumption of Theorem 2, then for the selfadjoint operators A_i and B_i with spectra $\operatorname{Sp}(A_i)$, $\operatorname{Sp}(B_i) \subset I$, and $p_i \ge 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \otimes \left(\sum_{i=1}^n p_i f^{2\nu}(B_i)\, g^{2(1-\nu)}(B_i)\right) \\
&\le (1-\nu)\left(\sum_{i=1}^n p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^2(B_i)\right) \\
&+ \nu\left(\sum_{i=1}^n p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i f^2(B_i)\right) \\
&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \\
&\otimes \left(\sum_{i=1}^n p_i f^{2\nu}(B_i)\, g^{2(1-\nu)}(B_i)\right)
\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}
0 &\le (1-\nu)\left(\sum_{i=1}^n p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^2(B_i)\right) \\
&+ \nu\left(\sum_{i=1}^n p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i f^2(B_i)\right) \\
&- \left(\sum_{i=1}^n p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \otimes \left(\sum_{i=1}^n p_i f^{2\nu}(B_i)\, g^{2(1-\nu)}(B_i)\right) \\
&\le L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^n p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^2(B_i)\right).
\end{aligned} \tag{2.40}$$

If take $B_i = A_i$ and consider the Hadamard product version, then we get the simpler inequalities

$$\left(\sum_{i=1}^{n} p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i f^{2\nu}(A_i)\, g^{2(1-\nu)}(A_i)\right) \tag{2.41}$$
$$\leq \left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i g^2(A_i)\right)$$
$$\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right)$$
$$\circ \left(\sum_{i=1}^{n} p_i f^{2\nu}(A_i)\, g^{2(1-\nu)}(A_i)\right)$$

and

$$0 \leq \left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \tag{2.42}$$
$$- \left(\sum_{i=1}^{n} p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i f^{2\nu}(A_i)\, g^{2(1-\nu)}(A_i)\right)$$
$$\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i g^2(A_i)\right).$$

3. Some Related Results

Further on, observe that if $a, b > 0$ and

$$0 < l^{-1} \leq \frac{a}{b} \leq L < \infty, \tag{3.1}$$

for some $L, l > 0$ with $Ll > 1$, then

$$S\left(\frac{a}{b}\right) \leq \max\left\{S\left(l^{-1}\right), S(L)\right\} = \max\left\{S(l), S(L)\right\}$$

and by (1.2) we have

$$(1-\nu)\, a + \nu b \leq \max\left\{S(l), S(L)\right\} a^{1-\nu} b^{\nu} \tag{3.2}$$

for every $\nu \in [0,1]$.

Theorem 3. *Assume that*

$$0 < m_1 \leq f(t) \leq M_1 < \infty,\ \ 0 < m_2 \leq g(t) \leq M_2 < \infty, \tag{3.3}$$

for $t \in I$. If $u(t), v(t) \geq 0$ and continuous on I, then for the selfadjoint operators A and B with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$,

$$(f(A)\, u(A)) \otimes (v(B)\, g(B)) \tag{3.4}$$
$$\leq \frac{1}{p}\left(u(A)\, f^p(A)\right) \otimes v(B) + \frac{1}{q} u(A) \otimes \left(g^q(B)\, v(B)\right)$$
$$\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} (f(A)\, u(A)) \otimes (v(B)\, g(B)).$$

In particular, for $p=q=2$

$$(f(A)u(A))\otimes(v(B)g(B)) \tag{3.5}$$
$$\leq \frac{1}{2}\left[\left(u(A)f^{2}(A)\right)\otimes v(B)+u(A)\otimes\left(g^{2}(B)v(B)\right)\right]$$
$$\leq \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)u(A))\otimes(v(B)g(B)).$$

Proof. Now, if we write the inequality (3.2) for $l=\frac{M_2^q}{m_1^p}$, $L=\frac{M_1^p}{m_2^q}$, $a=f^p(t)$, $b=g^q(s)$ and $\nu=\frac{1}{q}$, and use Young's inequality, then we get

$$f(t)g(s)\leq \frac{1}{p}f^p(t)+\frac{1}{q}g^q(s)\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}f(t)g(s) \tag{3.6}$$

for any t, $s\in I$.

