TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES VIA A RESULT OF KITTANEH AND MANASRAH

SILVESTRU SEVER DRAGOMIR[1,2]

ABSTRACT. Let H be a Hilbert space. In this paper we show among others that, if the selfadjoint operators A and B are nonnegative and $\nu \in [0,1]$, then

$$\begin{aligned} 0 &\leq r\left(A\otimes 1+1\otimes B-2A^{1/2}\otimes B^{1/2}\right) \\ &\leq (1-\nu)\, A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu} \\ &\leq R\left(A\otimes 1+1\otimes B-2A^{1/2}\otimes B^{1/2}\right), \end{aligned}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. We also have the following inequalities for the Hadamard product

$$\begin{aligned} 0 &\leq 2r\left(\left(\frac{A+B}{2}\right)\circ 1-A^{1/2}\circ B^{1/2}\right) \\ &\leq [(1-\nu)\, A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu} \\ &\leq 2R\left(\left(\frac{A+B}{2}\right)\circ 1-A^{1/2}\circ B^{1/2}\right), \end{aligned}$$

where $\nu \in [0,1]$.

1. Introduction

As is known to all, the famous *Young inequality* for scalars says that if $a,b>0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\, a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [9], [10] provided a refinement and a reverse for Young inequality as follows:

$$0 \leq r\left(\sqrt{a}-\sqrt{b}\right)^2 \leq (1-\nu)\, a+\nu b-a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.2}$$

where $a,b>0$, $\nu \in [0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We observe that, if $a,b\in [m,M]\subset (0,\infty)$, then $\left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{M}-\sqrt{m}$ and by (1.2) we obtain the following reverse of Young inequality

$$(1-\nu)\, a+\nu b-a^{1-\nu}b^{\nu} \leq R\left(\sqrt{M}-\sqrt{m}\right)^2. \tag{1.3}$$

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.

RGMIA Res. Rep. Coll. **25** (2022), Art. 75, 14 pp. Received 26/08/22

We can give a simple direct proof for (1.2) as follows.

Recall the following result obtained by Dragomir in 2006 [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$
\begin{aligned}
(1.4)\qquad 0 &\le n \min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right]\\
&\le \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)\\
&\le n \max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right],
\end{aligned}
$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n=\sum_{j=1}^{n}p_j>0$.

For $n=2$, we deduce from (1.4) that

$$
\begin{aligned}
(1.5)\qquad 0 &\le 2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\
&\le \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]\\
&\le 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$.

If we take $\Phi(x)=\exp(x)$, then we get from (1.5)

$$
\begin{aligned}
(1.6)\qquad 0 &\le 2\min\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]\\
&\le \nu\exp(x)+(1-\nu)\exp(y)-\exp[\nu x+(1-\nu)y]\\
&\le 2\max\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$. Further, denote $\exp(x)=a$, $\exp(y)=b$ with $a,b>0$, then from (1.6) we obtain the inequality (1.2).

Let $I_1,\dots,I_k$ be intervals from $\mathbb{R}$ and let $f : I_1\times\dots\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,\dots,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,\dots,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,\dots,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,\dots,k$; by following [2], we define

$$(1.7)\qquad f(A_1,\dots,A_k):=\int_{I_1}\dots\int_{I_k}f(\lambda_1,\dots,\lambda_1)\,dE_1(\lambda_1)\otimes\dots\otimes dE_k(\lambda_k)$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes\dots\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [7, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.8}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) \, dE(t) \otimes dF(s) \tag{1.9}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [13] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} \left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.10}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.11}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If *f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [7, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.12}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.13}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [8] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if the selfadjoint operators A and B are nonnegative and $\nu \in [0,1]$, then

$$\begin{aligned} 0 &\leq r \left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2} \right) \\ &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\leq R \left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2} \right), \end{aligned}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. We also have the following inequalities for the Hadamard product

$$\begin{aligned} 0 &\leq 2r \left(\left(\frac{A+B}{2} \right) \circ 1 - A^{1/2} \circ B^{1/2} \right) \\ &\leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \\ &\leq 2R \left(\left(\frac{A+B}{2} \right) \circ 1 - A^{1/2} \circ B^{1/2} \right), \end{aligned}$$

where $\nu \in [0,1]$.

2. Main Results

The first main result is as follows:

Theorem 1. *If the selfadjoint operators A and B are nonnegative and $\nu \in [0,1]$, then*

$$\begin{aligned} 0 &\leq r \left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2} \right) \\ &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\leq R \left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2} \right), \end{aligned} \tag{2.1}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Proof. From (1.2) we have

$$r \left(t - 2t^{1/2} s^{1/2} + s \right) \leq (1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \leq R \left(t - 2t^{1/2} s^{1/2} + s \right) \tag{2.2}$$

for all $t,\ s \geq 0$ and $\nu \in [0,1]$.

If

$$A=\int_{[0,\infty)} tdE(t) \text{ and } B=\int_{[0,\infty)} sdF(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_{[0,\infty)}\int_{[0,\infty)}$ over $dE(t)\otimes dF(s)$ in (2.2), we derive that

$$\begin{aligned}
&r\int_{[0,\infty)}\int_{[0,\infty)}\left(t-2t^{1/2}s^{1/2}+s\right)dE(t)\otimes dF(s)\\
&\leq \int_{[0,\infty)}\int_{[0,\infty)}\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s)\\
&\leq R\int_{[0,\infty)}\int_{[0,\infty)}\left(t-2t^{1/2}s^{1/2}+s\right)dE(t)\otimes dF(s).
\end{aligned}\tag{2.3}$$

Observe that

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)}\left(t+s-2t^{1/2}s^{1/2}\right)dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)} tdE(t)\otimes dF(s)+\int_{[0,\infty)}\int_{[0,\infty)} sdE(t)\otimes dF(s)\\
&-2\int_{[0,\infty)}\int_{[0,\infty)} t^{1/2}s^{1/2}dE(t)\otimes dF(s)\\
&=A\otimes 1+1\otimes B-2A^{1/2}\otimes B^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)}\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s)\\
&=(1-\nu)\int_{[0,\infty)}\int_{[0,\infty)} tdE(t)\otimes dF(s)+\nu\int_{[0,\infty)}\int_{[0,\infty)} sdE(t)\otimes dF(s)\\
&-\int_{[0,\infty)}\int_{[0,\infty)} t^{1-\nu}s^{\nu}dE(t)\otimes dF(s)\\
&=(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}
\end{aligned}$$

and by (2.3) we obtain (2.1). □

Remark 1. *If $0<m\leq A\leq M$ for some constants m and M, then*

$$\begin{aligned}
0&\leq r\left(A\otimes 1+1\otimes A-2A^{1/2}\otimes A^{1/2}\right)\\
&\leq (1-\nu)A\otimes 1+\nu 1\otimes A-A^{1-\nu}\otimes A^{\nu}\\
&\leq R\left(A\otimes 1+1\otimes A-2A^{1/2}\otimes A^{1/2}\right).
\end{aligned}\tag{2.4}$$

The following result for sums also holds:

Corollary 1. *Assume that the selfadjoint operators A_i and B_i are nonnegative and $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then*

$$
\begin{aligned}
0 &\leq r \left[\left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i B_i \right) \right. \\
&\left. -2 \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{1/2} \right) \right] \\
&\leq (1-\nu) \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i B_i \right) \\
&- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{\nu} \right) \\
&\leq R \left[\left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i B_i \right) \right. \\
&\left. -2 \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{1/2} \right) \right].
\end{aligned} \tag{2.5}
$$

Proof. From (2.1) we have

$$
\begin{aligned}
& r \left(A_i \otimes 1 + 1 \otimes B_j - 2 A_i^{1/2} \otimes B_j^{1/2} \right) \\
& \leq (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \\
& \leq R \left(A_i \otimes 1 + 1 \otimes B_j - 2 A_i^{1/2} \otimes B_j^{1/2} \right),
\end{aligned}
$$

for $i, j \in \{1, ..., n\}$.

If we multiply by $p_i p_j \geq 0$, then we get

$$
\begin{aligned}
& r \sum_{i,j=1}^{n} p_i p_j \left(A_i \otimes 1 + 1 \otimes B_j - 2 A_i^{1/2} \otimes B_j^{1/2} \right) \\
& \leq \sum_{i,j=1}^{n} p_i p_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \right] \\
& \leq R \sum_{i,j=1}^{n} p_i p_j \left(A_i \otimes 1 + 1 \otimes B_j - 2 A_i^{1/2} \otimes B_j^{1/2} \right),
\end{aligned}
$$

which is equivalent to (2.5). □

Remark 2. *If we take $B_i = A_i$, $i \in \{1, ..., n\}$ in (2.5), then we get*

$$0 \le r\left[\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right) -2\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{i=1}^{n} p_i A_j^{1/2}\right)\right] \le (1-\nu)\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right) - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \le R\left[\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right) -2\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{i=1}^{n} p_i A_j^{1/2}\right)\right]. \tag{2.6}$$

We have the following result for the Hadamard product:

Corollary 2. *Assume that A, $B \ge 0$ and $\nu \in [0,1]$. Then*

$$0 \le 2r\left(\left(\frac{A+B}{2}\right) \circ 1 - A^{1/2} \circ B^{1/2}\right) \le [(1-\nu)A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \le 2R\left(\left(\frac{A+B}{2}\right) \circ 1 - A^{1/2} \circ B^{1/2}\right), \tag{2.7}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, for $B = A$,

$$0 \le 2r\left(A \circ 1 - A^{1/2} \circ A^{1/2}\right) \le A \circ 1 - A^{1-\nu} \circ A^{\nu} \le 2R\left(A \circ 1 - A^{1/2} \circ B^{1/2}\right), \tag{2.8}$$

Proof. We have the representation

$$X \circ Y = \mathcal{U}^* (X \otimes Y) \mathcal{U},$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

Now, if we apply $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.1), then we get

$$r\mathcal{U}^*\left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2}\right)\mathcal{U} \le \mathcal{U}^*\left[(1-\nu)A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}\right]\mathcal{U} \le R\mathcal{U}^*\left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2}\right)\mathcal{U},$$

which gives that

$$\begin{aligned}
& r\left[\mathcal{U}^*(A\otimes 1)\mathcal{U}+\mathcal{U}^*(1\otimes B)\mathcal{U}-2\mathcal{U}^*\left(A^{1/2}\otimes B^{1/2}\right)\mathcal{U}\right] \\
& \leq (1-\nu)\mathcal{U}^*(A\otimes 1)\mathcal{U}+\nu\mathcal{U}^*(1\otimes B)\mathcal{U}-\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U} \\
& \leq R\left[\mathcal{U}^*(A\otimes 1)\mathcal{U}+\mathcal{U}^*(1\otimes B)\mathcal{U}-2\mathcal{U}^*\left(A^{1/2}\otimes B^{1/2}\right)\mathcal{U}\right],
\end{aligned}$$

namely

$$\begin{aligned}
r\left(A\circ 1+1\circ B-2A^{1/2}\circ B^{1/2}\right) & \leq (1-\nu)A\circ 1+\nu 1\circ B-A^{1-\nu}\circ B^{\nu} \\
& \leq R\left(A\circ 1+1\circ B-2A^{1/2}\circ B^{1/2}\right),
\end{aligned}$$

which is equivalent to (2.7). □

Remark 3. *Assume that the selfadjoint operators A_i and B_i are nonnegative and $p_i \geq 0$ for $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$, then*

$$\begin{aligned}
(2.9)\qquad 0 & \leq 2r\left[\left(\sum_{i=1}^n p_i \frac{A_i+B_i}{2}\right)\circ 1-\left(\sum_{i=1}^n p_i A_i^{1/2}\right)\circ\left(\sum_{i=1}^n p_i B_i^{1/2}\right)\right] \\
& \leq \left(\sum_{i=1}^n p_i\left[(1-\nu)A_i+\nu B_i\right]\right)\circ 1-\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right)\circ\left(\sum_{i=1}^n p_i B_i^{\nu}\right) \\
& \leq 2R\left[\left(\sum_{i=1}^n p_i \frac{A_i+B_i}{2}\right)\circ 1-\left(\sum_{i=1}^n p_i A_i^{1/2}\right)\circ\left(\sum_{i=1}^n p_i B_i^{1/2}\right)\right].
\end{aligned}$$

In particular, for $A_i = B_i$ where $i \in \{1,...,n\}$, then

$$\begin{aligned}
(2.10)\qquad 0 & \leq 2r\left[\left(\sum_{i=1}^n p_i A_i\right)\circ 1-\left(\sum_{i=1}^n p_i A_i^{1/2}\right)\circ\left(\sum_{i=1}^n p_i A_i^{1/2}\right)\right] \\
& \leq \left(\sum_{i=1}^n p_i A_i\right)\circ 1-\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right)\circ\left(\sum_{i=1}^n p_i A_i^{\nu}\right) \\
& \leq 2R\left[\left(\sum_{i=1}^n p_i A_i\right)\circ 1-\left(\sum_{i=1}^n p_i A_i^{1/2}\right)\circ\left(\sum_{i=1}^n p_i A_i^{1/2}\right)\right].
\end{aligned}$$

We have the following result for tensorial product of continuous functions of operators:

Theorem 2. *Assume that f, g are continuous and nonnegative on the interval I and selfadjoint operators A and B with spectra* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$,

$$\begin{aligned}
(2.11)\quad 0 & \leq r\left[f^2(A)\otimes g^2(B)+g^2(A)\otimes f^2(B)-2\left(f(A)g(A)\right)\otimes\left(f(B)g(B)\right)\right] \\
& \leq (1-\nu)f^2(A)\otimes g^2(B)+\nu g^2(A)\otimes f^2(B) \\
& -\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)\otimes\left(f^{2\nu}(B)g^{2(1-\nu)}(B)\right) \\
& \leq R\left[f^2(A)\otimes g^2(B)+g^2(A)\otimes f^2(B)-2\left(f(A)g(A)\right)\otimes\left(f(B)g(B)\right)\right]
\end{aligned}$$

for all $\nu \in [0,1]$, where $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

In particular, for $B = A$ we get

$$\begin{aligned}
(2.12)\quad 0 &\le r\left[f^2(A)\otimes g^2(A) + g^2(A)\otimes f^2(A) - 2\left(f(A)g(A)\right)\otimes\left(f(A)g(A)\right)\right]\\
&\le (1-\nu) f^2(A)\otimes g^2(A) + \nu g^2(A)\otimes f^2(A)\\
&- \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right)\otimes\left(f^{2\nu}(A)\, g^{2(1-\nu)}(A)\right)\\
&\le R\left[f^2(A)\otimes g^2(A) + g^2(A)\otimes f^2(A) - 2\left(f(A)g(A)\right)\otimes\left(f(A)g(A)\right)\right]
\end{aligned}$$

Proof. If we use the inequality (1.2) for

$$a = \frac{f^2(t)}{g^2(t)},\ b = \frac{f^2(s)}{g^2(s)},$$

then we get

$$\begin{aligned}
(2.13)\qquad 0 &\le r\left(\frac{f^2(t)}{g^2(t)} - 2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)} + \frac{f^2(s)}{g^2(s)}\right)\\
&\le (1-\nu)\frac{f^2(t)}{g^2(t)} + \nu b - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu}\\
&\le R\left(\frac{f^2(t)}{g^2(t)} - 2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)} + \frac{f^2(s)}{g^2(s)}\right)
\end{aligned}$$

for all $t,\ s \in I$.

Now, if we multiply (2.11) by $g^2(t)\, g^2(s) > 0$, then we get

$$\begin{aligned}
(2.14)\qquad 0 &\le r\left(f^2(t)\, g^2(s) - 2f(t)\, g(t)\, f(s)\, g(s) + f^2(s)\, g^2(t)\right)\\
&\le (1-\nu) f^2(t)\, g^2(s) + \nu g^2(t)\, f^2(s)\\
&- f^{2(1-\nu)}(t)\, g^{2\nu}(t)\, f^{2\nu}(s)\, g^{2(1-\nu)}(s)\\
&\le R\left(f^2(t)\, g^2(s) - 2f(t)\, g(t)\, f(s)\, g(s) + f^2(s)\, g^2(t)\right)
\end{aligned}$$

for any $t,\ s \in I$.

Assume that

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B.

Further on, if we take the double integral $\int_I \int_I$ over $dE(t)\otimes dF(s)$ in (2.13), then we get

$$\begin{aligned}
(2.15)\qquad 0 &\le r\int_I\int_I \left(f^2(t)\, g^2(s) - 2f(t)\, g(t)\, f(s)\, g(s) + f^2(s)\, g^2(t)\right)\\
&\times dE(t)\otimes dF(s)\\
&\le \int_I\int_I \left[(1-\nu) f^2(t)\, g^2(s) + \nu g^2(t)\, f^2(s)\right.\\
&\left.- f^{2(1-\nu)}(t)\, g^{2\nu}(t)\, f^{2\nu}(s)\, g^{2(1-\nu)}(s)\right] dE(t)\otimes dF(s)\\
&\le R\int_I\int_I \left(f^2(t)\, g^2(s) - 2f(t)\, g(t)\, f(s)\, g(s) + f^2(s)\, g^2(t)\right)\\
&\times dE(t)\otimes dF(s).
\end{aligned}$$

Observe that, by (1.7)

$$\begin{aligned}
&\int_I \int_I \left(f^2(t)\, g^2(s) - 2f(t)\, g(t)\, f(s)\, g(s) + f^2(s)\, g^2(t)\right) dE(t) \otimes dF(s) \\
&= \int_I \int_I f^2(t)\, g^2(s)\, dE(t) \otimes dF(s) + \int_I \int_I f^2(s)\, g^2(t)\, dE(t) \otimes dF(s) \\
&- 2 \int_I \int_I f(t)\, g(t)\, f(s)\, g(s)\, dE(t) \otimes dF(s) \\
&= f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B) - 2\left(f(A)\, g(A)\right) \otimes \left(f(B)\, g(B)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_I \left[(1-\nu)\, f^2(t)\, g^2(s) + \nu g^2(t)\, f^2(s)\right. \\
&\left. -f^{2(1-\nu)}(t)\, g^{2\nu}(t)\, f^{2\nu}(s)\, g^{2(1-\nu)}(s)\right] dE(t) \otimes dF(s) \\
&= (1-\nu) \int_I \int_I f^2(t)\, g^2(s)\, dE(t) \otimes dF(s) \\
&+ \nu \int_I \int_I g^2(t)\, f^2(s)\, dE(t) \otimes dF(s) \\
&- \int_I \int_I f^{2(1-\nu)}(t)\, g^{2\nu}(t)\, f^{2\nu}(s)\, g^{2(1-\nu)}(s)\, dE(t) \otimes dF(s) \\
&= (1-\nu)\, f^2(A) \otimes g^2(B) + \nu g^2(A) \otimes f^2(B) \\
&- \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B)\, g^{2(1-\nu)}(B)\right).
\end{aligned}$$

Therefore, by (2.15) we derive the desired result (2.11). □

Corollary 3. *With the assumptions of Theorem 2 we have the following inequalities for the Hadamard product*

$$\begin{aligned}
0 &\le r\left[f^2(A) \circ g^2(B) + g^2(A) \circ f^2(B) - 2\left(f(A)\, g(A)\right) \circ \left(f(B)\, g(B)\right)\right] \\
&\le (1-\nu)\, f^2(A) \circ g^2(B) + \nu g^2(A) \circ f^2(B) \\
&- \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \circ \left(f^{2\nu}(B)\, g^{2(1-\nu)}(B)\right) \\
&\le R\left[f^2(A) \circ g^2(B) + g^2(A) \circ f^2(B) - 2\left(f(A)\, g(A)\right) \circ \left(f(B)\, g(B)\right)\right]
\end{aligned} \tag{2.16}$$

for all $\nu \in [0,1]$.

In particular, for $B = A$ *we get*

$$\begin{aligned}
0 &\le 2r\left[f^2(A) \circ g^2(A) - \left(f(A)\, g(A)\right) \circ \left(f(A)\, g(A)\right)\right] \\
&\le f^2(A) \circ g^2(A) - \left(f^{2(1-\nu)}(A)\, g^{2\nu}(A)\right) \circ \left(f^{2\nu}(A)\, g^{2(1-\nu)}(A)\right) \\
&\le 2R\left[f^2(A) \circ g^2(A) - \left(f(A)\, g(A)\right) \circ \left(f(A)\, g(A)\right)\right].
\end{aligned} \tag{2.17}$$

We have the following result for sums of operators:

Corollary 4. *Assume that the selfadjoint operators A_i and B_i are nonnegative and $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$, then*

$$
\begin{aligned}
0 \leq\; & r\left[\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g^2(B_i)\right)\right. \\
& + \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^2(B_i)\right) \\
& \left. -2\left(\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f(B_i)\, g(B_i)\right)\right] \\
\leq\; & (1-\nu)\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g^2(B_i)\right) \\
& + \nu\left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^2(B_i)\right) \\
& - \left(\sum_{i=1}^{n} p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^{2\nu}(B_i)\, g^{2(1-\nu)}(B_i)\right) \\
\leq\; & R\left[\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g^2(B_i)\right)\right. \\
& + \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^2(B_i)\right) \\
& \left. -2\left(\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f(B_i)\, g(B_i)\right)\right].
\end{aligned} \tag{2.18}
$$

In particular,

$$
\begin{aligned}
0 \leq\; & r\left[\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g^2(A_i)\right)\right. \\
& + \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \\
& \left. -2\left(\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right)\right] \\
\leq\; & (1-\nu)\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \\
& + \nu\left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \\
& - \left(\sum_{i=1}^{n} p_i f^{2(1-\nu)}(A_i)\, g^{2\nu}(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^{2\nu}(A_i)\, g^{2(1-\nu)}(A_i)\right)
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
&\le R\left[\left(\sum_{i=1}^{n}p_if^2(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_ig^2(A_i)\right)\right.\\
&+\left(\sum_{i=1}^{n}p_ig^2(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_if^2(A_i)\right)\\
&\left.-2\left(\sum_{i=1}^{n}p_if(A_i)g(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_if(A_i)g(A_i)\right)\right],
\end{aligned}
$$

for $\nu\in[0,1]$.

Remark 4. *The corresponding Hadamard inequality we can get from (2.19) is*

$$
\begin{aligned}
(2.20)\qquad 0 &\le 2r\left[\left(\sum_{i=1}^{n}p_if^2(A_i)\right)\circ\left(\sum_{i=1}^{n}p_ig^2(A_i)\right)\right.\\
&\left.-\left(\sum_{i=1}^{n}p_if(A_i)g(A_i)\right)\circ\left(\sum_{i=1}^{n}p_if(A_i)g(A_i)\right)\right]\\
&\le\left(\sum_{i=1}^{n}p_if^2(A_i)\right)\circ\left(\sum_{i=1}^{n}p_ig^2(A_i)\right)\\
&-\left(\sum_{i=1}^{n}p_if^{2(1-\nu)}(A_i)g^{2\nu}(A_i)\right)\circ\left(\sum_{i=1}^{n}p_if^{2\nu}(A_i)g^{2(1-\nu)}(A_i)\right)\\
&\le 2R\left[\left(\sum_{i=1}^{n}p_if^2(A_i)\right)\circ\left(\sum_{i=1}^{n}p_ig^2(A_i)\right)\right.\\
&\left.-\left(\sum_{i=1}^{n}p_if(A_i)g(A_i)\right)\circ\left(\sum_{i=1}^{n}p_if(A_i)g(A_i)\right)\right]
\end{aligned}
$$

for $\nu\in[0,1]$.

3. Some Examples

Consider the functions $f(t)=t^p$ and $g(t)=t^q$ for $t>0$ and $p,q\neq 0$. If the selfadjoint operators A and B are nonnegative and $\nu\in[0,1]$, then by Theorem 2

$$
\begin{aligned}
(3.1)\qquad 0 &\le r\left(A^{2p}\otimes B^{2q}+A^{2q}\otimes B^{2p}-2A^{p+q}\otimes B^{p+q}\right)\\
&\le(1-\nu)A^{2p}\otimes B^{2q}+\nu A^{2q}\otimes B^{2p}-A^{2(1-\nu)p+2\nu q}\otimes B^{2\nu p+2(1-\nu)q}\\
&\le R\left(A^{2p}\otimes B^{2q}+A^{2q}\otimes B^{2p}-2A^{p+q}\otimes B^{p+q}\right)
\end{aligned}
$$

for all $\nu\in[0,1]$, where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, for $B=A$ we get

$$
\begin{aligned}
(3.2)\qquad 0 &\le r\left(A^{2p}\otimes A^{2q}+A^{2q}\otimes A^{2p}-2A^{p+q}\otimes A^{p+q}\right)\\
&\le(1-\nu)A^{2p}\otimes A^{2q}+\nu A^{2q}\otimes A^{2p}-A^{2(1-\nu)p+2\nu q}\otimes A^{2\nu p+2(1-\nu)q}\\
&\le R\left(A^{2p}\otimes A^{2q}+A^{2q}\otimes A^{2p}-2A^{p+q}\otimes A^{p+q}\right)
\end{aligned}
$$

for all $\nu\in[0,1]$.

If we take $p=1/2$, $q=-1/2$ in (3.2), then we get

$$\begin{aligned}0 &\le r\left(A\otimes A^{-1}+A^{-1}\otimes A-2\right)\\ &\le (1-\nu)\,A\otimes A^{-1}+\nu A^{-1}\otimes A-A^{1-2\nu}\otimes A^{2\nu-1}\\ &\le R\left(A\otimes A^{-1}+A^{-1}\otimes A-2\right)\end{aligned}\tag{3.3}$$

for all $\nu\in[0,1]$.

We also have the inequalities for the Hadamard product

$$\begin{aligned}0 &\le r\left(A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}-2A^{p+q}\circ B^{p+q}\right)\\ &\le (1-\nu)\,A^{2p}\circ B^{2q}+\nu A^{2q}\circ B^{2p}-A^{2(1-\nu)p+2\nu q}\circ B^{2\nu p+2(1-\nu)q}\\ &\le R\left(A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}-2A^{p+q}\circ B^{p+q}\right)\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}0 &\le 2r\left(A^{2p}\circ A^{2q}-A^{p+q}\circ A^{p+q}\right)\\ &\le A^{2p}\circ A^{2q}-A^{2(1-\nu)p+2\nu q}\circ A^{2\nu p+2(1-\nu)q}\\ &\le 2R\left(A^{2p}\circ A^{2q}-A^{p+q}\circ A^{p+q}\right)\end{aligned}\tag{3.5}$$

for all $\nu\in[0,1]$.

If we take $p=1/2$, $q=-1/2$ in (3.5), then we get

$$\begin{aligned}0 &\le 2r\left(A\circ A^{-1}-1\right)\le A\circ A^{-1}-A^{1-2\nu}\circ A^{2\nu-1}\\ &\le 2R\left(A\circ A^{-1}-1\right)\end{aligned}\tag{3.6}$$

for all $\nu\in[0,1]$.

Now, assume that $\mathrm{Sp}\,(A_i)\subseteq[m,M]\subset(0,\infty)$ and $p_i\ge 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$, then from (2.20) we derive

$$\begin{aligned}0 &\le 2r\left[\left(\sum_{i=1}^n p_iA_i^{2p}\right)\circ\left(\sum_{i=1}^n p_iA_i^{2q}\right)-\left(\sum_{i=1}^n p_iA_i^{p+q}\right)\circ\left(\sum_{i=1}^n p_iA_i^{p+q}\right)\right]\\ &\le \left(\sum_{i=1}^n p_iA_i^{2p}\right)\circ\left(\sum_{i=1}^n p_iA_i^{2q}\right)\\ &\quad-\left(\sum_{i=1}^n p_iA_i^{2(1-\nu)p+2\nu q}\right)\circ\left(\sum_{i=1}^n p_iA_i^{2\nu p+2(1-\nu)q}\right)\\ &\le 2R\left[\left(\sum_{i=1}^n p_iA_i^{2p}\right)\circ\left(\sum_{i=1}^n p_iA_i^{2q}\right)-\left(\sum_{i=1}^n p_iA_i^{p+q}\right)\circ\left(\sum_{i=1}^n p_iA_i^{p+q}\right)\right]\end{aligned}\tag{3.7}$$

for all $\nu\in[0,1]$.

If we take $p=1/2$, $q=-1/2$ in (3.7), then we get

$$\begin{aligned}0 &\le 2r\left[\left(\sum_{i=1}^n p_iA_i\right)\circ\left(\sum_{i=1}^n p_iA_i^{-1}\right)-1\right]\\ &\le \left(\sum_{i=1}^n p_iA_i\right)\circ\left(\sum_{i=1}^n p_iA_i^{-1}\right)-\left(\sum_{i=1}^n p_iA_i^{1-2\nu}\right)\circ\left(\sum_{i=1}^n p_iA_i^{2\nu-1}\right)\\ &\le 2R\left[\left(\sum_{i=1}^n p_iA_i\right)\circ\left(\sum_{i=1}^n p_iA_i^{-1}\right)-1\right]\end{aligned}\tag{3.8}$$

for all $\nu \in [0,1]$.

The interested reader may also consider the examples $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha \neq \beta$ and $t \in \mathbb{R}$.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[5] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[6] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[7] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[8] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[9] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269

[10] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.

[11] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.

[12] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[13] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa