SOME TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES IN TERMS OF KANTOROVICH RATIO

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if A_i and B_i are selfadjoint operators with $0 \leq m \leq A_i$, $B_i \leq M$, $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $\nu \in [0, 1]$, then

$$\begin{aligned} & \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{\nu} \right) \\ & \leq (1-\nu) \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i B_i \right) \\ & \leq K^R \left(\frac{M}{m} \right) \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{\nu} \right), \end{aligned}$$

where $K(\cdot)$ is Kantorovich ration and $R = \max\{1-\nu, \nu\}$. We also have the following inequalities for the Hadamard product

$$\begin{aligned} \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^{n} p_i B_i^{\nu} \right) & \leq \sum_{i=1}^{n} p_i \left[(1-\nu) A_i + \nu B_i \right] \circ 1 \\ & \leq K^R \left(\frac{M}{m} \right) \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^{n} p_i B_i^{\nu} \right), \end{aligned}$$

where $\nu \in [0, 1]$.

1. Introduction

The famous *Young inequality* for scalars says that, if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [8], [9] provided a refinement and an additive reverse for Young inequality as follows:

$$r \left(\sqrt{a} - \sqrt{b} \right)^2 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R \left(\sqrt{a} - \sqrt{b} \right)^2 \tag{1.2}$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

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We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function S is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \leq (1-\nu)a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.4}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.4) is due to Tominaga [14] while the first one is due to Furuichi [7].

It is an open question for the author if in the right hand side of (1.4) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R = \max\{1-\nu, \nu\}$.

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leq (1-\nu)a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zuo et al. in [16] while the second by Liao et al. [12].

In [16] the authors also showed that

$$K^r(h) \geq S(h^r) \text{ for } h > 0 \text{ and } r \in \left[0, \frac{1}{2}\right]$$

implying that the lower bound in (1.6) is better than the lower bound from (1.4).

We can give a simple direct proof for (1.6) as follows.

Recall the following result obtained by the author in 2006 [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}
(1.7)\qquad & n \min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right]\\
&\leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)\\
&\leq n \max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right],
\end{aligned}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (1.7) that

$$\begin{aligned}
(1.8)\qquad & 2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\
&\leq \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]\\
&\leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]
\end{aligned}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Now, if we write the inequality (1.8) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get (1.6).

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$(1.9)\qquad f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [6, p. 173]

$$f(A\otimes B)\geq(\leq)f(A)\otimes f(B)\text{ for all }A,\ B\geq 0. \tag{1.10}$$

This follows by observing that, if

$$A=\int_{[0,\infty)}tdE(t)\text{ and }B=\int_{[0,\infty)}sdF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)}f(st)\,dE(t)\otimes dF(s) \tag{1.11}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A\text{ and }(A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B)&\leq\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\leq\frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.12}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle(A\circ B)\,e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A\circ B=\mathcal{U}^*(A\otimes B)\,\mathcal{U} \tag{1.13}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [6, p. 173]

$$f(A\circ B)\geq(\leq)f(A)\circ f(B)\text{ for all }A,\ B\geq 0. \tag{1.14}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1\text{ for }A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1\text{ for }A>0. \tag{1.15}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [10] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if A_i and B_i are selfadjoint operators with $0 \leq m \leq A_i$, $B_i \leq M$, $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$ and $\nu \in [0,1]$, then

$$\begin{aligned} &\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right) \\ &\leq (1-\nu)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^n p_i B_i\right) \\ &\leq K^R\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right). \end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned} \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^n p_i B_i^{\nu}\right) &\leq \sum_{i=1}^n p_i \left[(1-\nu) A_i + \nu B_i\right] \circ 1 \\ &\leq K^R\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^n p_i B_i^{\nu}\right), \end{aligned}$$

where $\nu \in [0,1]$.

2. Main Results

We have the following result for the tensorial product:

Theorem 1. *Assume that A and B are selfadjoint operators with $0 \leq m \leq A$, $B \leq M$ for some constants $m < M$, then for all $\nu \in [0,1]$*

$$A^{1-\nu} \otimes B^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq K^R\left(\frac{M}{m}\right) A^{1-\nu} \otimes B^{\nu} \tag{2.1}$$

and, in particular

$$A^{1-\nu} \otimes A^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes A \leq K^R\left(\frac{M}{m}\right) A^{1-\nu} \otimes A^{\nu}. \tag{2.2}$$

For $\nu = 1/2$ we derive that

$$A^{1/2} \otimes B^{1/2} \leq \frac{1}{2}(A \otimes 1 + 1 \otimes B) \leq \frac{M+m}{2\sqrt{mM}} A^{1/2} \otimes B^{1/2} \tag{2.3}$$

and, in particular

$$A^{1/2} \otimes A^{1/2} \leq \frac{1}{2}(A \otimes 1 + 1 \otimes A) \leq \frac{M+m}{2\sqrt{mM}} A^{1/2} \otimes A^{1/2}. \tag{2.4}$$

Proof. Let $t,s\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\le\frac{t}{s}\le\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{t}{s}\in\left[\frac{m}{M},1\right)$ then $K\left(\frac{t}{s}\right)\le K\left(\frac{m}{M}\right)=K\left(\frac{M}{m}\right)$. If $\frac{t}{s}\in\left(1,\frac{M}{m}\right]$ then also $K\left(\frac{t}{s}\right)\le K\left(\frac{M}{m}\right)$. Therefore for any $t,s\in[m,M]$ we have from (1.6) that

$$t^{1-\nu}s^{\nu}\le(1-\nu)\,t+\nu s\le K^{R}\left(\frac{M}{m}\right)t^{1-\nu}s^{\nu},\tag{2.5}$$

where $R=\max\{1-\nu,\nu\}$.

If

$$A=\int_m^M tdE\,(t)\text{ and }B=\int_m^M sdF\,(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_m^M\int_m^M$ over $dE\,(t)\otimes dF\,(s)$ in (2.5), we derive that

$$\begin{aligned}&\int_m^M\int_m^M\left[(1-\nu)\,t+\nu s\right]dE\,(t)\otimes dF\,(s)\\&\le K^{R}\left(\frac{M}{m}\right)\int_m^M\int_m^M t^{1-\nu}s^{\nu}dE\,(t)\otimes dF\,(s)\,.\end{aligned}\tag{2.6}$$

Observe, by (1.9), that

$$\begin{aligned}&\int_m^M\int_m^M\left[(1-\nu)\,t+\nu s\right]dE\,(t)\otimes dF\,(s)\\&=(1-\nu)\int_m^M\int_m^M tdE\,(t)\otimes dF\,(s)+\nu\int_m^M\int_m^M sdE\,(t)\otimes dF\,(s)\\&=(1-\nu)\,A\otimes 1+\nu 1\otimes B\end{aligned}$$

and

$$\int_m^M\int_m^M t^{1-\nu}s^{\nu}dE\,(t)\otimes dF\,(s)=A^{1-\nu}\otimes B^{\nu}$$

and by (2.6) we derive (2.1). □

Corollary 1. *Assume that A_i and B_i are selfadjoint operators with $0\le m\le A_i$, $B_i\le M$, $p_i\ge 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $\nu\in[0,1]$, then*

$$\begin{aligned}&\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{i=1}^n p_iB_i^{\nu}\right)\\&\le(1-\nu)\left(\sum_{i=1}^n p_iA_i\right)\otimes 1+\nu 1\otimes\left(\sum_{i=1}^n p_iB_i\right)\\&\le K^{R}\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{i=1}^n p_iB_i^{\nu}\right).\end{aligned}\tag{2.7}$$

In particular,

$$\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \tag{2.8}$$
$$\leq (1-\nu)\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right)$$
$$\leq K^R\left(\frac{M}{m}\right)\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right).$$

Proof. From (2.1) we get

$$A_i^{1-\nu} \otimes B_j^{\nu} \leq (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j \leq K^R\left(\frac{M}{m}\right) A_i^{1-\nu} \otimes B_j^{\nu} \tag{2.9}$$

for all $i, j \in \{1, ..., n\}$.

If we multiply by $p_i p_j$ and sum, then we get

$$\sum_{i,j=1}^{n} p_i p_j \left(A_i^{1-\nu} \otimes B_j^{\nu}\right) \leq \sum_{i,j=1}^{n} p_i p_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j\right]$$
$$\leq K^R\left(\frac{M}{m}\right) \sum_{i,j=1}^{n} p_i p_j \left(A_i^{1-\nu} \otimes B_j^{\nu}\right),$$

that is equivalent to (2.7). □

Remark 1. *We observe that, if in Corollary 1 we take $\nu = 1/2$, then we get*

$$\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{1/2}\right) \tag{2.10}$$
$$\leq \frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i B_i\right)\right]$$
$$\leq \frac{M+m}{2\sqrt{mM}}\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{i=1}^{n} p_i B_i^{1/2}\right).$$

In particular,

$$\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \tag{2.11}$$
$$\leq \frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i\right)\right]$$
$$\leq \frac{M+m}{2\sqrt{mM}}\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{1/2}\right).$$

Corollary 2. *With the assumptions of Theorem 1 we have*

$$A^{1-\nu} \circ B^{\nu} \leq \left[(1-\nu) A + \nu B\right] \circ 1 \leq K^R\left(\frac{M}{m}\right) A^{1-\nu} \circ B^{\nu} \tag{2.12}$$

and, in particular

$$A^{1-\nu} \circ A^{\nu} \leq A \circ 1 \leq K^{R}\left(\frac{M}{m}\right) A^{1-\nu} \circ A^{\nu}. \tag{2.13}$$

For $\nu = 1/2$ we derive that

$$A^{1/2} \circ B^{1/2} \leq \frac{A+B}{2} \circ 1 \leq \frac{M+m}{2\sqrt{mM}} A^{1/2} \circ B^{1/2} \tag{2.14}$$

and, in particular

$$A^{1/2} \circ A^{1/2} \leq A \circ 1 \leq \frac{M+m}{2\sqrt{mM}} A^{1/2} \circ A^{1/2}. \tag{2.15}$$

Proof. We have the representation

$$X \circ Y = \mathcal{U}^{*}\left(X \otimes Y\right)\mathcal{U}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take $\mathcal{U}^{*}$ at the left and $\mathcal{U}$ at the right in (2.1), then we get

$$\begin{aligned}\mathcal{U}^{*}\left(A^{1-\nu} \otimes B^{\nu}\right)\mathcal{U} &\leq (1-\nu)\,\mathcal{U}^{*}\left(A \otimes 1\right)\mathcal{U} + \nu\,\mathcal{U}^{*}\left(1 \otimes B\right)\mathcal{U} \\ &\leq K^{R}\left(\frac{M}{m}\right)\mathcal{U}^{*}\left(A^{1-\nu} \otimes B^{\nu}\right)\mathcal{U},\end{aligned}$$

which gives

$$A^{1-\nu} \circ B^{\nu} \leq (1-\nu)\, A \circ 1 + \nu 1 \circ B \leq K^{R}\left(\frac{M}{m}\right) A^{1-\nu} \circ B^{\nu}$$

that is equivalent to (2.12). □

Remark 2. *Assume that A_i and B_i are selfadjoint operators with $0 \leq m \leq A_i$, $B_i \leq M$, $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $\nu \in [0,1]$, then by (2.12) we get*

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i B_i^{\nu}\right) &\leq \left[(1-\nu)\sum_{i=1}^{n} p_i A_i + \nu \sum_{i=1}^{n} p_i B_i\right] \circ 1 \\ &\leq K^{R}\left(\frac{M}{m}\right)\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i B_i^{\nu}\right)\end{aligned} \tag{2.16}$$

and, in particular,

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) &\leq \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 \\ &\leq K^{R}\left(\frac{M}{m}\right)\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right).\end{aligned} \tag{2.17}$$

For $\nu = 1/2$ we get

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} p_i B_i^{1/2}\right) &\leq \left(\sum_{i=1}^{n} p_i \frac{A_i + B_i}{2}\right) \circ 1 \\ &\leq \frac{M+m}{2\sqrt{mM}}\left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} p_i B_i^{1/2}\right)\end{aligned} \tag{2.18}$$

and

$$\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\le\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1 \le\frac{M+m}{2\sqrt{mM}}\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right). \tag{2.19}$$

We also have the following bounds when the spectra of the operators are not located in the same interval:

Theorem 2. *Assume that the selfadjoint operators A and B satisfy the conditions $0<m_1\le A\le M_1$ and $0<m_2\le B\le M_2$. Define*

$$U(m_1,M_1,m_2,M_2):=\begin{cases} K\left(\frac{M_1}{m_2}\right) & \text{if } 1\le\frac{m_1}{M_2},\\ \max\left\{K\left(\frac{M_1}{m_2}\right),K\left(\frac{m_1}{M_2}\right)\right\} & \text{if } \frac{m_1}{M_2}<1<\frac{M_1}{m_2},\\ K\left(\frac{m_1}{M_2}\right) & \text{if } \frac{M_1}{m_2}\le 1,\end{cases} \tag{2.20}$$

and

$$u(m_1,M_1,m_2,M_2):=\begin{cases} K\left(\frac{m_1}{M_2}\right) & \text{if } 1\le\frac{m_1}{M_2},\\ 1 & \text{if } \frac{m_1}{M_2}<1<\frac{M_1}{m_2},\\ K\left(\frac{M_1}{m_2}\right) & \text{if } \frac{M_1}{m_2}\le 1,\end{cases} \tag{2.21}$$

then for all $\nu\in[0,1]$

$$u^r(m_1,M_1,m_2,M_2)\,A^{1-\nu}\otimes B^{\nu}\le(1-\nu)\,A\otimes 1+\nu 1\otimes B \le U^R(m_1,M_1,m_2,M_2)\,A^{1-\nu}\otimes B^{\nu}, \tag{2.22}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, we have

$$u^{1/2}(m_1,M_1,m_2,M_2)\,A^{1/2}\otimes B^{1/2}\le\frac{1}{2}(A\otimes 1+1\otimes B) \le U^{1/2}(m_1,M_1,m_2,M_2)\,A^{1/2}\otimes B^{1/2}. \tag{2.23}$$

Proof. If $t\in[m_1,M_1]\subset(0,\infty)$ and $s\in[m_2,M_2]\subset(0,\infty)$, then

$$\frac{t}{s}\in\left[\frac{m_1}{M_2},\frac{M_1}{m_2}\right]\subset(0,\infty).$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, then we observe that

$$\max_{\tau\in\left[\frac{m_1}{M_2},\frac{M_1}{m_2}\right]} K(\tau)=\begin{cases} K\left(\frac{M_1}{m_2}\right) \text{ if } 1\le \frac{m_1}{M_2}, \\ \\ \max\left\{K\left(\frac{M_1}{m_2}\right),K\left(\frac{m_1}{M_2}\right)\right\} \\ \text{if } \frac{m_1}{M_2}<1<\frac{M_1}{m_2}, \\ \\ K\left(\frac{m_1}{M_2}\right) \text{ if } \frac{M_1}{m_2}\le 1, \end{cases} = U(m_1,M_1,m_2,M_2)$$

and

$$\min_{\tau\in\left[\frac{m_1}{M_2},\frac{M_1}{m_2}\right]} K(\tau)=\begin{cases} K\left(\frac{m_1}{M_2}\right) \text{ if } 1\le \frac{m_1}{M_2}, \\ \\ 1 \text{ if } \frac{m_1}{M_2}<1<\frac{M_1}{m_2}, \\ \\ K\left(\frac{M_1}{m_2}\right) \text{ if } \frac{M_1}{m_2}\le 1, \end{cases} = u(m_1,M_1,m_2,M_2).$$

By (1.6) we then get

$$\begin{aligned} &u^r(m_1,M_1,m_2,M_2)\,t^{1-\nu}s^\nu \\ &\le K^r\left(\frac{t}{s}\right)t^{1-\nu}s^\nu\le(1-\nu)\,t+\nu s \\ &\le K^R\left(\frac{t}{s}\right)t^{1-\nu}s^\nu\le U^R(m_1,M_1,m_2,M_2)\,t^{1-\nu}s^\nu, \end{aligned}\tag{2.24}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

If

$$A=\int_{m_1}^{M_1} t dE(t) \text{ and } B=\int_{m_2}^{M_2} s dF(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_{m_1}^{M_1}\int_{m_2}^{M_2}$ over $dE(t)\otimes dF(s)$ in (2.24), we derive that

$$\begin{aligned} &u^r(m_1,M_1,m_2,M_2)\int_{m_1}^{M_1}\int_{m_2}^{M_2} t^{1-\nu}s^\nu dE(t)\otimes dF(s) \\ &\le \int_{m_1}^{M_1}\int_{m_2}^{M_2}[(1-\nu)\,t+\nu s]\,dE(t)\otimes dF(s) \\ &\le U^R(m_1,M_1,m_2,M_2)\int_{m_1}^{M_1}\int_{m_2}^{M_2} t^{1-\nu}s^\nu dE(t)\otimes dF(s), \end{aligned}$$

which, by (1.9), gives (2.22). □

Corollary 3. *With the assumptions of Theorem 2, we have the Hadamard product inequalities for all* $\nu\in[0,1]$

$$\begin{aligned} u^r(m_1,M_1,m_2,M_2)\,A^{1-\nu}\circ B^\nu &\le [(1-\nu)\,A+\nu B]\circ 1 \\ &\le U^R(m_1,M_1,m_2,M_2)\,A^{1-\nu}\circ B^\nu, \end{aligned}\tag{2.25}$$

where $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

In particular, we have

$$u^{1/2}(m_1,M_1,m_2,M_2)\,A^{1/2}\circ B^{1/2}\le\frac{A+B}{2}\circ 1\le U^{1/2}(m_1,M_1,m_2,M_2)\,A^{1/2}\circ B^{1/2}. \tag{2.26}$$

Corollary 4. *Assume that the selfadjoint operators A_i and B_i satisfy the conditions $0<m_1\le A_i\le M_1$ and $0<m_2\le B_i\le M_2$, $p_i\ge 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $\nu\in[0,1]$. Then*

$$\begin{aligned}&u^r(m_1,M_1,m_2,M_2)\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{i=1}^n p_iB_i^{\nu}\right)\\&\le(1-\nu)\left(\sum_{i=1}^n p_iA_i\right)\otimes 1+\nu 1\otimes\left(\sum_{i=1}^n p_iB_i\right)\\&\le U^R(m_1,M_1,m_2,M_2)\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{i=1}^n p_iB_i^{\nu}\right)\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}&u^r(m_1,M_1,m_2,M_2)\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^n p_iB_i^{\nu}\right)\\&\le\left(\sum_{i=1}^n p_i\left[(1-\nu)A_i+\nu B_i\right]\right)\circ 1\\&\le U^R(m_1,M_1,m_2,M_2)\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^n p_iB_i^{\nu}\right).\end{aligned} \tag{2.28}$$

3. Related Results

We also have:

Theorem 3. *Assume that A, $B>0$ and $\nu\in[0,1]$, then*

$$(1-\nu)A^{\nu}\otimes B^{-\nu}+\nu A^{\nu-1}\otimes B^{1-\nu}\le\frac{1}{4}\left(A\otimes B^{-1}+A^{-1}\otimes B+2\right). \tag{3.1}$$

In particular

$$(1-\nu)A^{\nu}\otimes A^{-\nu}+\nu A^{\nu-1}\otimes A^{1-\nu}\le\frac{1}{4}\left(A\otimes A^{-1}+A^{-1}\otimes A+2\right). \tag{3.2}$$

For $\nu=1/2$ we obtain

$$A^{1/2}\otimes B^{-1/2}+A^{-1/2}\otimes B^{1/2}\le\frac{1}{2}\left(A\otimes B^{-1}+A^{-1}\otimes B+2\right) \tag{3.3}$$

and

$$A^{1/2}\otimes A^{-1/2}+A^{-1/2}\otimes A^{1/2}\le\frac{1}{2}\left(A\otimes A^{-1}+A^{-1}\otimes A+2\right). \tag{3.4}$$

Proof. We have from (1.6) that

$$\frac{(1-\nu)t+\nu s}{t^{1-\nu}s^{\nu}}\le K^R\left(\frac{t}{s}\right)\le K\left(\frac{t}{s}\right)=\frac{(t+s)^2}{4ts}=\frac{1}{4}\left(\frac{t}{s}+\frac{s}{t}+2\right),$$

namely

$$(1-\nu)\,t^{\nu}s^{-\nu}+\nu t^{\nu-1}s^{1-\nu}\leq\frac{1}{4}\left(ts^{-1}+t^{-1}s+2\right) \tag{3.5}$$

for $t,s>0$ and $\nu\in[0,1]$.

If

$$A=\int_{0}^{\infty}tdE\,(t)\ \text{ and } B=\int_{0}^{\infty}sdF\,(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_{0}^{\infty}\int_{0}^{\infty}$ over $dE\,(t)\otimes dF\,(s)$ in (3.5) we get

$$\begin{aligned}&\int_{0}^{\infty}\int_{0}^{\infty}\left[(1-\nu)\,t^{\nu}s^{-\nu}+\nu t^{\nu-1}s^{1-\nu}\right]dE\,(t)\otimes dF\,(s)\\&\leq\frac{1}{4}\int_{0}^{\infty}\int_{0}^{\infty}\left(ts^{-1}+t^{-1}s+2\right)dE\,(t)\otimes dF\,(s)\,.\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}&\int_{0}^{\infty}\int_{0}^{\infty}\left[(1-\nu)\,t^{\nu}s^{-\nu}+\nu t^{\nu-1}s^{1-\nu}\right]dE\,(t)\otimes dF\,(s)\\&=(1-\nu)\int_{0}^{\infty}\int_{0}^{\infty}t^{\nu}s^{-\nu}dE\,(t)\otimes dF\,(s)\\&+\nu\int_{0}^{\infty}\int_{0}^{\infty}t^{\nu-1}s^{1-\nu}dE\,(t)\otimes dF\,(s)\\&=(1-\nu)\,A^{\nu}\otimes B^{-\nu}+\nu A^{\nu-1}\otimes B^{1-\nu}\end{aligned}$$

and

$$\begin{aligned}&\int_{0}^{\infty}\int_{0}^{\infty}\left(ts^{-1}+t^{-1}s+2\right)dE\,(t)\otimes dF\,(s)\\&=A\otimes B^{-1}+A^{-1}\otimes B+2,\end{aligned}$$

hence by (3.6) we obtain (3.1). □

Corollary 5. *With the assumptions of Theorem 2, then*

$$(1-\nu)\,A^{\nu}\circ B^{-\nu}+\nu A^{\nu-1}\circ B^{1-\nu}\leq\frac{1}{4}\left(A\circ B^{-1}+A^{-1}\circ B+2\right). \tag{3.7}$$

In particular

$$(1-\nu)\,A^{\nu}\circ A^{-\nu}+\nu A^{\nu-1}\circ A^{1-\nu}\leq\frac{1}{2}\left(A\circ A^{-1}+1\right). \tag{3.8}$$

For $\nu=1/2$ we obtain

$$A^{1/2}\circ B^{-1/2}+A^{-1/2}\circ B^{1/2}\leq\frac{1}{2}\left(A\circ B^{-1}+A^{-1}\circ B+2\right) \tag{3.9}$$

and

$$A^{1/2}\circ A^{-1/2}\leq\frac{1}{2}\left(A\circ A^{-1}+1\right). \tag{3.10}$$

Theorem 4. *Assume that the selfadjoint operators A and B satisfy the conditions $0<m_1\leq A\leq M_1$ and $0<m_2\leq B\leq M_2$. Then*

$$\begin{aligned}u^{r}\left(m_1,M_1,m_2,M_2\right)&\leq(1-\nu)\,A^{\nu}\otimes B^{-\nu}+\nu A^{\nu-1}\otimes B^{1-\nu}\\&\leq U^{R}\left(m_1,M_1,m_2,M_2\right),\end{aligned} \tag{3.11}$$

where $U(m_1, M_1, m_2, M_2)$ *is defined by (2.20),* $u(m_1, M_1, m_2, M_2)$ *is given by (2.21),* $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

In particular,

$$u^{1/2}(m_1, M_1, m_2, M_2) \le (1-\nu) A^{1/2} \otimes B^{-1/2} + \nu A^{-1/2} \otimes B^{1/2} \le U^{1/2}(m_1, M_1, m_2, M_2). \tag{3.12}$$

Proof. From (2.24) we get

$$u^r(m_1, M_1, m_2, M_2) \le \frac{(1-\nu)t + \nu s}{t^{1-\nu}s^{\nu}} \le U^R(m_1, M_1, m_2, M_2),$$

namely

$$u^r(m_1, M_1, m_2, M_2) \le (1-\nu) t^{\nu} s^{-\nu} + \nu t^{\nu-1} s^{1-\nu} \le U^R(m_1, M_1, m_2, M_2), \tag{3.13}$$

for $t \in [m_1, M_1]$ and $s \in [m_2, M_2]$.

If

$$A = \int_{m_1}^{M_1} t dE(t) \text{ and } B = \int_{m_2}^{M_2} s dF(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_{m_1}^{M_1} \int_{m_2}^{M_2}$ over $dE(t) \otimes dF(s)$ in (3.13) we get (3.11). □

Corollary 6. *With the assumptions of Theorem 4, we have*

$$\begin{aligned} u^r(m_1, M_1, m_2, M_2) &\le (1-\nu) A^{\nu} \circ B^{-\nu} + \nu A^{\nu-1} \circ B^{1-\nu} \\ &\le U^R(m_1, M_1, m_2, M_2) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} u^{1/2}(m_1, M_1, m_2, M_2) &\le (1-\nu) A^{1/2} \circ B^{-1/2} + \nu A^{-1/2} \circ B^{-1/2} \\ &\le U^{1/2}(m_1, M_1, m_2, M_2). \end{aligned} \tag{3.15}$$

Similar inequalities may be stated for the weighted sums of operators, however the details are omitted.

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