If we multiply (3.6) by $u(t)v(s)\geq 0$, then we get

$$f(t)u(t)v(s)g(s)\leq \frac{1}{p}u(t)f^p(t)v(s)+\frac{1}{q}u(t)g^q(s)v(s) \tag{3.7}$$
$$\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}f(t)u(t)v(s)g(s)$$

for any t, $s\in I$.

If we take the double integral $\int_I\int_I$ over $dE(t)\otimes dF(s)$ in (3.7) and use a similar argument as above, we deduce the desired result (2.24). □

Corollary 5. *With the assumptions of Theorem 3 we have the tensorial inequalities*

$$(f(A)g(A))\otimes(f(B)g(B)) \tag{3.8}$$
$$\leq \frac{1}{p}(g(A)f^p(A))\otimes f(B)+\frac{1}{q}g(A)\otimes(g^q(B)f(B))$$
$$\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}(f(A)g(A))\otimes(f(B)g(B))$$

and

$$\left(f^2(A)\right)\otimes\left(g^2(B)\right) \tag{3.9}$$
$$\leq \frac{1}{p}\left(f^{p+1}(A)\right)\otimes g(B)+\frac{1}{q}f(A)\otimes\left(g^{q+1}(B)\right)$$
$$\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}\left(f^2(A)\right)\otimes\left(g^2(B)\right).$$

For $p=q=2$, *we obtain*

$$(f(A)g(A))\otimes(f(B)g(B)) \tag{3.10}$$
$$\leq \frac{1}{2}\left[\left(g(A)f^2(A)\right)\otimes f(B)+g(A)\otimes\left(g^2(B)f(B)\right)\right]$$
$$\leq \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)g(A))\otimes(f(B)g(B))$$

and

$$\begin{aligned}(3.11)\qquad & \left(f^2(A)\right)\otimes\left(g^2(B)\right)\\ & \leq\frac{1}{2}\left[\left(f^3(A)\right)\otimes g(B)+f(A)\otimes\left(g^3(B)\right)\right]\\ & \leq\max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right),S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}\left(f^2(A)\right)\otimes\left(g^2(B)\right).\end{aligned}$$

We have the following results for Hadamard product:

Corollary 6. *With the assumptions of Theorem 3,*

$$\begin{aligned}(3.12)\qquad & (f(A)u(A))\circ(v(B)g(B))\\ & \leq\frac{1}{p}(u(A)f^p(A))\circ v(B)+\frac{1}{q}u(A)\circ(g^q(B)v(B))\\ & \leq\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}(f(A)u(A))\circ(v(B)g(B)).\end{aligned}$$

In particular, for $p=q=2$

$$\begin{aligned}(3.13)\qquad & (f(A)u(A))\circ(v(B)g(B))\\ & \leq\frac{1}{2}\left[\left(u(A)f^2(A)\right)\circ v(B)+u(A)\circ\left(g^2(B)v(B)\right)\right]\\ & \leq\max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right),S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)u(A))\circ(v(B)g(B)).\end{aligned}$$

For $v=u$, we get

$$\begin{aligned}(3.14)\qquad & (f(A)u(A))\circ(u(B)g(B))\\ & \leq\frac{1}{p}(u(A)f^p(A))\circ u(B)+\frac{1}{q}u(A)\circ(g^q(B)u(B))\\ & \leq\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}(f(A)u(A))\circ(u(B)g(B)).\end{aligned}$$

In particular, for $p=q=2$

$$\begin{aligned}(3.15)\qquad & (f(A)u(A))\circ(u(B)g(B))\\ & \leq\frac{1}{2}\left[\left(u(A)f^2(A)\right)\circ u(B)+u(A)\circ\left(g^2(B)u(B)\right)\right]\\ & \leq\max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right),S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(f(A)u(A))\circ(u(B)g(B)).\end{aligned}$$

Moreover, if we take in (3.14) and (3.15) $B=A$, we get

$$\begin{aligned}(3.16)\qquad & (u(A)f(A))\circ(u(A)g(A))\\ & \leq\left(u(A)\left[\frac{1}{p}f^p(A)+\frac{1}{q}g^q(A)\right]\right)\circ u(A)\\ & \leq\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}(u(A)f(A))\circ(u(A)g(A)).\end{aligned}$$

In particular, for $p = q = 2$

$$(u(A)f(A)) \circ (u(A)g(A)) \le \left(u(A)\left[\frac{f^2(A)+g^2(A)}{2}\right]\right) \circ u(A) \le \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\}(u(A)f(A)) \circ (u(A)g(A)). \tag{3.17}$$

Moreover, if we take $g = f$ in (3.17), then we get

$$(u(A)f(A)) \circ (u(A)f(A)) \le \left(u(A)f^2(A)\right) \circ u(A) \le S\left(\left(\frac{M_1}{m_2}\right)^2\right)(u(A)f(A)) \circ (u(A)f(A)). \tag{3.18}$$

We also have the following inequalities for sums:

Corollary 7. *Assume that f and g satisfy the conditions (3.3). If $u(t)$, $v(t) \ge 0$ and continuous on I, then for the selfadjoint operators A_i and B_i with* $\operatorname{Sp}(A_i)$, $\operatorname{Sp}(B_i) \subset I$, *and $p_i \ge 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$,*

$$\begin{aligned} &\left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right) \\ &\le \frac{1}{p}\left(\sum_{i=1}^n p_i u(A_i) f^p(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i)\right) \\ &+ \frac{1}{q}\left(\sum_{i=1}^n p_i u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^q(B_i) v(B_i)\right) \\ &\le \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} \\ &\times \left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right). \end{aligned} \tag{3.19}$$

In particular, for $p = q = 2$

$$\begin{aligned} &\left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right) \\ &\le \frac{1}{2}\left[\left(\sum_{i=1}^n p_i u(A_i) f^2(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i)\right)\right. \\ &\left.+ \left(\sum_{i=1}^n p_i u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g^2(B_i) v(B_i)\right)\right] \\ &\le \max\left\{S\left(\left(\frac{M_2}{m_1}\right)^2\right), S\left(\left(\frac{M_1}{m_2}\right)^2\right)\right\} \\ &\times \left(\sum_{i=1}^n p_i f(A_i) u(A_i)\right) \otimes \left(\sum_{i=1}^n p_i v(B_i) g(B_i)\right). \end{aligned} \tag{3.20}$$

From (3.20) for $g = f$, $v = u$ and $B_i = A_i$, $i \in \{1, ..., n\}$, we get the inequality for the Hadamard product

$$
\begin{aligned}
(3.21)\qquad & \left(\sum_{i=1}^{n} p_i u(A_i) f(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i u(A_i) f(A_i)\right) \\
& \leq \left(\sum_{i=1}^{n} p_i u(A_i) f^2(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i u(A_i)\right) \\
& \leq S\left(\left(\frac{M_1}{m_2}\right)^2\right) \left(\sum_{i=1}^{n} p_i f(A_i) u(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i u(A_i) f(A_i)\right).
\end{aligned}
$$

4. Some Examples

Consider the functions $f(t) = t^p$ and $g(t) = t^q$ for $t > 0$ and $p, q \neq 0$. If A, B are selfadjoint operators with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m, M] \subset (0, \infty)$. Then

$$\frac{f(t)}{g(t)} = t^{p-q}, \text{ for } t > 0.$$

Therefore

$$m^{p-q} \leq \frac{f(t)}{g(t)} \leq M^{p-q} \text{ for } t \in [m, M] \text{ and } p > q$$

and

$$M^{p-q} \leq \frac{f(t)}{g(t)} \leq m^{p-q} \text{ for } t \in [m, M] \text{ and } p < q.$$

From Theorem 2 we get for $p > q$ that

$$
\begin{aligned}
(4.1)\qquad & A^{2(1-\nu)p+2\nu q} \otimes B^{2\nu p+2(1-\nu)q} \\
& \leq (1-\nu) A^{2p} \otimes B^{2q} + \nu A^{2q} \otimes B^{2p} \\
& \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2(1-\nu)p+2\nu q} \otimes B^{2\nu p+2(1-\nu)q}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.2)\qquad & 0 \leq (1-\nu) A^{2p} \otimes B^{2q} + \nu A^{2q} \otimes B^{2p} - A^{2(1-\nu)p+2\nu q} \otimes B^{2\nu p+2(1-\nu)q} \\
& \leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p} \otimes B^{2q}.
\end{aligned}
$$

In particular, for $\nu = 1/2$,

$$
\begin{aligned}
(4.3)\qquad A^{p+q} \otimes B^{p+q} & \leq \frac{1}{2}\left(A^{2p} \otimes B^{2q} + A^{2q} \otimes B^{2p}\right) \\
& \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{p+q} \otimes B^{p+q}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.4)\qquad & 0 \leq \frac{1}{2}\left(A^{2p} \otimes B^{2q} + A^{2q} \otimes B^{2p}\right) - A^{p+q} \otimes B^{p+q} \\
& \leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p} \otimes B^{2q}.
\end{aligned}
$$

We also have the inequalities for the Hadamard product

$$\begin{aligned} & A^{2(1-\nu)p+2\nu q}\circ B^{2\nu p+2(1-\nu)q} \\ & \leq (1-\nu)\, A^{2p}\circ B^{2q}+\nu A^{2q}\circ B^{2p} \\ & \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2(1-\nu)p+2\nu q}\circ B^{2\nu p+2(1-\nu)q} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} 0 & \leq (1-\nu)\, A^{2p}\circ B^{2q}+\nu A^{2q}\circ B^{2p}-A^{2(1-\nu)p+2\nu q}\circ B^{2\nu p+2(1-\nu)q} \\ & \leq L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p}\circ B^{2q}. \end{aligned} \tag{4.6}$$

In particular, for $\nu=1/2$,

$$\begin{aligned} A^{p+q}\circ B^{p+q} & \leq \frac{1}{2}\left(A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}\right) \\ & \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{p+q}\circ B^{p+q} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} 0 & \leq \frac{1}{2}\left(A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}\right)-A^{p+q}\circ B^{p+q} \\ & \leq L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p}\circ B^{2q}. \end{aligned} \tag{4.8}$$

Moreover, if we take $B=A$ in (4.5)-(4.8), then we get

$$\begin{aligned} A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q} & \leq A^{2p}\circ A^{2q} \\ & \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} 0 & \leq A^{2p}\circ A^{2q}-A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q} \\ & \leq L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p}\circ A^{2q}. \end{aligned} \tag{4.10}$$

In particular, for $\nu=1/2$,

$$A^{p+q}\circ A^{p+q}\leq A^{2p}\circ A^{2q}\leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{p+q}\circ A^{p+q} \tag{4.11}$$

and

$$\begin{aligned} 0 & \leq A^{2p}\circ A^{2q}-A^{p+q}\circ A^{p+q} \\ & \leq L\left(1,\left(\frac{M}{m}\right)^{2(p-q)}\right)\ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) A^{2p}\circ A^{2q}. \end{aligned} \tag{4.12}$$

Now, assume that $\operatorname{Sp}(A_i) \subseteq [m, M] \subset (0, \infty)$ and $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then from (2.41) and (2.42) we derive

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i A_i^{2(1-\nu)p+2\nu q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2\nu p+2(1-\nu)q}\right) \\ &\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{2(1-\nu)p+2\nu q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2\nu p+2(1-\nu)q}\right) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \\ &- \left(\sum_{i=1}^{n} p_i A_i^{2(1-\nu)p+2\nu q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2\nu p+2(1-\nu)q}\right) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \end{aligned} \tag{4.14}$$

for $\nu \in [0, 1]$.

In particular, for $\nu = 1/2$ we get

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \\ &\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) - \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{2q}\right) \end{aligned} \tag{4.16}$$

The interested reader may also consider the examples $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha \neq \beta$ and $t \in \mathbb{R}$.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.
[5] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535
[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[7] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.
[8] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.
[9] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.
[10] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa