SOME TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES VIA A CARTWRIGHT-FIELD RESULT

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if the operators A, B satisfy the conditions $0 < m \leq A$, $B \leq M$, then

$$\begin{aligned} 0 &\leq \frac{1}{M}\nu(1-\nu)\left(\frac{A^2\otimes 1 + 1\otimes B^2}{2} - A\otimes B\right) \\ &\leq (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \\ &\leq \frac{1}{m}\nu(1-\nu)\left(\frac{A^2\otimes 1 + 1\otimes B^2}{2} - A\otimes B\right) \end{aligned}$$

for all $\nu \in [0,1]$. We also have the following result for the Hadamard product

$$\begin{aligned} 0 &\leq \frac{1}{M}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \\ &\leq [(1-\nu)A + \nu B]\circ 1 - A^{1-\nu}\circ B^{\nu} \\ &\leq \frac{1}{m}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right). \end{aligned}$$

1. Introduction

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.1}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [4] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Since $\max\{a,b\}\min\{a,b\} = ab$ for $a, b > 0$, then by (1.1) we get

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)\min\{a,b\}\frac{(b-a)^2}{ab} &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)\max\{a,b\}\frac{(b-a)^2}{ab}, \end{aligned}$$

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namely

$$0 \le \frac{1}{2}\nu(1-\nu)\min\{a,b\}\left(\frac{a}{b}+\frac{b}{a}-2\right) \le (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\max\{a,b\}\left(\frac{a}{b}+\frac{b}{a}-2\right), \tag{1.2}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.3}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \ge (\le) f(s) f(t) \text{ for all } s, t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [7, p. 173]

$$f(A \otimes B) \ge (\le) f(A) \otimes f(B) \text{ for all } A,\ B \ge 0. \tag{1.4}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.5}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [9] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B)\leq\frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right]\leq\frac{1}{2}(A\otimes B+B\otimes A) \tag{1.6}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle(A\circ B)e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B=\mathcal{U}^*(A\otimes B)\mathcal{U} \tag{1.7}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [7, p. 173]

$$f(A\circ B)\geq(\leq)f(A)\circ f(B)\text{ for all }A,\ B\geq0. \tag{1.8}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ1\text{ for }A,\ B\geq0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq1\text{ for }A>0. \tag{1.9}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^2\circ1\right)^{1/2}\left(B^2\circ1\right)^{1/2}\text{ for }A,\ B\geq0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^2\circ B^2\right)^{1/2}\text{ for }A,\ B\geq0.$$

It has been shown in [8] that $\left(A^2\circ1\right)^{1/2}\left(B^2\circ1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if the operators A, B satisfy the conditions $0<m\leq A,\ B\leq M$, then

$$\begin{aligned}0&\leq\frac{1}{M}\nu(1-\nu)\left(\frac{A^2\otimes1+1\otimes B^2}{2}-A\otimes B\right)\\&\leq(1-\nu)A\otimes1+\nu1\otimes B-A^{1-\nu}\otimes B^{\nu}\\&\leq\frac{1}{m}\nu(1-\nu)\left(\frac{A^2\otimes1+1\otimes B^2}{2}-A\otimes B\right)\end{aligned}$$

for all $\nu \in [0,1]$. We also have the following result for the Hadamard product

$$\begin{aligned} 0 &\leq \frac{1}{M}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \\ &\leq [(1-\nu)A+\nu B]\circ 1 - A^{1-\nu}\circ B^{\nu} \\ &\leq \frac{1}{m}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right). \end{aligned}$$

2. Main Results

We start to the following result:

Theorem 1. *Assume that the operators A, B satisfy the conditions $0 < m \leq A$, $B \leq M$, then*

$$\begin{aligned} 0 &\leq \frac{1}{M}\nu(1-\nu)\left(\frac{A^2\otimes 1 + 1\otimes B^2}{2} - A\otimes B\right) \\ &\leq (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \\ &\leq \frac{1}{m}\nu(1-\nu)\left(\frac{A^2\otimes 1 + 1\otimes B^2}{2}\right) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} 0 &\leq m\nu(1-\nu)\left(\frac{A\otimes B^{-1} + A^{-1}\otimes B}{2} - 1\right) \\ &\leq (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \\ &\leq M\nu(1-\nu)\left(\frac{A\otimes B^{-1} + A^{-1}\otimes B}{2} - 1\right) \end{aligned} \tag{2.2}$$

for all $\nu \in [0,1]$.

In particular, we have

$$\begin{aligned} 0 &\leq \frac{1}{4M}\left(\frac{A^2\otimes 1 + 1\otimes B^2}{2} - A\otimes B\right) \\ &\leq \frac{1}{2}(A\otimes 1 + 1\otimes B) - A^{1/2}\otimes B^{1/2} \leq \frac{1}{4m}\left(\frac{A^2\otimes 1 + 1\otimes B^2}{2} - A\otimes B\right) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{4}m\left(\frac{A\otimes B^{-1} + A^{-1}\otimes B}{2} - 1\right) \\ &\leq \frac{1}{2}(A\otimes 1 + 1\otimes B) - A^{1/2}\otimes B^{1/2} \leq \frac{1}{4}M\left(\frac{A\otimes B^{-1} + A^{-1}\otimes B}{2} - 1\right). \end{aligned} \tag{2.4}$$

Proof. Now if t, $s \in [m,M] \subset (0,\infty)$, then we have from (1.1) and (1.2) the following two inequalities

$$\begin{aligned} 0 &\leq \frac{1}{2M}\nu(1-\nu)\left(t^2 - 2ts + s^2\right) \leq (1-\nu)t + \nu s - t^{1-\nu}s^{\nu} \\ &\leq \frac{1}{2m}\nu(1-\nu)\left(t^2 - 2ts + s^2\right) \end{aligned} \tag{2.5}$$

and

$$0 \le \frac{1}{2} m\nu (1-\nu) \left(\frac{t}{s} + \frac{s}{t} - 2 \right) \le (1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \le \frac{1}{2} M\nu (1-\nu) \left(\frac{t}{s} + \frac{s}{t} - 2 \right) \tag{2.6}$$

for $\nu \in [0,1]$.

If

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, we get

$$\begin{aligned} 0 &\le \frac{1}{2M} \nu (1-\nu) \int_m^M \int_m^M \left(t^2 - 2ts + s^2 \right) dE(t) \otimes dF(s) \\ &\le \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) \\ &\le \frac{1}{2m} \nu (1-\nu) \int_m^M \int_m^M \left(t^2 - 2ts + s^2 \right) dE(t) \otimes dF(s) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} 0 &\le \frac{1}{2} m\nu (1-\nu) \int_m^M \int_m^M \left(\frac{t}{s} + \frac{s}{t} - 2 \right) dE(t) \otimes dF(s) \\ &\le \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) \\ &\le \frac{1}{2} M\nu (1-\nu) \int_m^M \int_m^M \left(\frac{t}{s} + \frac{s}{t} - 2 \right) dE(t) \otimes dF(s). \end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned} &\int_m^M \int_m^M \left(t^2 - 2ts + s^2 \right) dE(t) \otimes dF(s) \\ &= \int_m^M \int_m^M t^2 dE(t) \otimes dF(s) + \int_m^M \int_m^M s^2 dE(t) \otimes dF(s) \\ &- 2 \int_m^M \int_m^M ts dE(t) \otimes dF(s) \\ &= A^2 \otimes 1 + 1 \otimes B^2 - 2A \otimes B, \end{aligned}$$

$$\begin{aligned} &\int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) \\ &= (1-\nu) \int_m^M \int_m^M t dE(t) \otimes dF(s) + \nu \int_m^M \int_m^M s dE(t) \otimes dF(s) \\ &- \int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) \\ &= (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \end{aligned}$$

and

$$\begin{aligned}
&\int_m^M \int_m^M \left(\frac{t}{s}+\frac{s}{t}-2\right) dE(t)\otimes dF(s) \\
&= \int_m^M \int_m^M \left(ts^{-1}+t^{-1}s-2\right) dE(t)\otimes dF(s) \\
&= \int_m^M \int_m^M ts^{-1} dE(t)\otimes dF(s) + \int_m^M \int_m^M t^{-1}s dE(t)\otimes dF(s) \\
&- 2\int_m^M \int_m^M dE(t)\otimes dF(s) \\
&= A\otimes B^{-1} + A^{-1}\otimes B - 2.
\end{aligned}$$

Then by (2.7) and (2.8) we deduce the desired results (2.1) and (2.2). □

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}
0 &\le \frac{1}{M}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \\
&\le [(1-\nu)A+\nu B]\circ 1 - A^{1-\nu}\circ B^{\nu} \\
&\le \frac{1}{m}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right)
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
0 &\le m\nu(1-\nu)\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2} - 1\right) \\
&\le [(1-\nu)A+\nu B]\circ 1 - A^{1-\nu}\circ B^{\nu} \\
&\le M\nu(1-\nu)\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2} - 1\right)
\end{aligned} \tag{2.10}$$

for all $\nu\in[0,1]$.

In particular, we have

$$\begin{aligned}
0 &\le \frac{1}{4M}\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \\
&\le \frac{1}{2}(A\circ 1 + 1\circ B) - A^{1/2}\circ B^{1/2} \le \frac{1}{4m}\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right)
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
0 &\le \frac{1}{4}m\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2} - 1\right) \\
&\le \frac{1}{2}(A\circ 1 + 1\circ B) - A^{1/2}\circ B^{1/2} \le \frac{1}{4}M\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2} - 1\right).
\end{aligned} \tag{2.12}$$

Proof. For X, Y we have the representation

$$X\circ Y = \mathcal{U}^*(X\otimes Y)\mathcal{U},$$

where $\mathcal{U}: H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j\otimes e_j$ for all $j\in\mathbb{N}$.

By (2.1) we derive

$$\begin{aligned}0 &\le \frac{1}{2M}\nu(1-\nu)\mathcal{U}^*\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\mathcal{U}\\ &\le \mathcal{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\right]\mathcal{U}\\ &\le \frac{1}{2m}\nu(1-\nu)\mathcal{U}^*\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\mathcal{U},\end{aligned}$$

namely

$$\begin{aligned}0 &\le \frac{1}{2M}\nu(1-\nu)\left[\mathcal{U}^*\left(A^2\otimes 1\right)\mathcal{U}+\mathcal{U}^*\left(1\otimes B^2\right)\mathcal{U}-2\mathcal{U}^*(A\otimes B)\mathcal{U}\right]\\ &\le (1-\nu)\mathcal{U}^*(A\otimes 1)\mathcal{U}+\nu\mathcal{U}^*(1\otimes B)\mathcal{U}-\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U}\\ &\le \frac{1}{2m}\nu(1-\nu)\left[\mathcal{U}^*\left(A^2\otimes 1\right)\mathcal{U}+\mathcal{U}^*\left(1\otimes B^2\right)\mathcal{U}-2\mathcal{U}^*(A\otimes B)\mathcal{U}\right],\end{aligned}$$

which is equivalent to (2.9). □

Remark 1. *If we take $B=A$ in Corollary 1, then we get for an operator $0<m\le A\le M$, that*

$$\begin{aligned}0 &\le \frac{1}{M}\nu(1-\nu)\left(A^2\circ 1-A\circ A\right)\\ &\le A\circ 1-A^{1-\nu}\circ A^{\nu}\le \frac{1}{m}\nu(1-\nu)\left(A^2\circ 1-A\circ A\right)\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}0 &\le m\nu(1-\nu)\left(A\circ A^{-1}-1\right)\\ &\le A\circ 1-A^{1-\nu}\circ A^{\nu}\le M\nu(1-\nu)\left(A\circ A^{-1}-1\right)\end{aligned}\tag{2.14}$$

for all $\nu\in[0,1]$.

In particular, we have

$$0\le \frac{1}{4M}\left(A^2\circ 1-A\circ A\right)\le A\circ 1-A^{1/2}\circ B^{1/2}\le \frac{1}{4m}\left(A^2\circ 1-A\circ A\right)\tag{2.15}$$

and

$$0\le \frac{1}{4}m\left(A\circ A^{-1}-1\right)\le A\circ 1-A^{1/2}\circ B^{1/2}\le \frac{1}{4}M\left(A\circ A^{-1}-1\right).\tag{2.16}$$

The second main result is as follows:

Theorem 2. *Assume that the finite intervals $I,\ J\subset(0,\infty)$ satisfy the condition $I/J:=\{x/y,\ x\in I,\ y\in J\}\subseteq[\gamma,\Gamma]\subset(0,\infty)$. If $A,\ B$ are selfadjoint operators with $\operatorname{Sp}(A)\subset I$ and $\operatorname{Sp}(B)\subset J$, then*

$$\begin{aligned}0 &\le \frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)\\ &\le \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-2}+1}{2}-A\otimes B^{-1}\right)\\ &\le 1-\nu+\nu A\otimes B^{-1}-A^{\nu}\otimes B^{-\nu}\\ &\le \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-2}+1}{2}-A\otimes B^{-1}\right)\\ &\le \frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma),\end{aligned}\tag{2.17}$$

where

$$c(\gamma,\Gamma) := \begin{cases} (\Gamma-1)^2 & \textit{if } \Gamma<1, \\ 0 \textit{ if } \gamma \le 1 \le \Gamma, & \\ \frac{(\gamma-1)^2}{\Gamma} & \textit{if } 1<\gamma \end{cases}$$

and

$$C(\gamma,\Gamma) := \begin{cases} \frac{(\gamma-1)^2}{\gamma} & \textit{if } \Gamma<1, \\ \frac{1}{\gamma}\max\left\{(\gamma-1)^2,(\Gamma-1)^2\right\} & \textit{if } \gamma\le 1\le\Gamma, \\ (\Gamma-1)^2 & \textit{if } 1<\gamma. \end{cases}$$

We also have

$$\begin{aligned} 0 &\le \frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)(1\otimes B) \\ &\le \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-1}+1\otimes B}{2}-A\otimes 1\right) \\ &\le (1-\nu)(1\otimes B)+\nu A\otimes 1-A^{\nu}\otimes B^{1-\nu} \\ &\le \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-1}+1\otimes B}{2}-A\otimes 1\right) \\ &\le \frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma)(1\otimes B). \end{aligned} \tag{2.18}$$

Proof. If we write the inequality (1.1) for $a=1$ and $b=x$ we get

$$\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{x,1\}} \le 1-\nu+\nu x-x^{\nu} \le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{x,1\}} \tag{2.19}$$

for any $x>0$ and for any $\nu\in[0,1]$.

If $x\in[\gamma,\Gamma]\subset(0,\infty)$, then $\max\{x,1\}\le\max\{\Gamma,1\}$ and $\min\{\gamma,1\}\le\min\{x,1\}$ and by (2.19) we get

$$\begin{aligned} 0 &\le \frac{1}{2}\nu(1-\nu)\frac{\min_{x\in[\gamma,\Gamma]}(x-1)^2}{\max\{\Gamma,1\}} \le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{\Gamma,1\}} \\ &\le 1-\nu+\nu x-x^{\nu} \le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{\gamma,1\}} \\ &\le \frac{1}{2}\nu(1-\nu)\frac{\max_{x\in[\gamma,\Gamma]}(x-1)^2}{\min\{\gamma,1\}} \end{aligned} \tag{2.20}$$

for any $x\in[\gamma,\Gamma]$ and for any $\nu\in[0,1]$.

Observe that

$$\min_{x\in[\gamma,\Gamma]}(x-1)^2 = \begin{cases} (\Gamma-1)^2 & \text{if } \Gamma<1, \\ 0 \text{ if } \gamma\le 1\le\Gamma, & \\ (\gamma-1)^2 & \text{if } 1<\gamma \end{cases}$$

and

$$\max_{x\in[\gamma,\Gamma]}(x-1)^2 = \begin{cases} (\gamma-1)^2 & \text{if } \Gamma<1, \\ \max\left\{(\gamma-1)^2,(\Gamma-1)^2\right\} & \text{if } \gamma\le 1\le\Gamma, \\ (\Gamma-1)^2 & \text{if } 1<\gamma. \end{cases}$$

Then

$$\frac{\min_{x\in[\gamma,\Gamma]}(x-1)^2}{\max\{\Gamma,1\}} = c(\gamma,\Gamma)$$

and

$$\frac{\max_{x\in[\gamma,\Gamma]}(x-1)^2}{\min\{\gamma,1\}}=C(\gamma,\Gamma).$$

Using the inequality (2.20) we have

$$0\le\frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)\le\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{\Gamma,1\}}\tag{2.21}$$
$$\le 1-\nu+\nu x-x^{\nu}\le\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{\gamma,1\}}$$
$$\le\frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma)$$

for any $x\in[\gamma,\Gamma]$ and for any $\nu\in[0,1]$.

Let t, $s>0$ such that $\frac{t}{s}\in[\gamma,\Gamma]$, then by (2.21) we get for $x=\frac{t}{s}$ that

$$0\le\frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)\le\frac{1}{2\max\{\Gamma,1\}}\nu(1-\nu)\left(t^2s^{-2}+1-2ts^{-1}\right)\tag{2.22}$$
$$\le 1-\nu+\nu ts^{-1}-t^{\nu}s^{-\nu}$$
$$\le\frac{1}{2\min\{\gamma,1\}}\nu(1-\nu)\left(t^2s^{-2}+1-2ts^{-1}\right)\le\frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma).$$

If

$$A=\int_I tdE(t)\text{ and }B=\int_J sdF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_I\int_J$ over $dE(t)\otimes dF(s)$, we get

$$0\le\frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)\int_I\int_J dE(t)\otimes dF(s)$$
$$\le\frac{1}{2\max\{\Gamma,1\}}\nu(1-\nu)\int_I\int_J\left(t^2s^{-2}+1-2ts^{-1}\right)dE(t)\otimes dF(s)$$
$$\le\int_I\int_J\left[1-\nu+\nu ts^{-1}-t^{\nu}s^{-\nu}\right]dE(t)\otimes dF(s)$$
$$\le\frac{1}{2\min\{\gamma,1\}}\nu(1-\nu)\int_I\int_J\left(t^2s^{-2}+1-2ts^{-1}\right)dE(t)\otimes dF(s)$$
$$\le\frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma)\int_I\int_J dE(t)\otimes dF(s),$$

which is equivalent to (2.17).

Now, if we multiply by s in the inequality (2.22), then we get

$$0\le\frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)s\le\frac{1}{2\max\{\Gamma,1\}}\nu(1-\nu)\left(t^2s^{-1}+s-2t\right)$$
$$\le(1-\nu)s+\nu t-t^{\nu}s^{1-\nu}$$
$$\le\frac{1}{2\min\{\gamma,1\}}\nu(1-\nu)\left(t^2s^{-1}+s-2t\right)$$
$$\le\frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma)s,$$

for $t\in I$, $s\in J$, which, by a similar argument as above, gives the desired tensorial inequality (2.18). □

Remark 2. *We observe that if* $0 < \gamma_1 \leq A \leq \Gamma_1$ *and* $0 < \gamma_2 \leq B \leq \Gamma_2$ *then we can take* $\gamma = \frac{\gamma_1}{\Gamma_2}$ *and* $\Gamma = \frac{\Gamma_1}{\gamma_2}$ *in the above inequalities (2.17) and (2.18). If* $\gamma_2 = \gamma_1 = m$ *and* $\Gamma_2 = \Gamma_1 = M$ *then we can take* $\gamma = \frac{m}{M} \leq 1$ *and* $\Gamma = \frac{M}{m} \geq 1$ *in (2.17) and (2.18).*

Corollary 2. *With the assumptions of Theorem 2, we have the following inequalities for the Hadamard product*

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu)\,c(\gamma,\Gamma) \\ &\leq \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\circ B^{-2}+1}{2} - A\circ B^{-1}\right) \\ &\leq 1-\nu+\nu A\circ B^{-1} - A^{\nu}\circ B^{-\nu} \\ &\leq \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\circ B^{-2}+1}{2} - A\circ B^{-1}\right) \\ &\leq \frac{1}{2}\nu(1-\nu)\,C(\gamma,\Gamma), \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu)\,c(\gamma,\Gamma)(1\circ B) \\ &\leq \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2} - A\circ 1\right) \\ &\leq (1-\nu)(1\circ B)+\nu A\circ 1 - A^{\nu}\circ B^{1-\nu} \\ &\leq \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2} - A\circ 1\right) \\ &\leq \frac{1}{2}\nu(1-\nu)\,C(\gamma,\Gamma)(1\circ B) \end{aligned} \tag{2.24}$$

for all $\nu \in [0,1]$.

Remark 3. *We observe that, if* $0 < m \leq A, B \leq M$*, then by Corollary 2 we get*

$$\begin{aligned} 0 &\leq \frac{m}{M}\nu(1-\nu)\left(\frac{A^2\circ B^{-2}+1}{2} - A\circ B^{-1}\right) \\ &\leq 1-\nu+\nu A\circ B^{-1} - A^{\nu}\circ B^{-\nu} \\ &\leq \frac{M}{m}\nu(1-\nu)\left(\frac{A^2\circ B^{-2}+1}{2} - A\circ B^{-1}\right) \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} 0 &\leq \frac{m}{M}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2} - A\circ 1\right) \\ &\leq [(1-\nu)B+\nu A]\circ 1 - A^{\nu}\circ B^{1-\nu} \end{aligned} \tag{2.26}$$

$$\leq \frac{M}{m}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right)$$
$$\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2(1\circ B).$$

In particular, for $\nu=1/2$, we derive

$$0\leq \frac{m}{4M}\left(\frac{A^2\circ B^{-2}+1}{2}-A\circ B^{-1}\right) \tag{2.27}$$
$$\leq \frac{1}{2}\left(1+A\circ B^{-1}\right)-A^{1/2}\circ B^{-1/2}$$
$$\leq \frac{M}{4m}\left(\frac{A^2\circ B^{-2}+1}{2}-A\circ B^{-1}\right)\leq \frac{M}{8m}\left(\frac{M}{m}-1\right)^2,$$

and

$$0\leq \frac{m}{4M}\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right) \tag{2.28}$$
$$\leq \frac{A+B}{2}\circ 1-A^{\nu}\circ B^{1-\nu}$$
$$\leq \frac{M}{4m}\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right)\leq \frac{1}{8}\frac{M}{m}\left(\frac{M}{m}-1\right)^2(1\circ B).$$

Moreover, if $0<m\leq A\leq M$, then by taking $B=A$ in (2.25)-(2.28), we get

$$0\leq \frac{m}{M}\nu(1-\nu)\left(\frac{A^2\circ A^{-2}+1}{2}-A\circ A^{-1}\right) \tag{2.29}$$
$$\leq 1-\nu+\nu A\circ A^{-1}-A^{\nu}\circ A^{-\nu}$$
$$\leq \frac{M}{m}\nu(1-\nu)\left(\frac{A^2\circ A^{-2}+1}{2}-A\circ A^{-1}\right)\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2,$$

and

$$0\leq \frac{m}{2M}\nu(1-\nu)\left(A^2\circ B^{-1}-A\circ 1\right)\leq A\circ 1-A^{\nu}\circ A^{1-\nu} \tag{2.30}$$
$$\leq \frac{M}{2m}\nu(1-\nu)\left(A^2\circ B^{-1}-A\circ 1\right)$$
$$\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2(1\circ A).$$

In particular, for $\nu=1/2$, we derive

$$0\leq \frac{m}{4M}\left(\frac{A^2\circ A^{-2}+1}{2}-A\circ A^{-1}\right)\leq \frac{1+A\circ A^{-1}}{2}-A^{1/2}\circ A^{-1/2} \tag{2.31}$$
$$\leq \frac{M}{4m}\left(\frac{A^2\circ A^{-2}+1}{2}-A\circ A^{-1}\right)\leq \frac{M}{8m}\left(\frac{M}{m}-1\right)^2,$$

and

$$0\leq \frac{m}{8M}\left(A^2\circ A^{-1}-A\circ 1\right)\leq A\circ 1-A^{1/2}\circ A^{-1/2} \tag{2.32}$$
$$\leq \frac{M}{8m}\left(A^2\circ A^{-1}-A\circ 1\right)\leq \frac{1}{8}\frac{M}{m}\left(\frac{M}{m}-1\right)^2(1\circ A).$$

3. Some Inequalities for Sums

We have:

Proposition 1. *Assume that the operators* A_i, B_i *satisfy the conditions* $0 < m \le A_i,\ B_i \le M$ *and* $p_i \ge 0$ *for* $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^n p_i = 1$, *then*

$$\begin{aligned}(3.1)\quad & 0 \le \frac{1}{M}\nu(1-\nu) \\ & \times \left(\frac{\left(\sum_{i=1}^n p_i A_i^2\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^n p_i B_i^2\right)}{2} - \left(\sum_{i=1}^n p_i A_i\right) \otimes \left(\sum_{i=1}^n p_i B_i\right)\right) \\ & \le (1-\nu)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^n p_i B_i\right) \\ & - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right) \\ & \le \frac{1}{m}\nu(1-\nu) \\ & \times \left(\frac{\left(\sum_{i=1}^n p_i A_i^2\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^n p_i B_i^2\right)}{2} - \left(\sum_{i=1}^n p_i A_i\right) \otimes \left(\sum_{i=1}^n p_i B_i\right)\right)\end{aligned}$$

and

$$\begin{aligned}(3.2)\quad & 0 \le m\nu(1-\nu) \\ & \times \left(\frac{\left(\sum_{i=1}^n p_i A_i\right) \otimes \left(\sum_{i=1}^n p_i B_i^{-1}\right) + \left(\sum_{i=1}^n p_i A_i^{-1}\right) \otimes \left(\sum_{i=1}^n p_i B_i\right)}{2} - 1\right) \\ & \le (1-\nu)\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu 1 \otimes \left(\sum_{i=1}^n p_i B_i\right) \\ & - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{i=1}^n p_i B_i^{\nu}\right) \\ & \le M\nu(1-\nu) \\ & \times \left(\frac{\left(\sum_{i=1}^n p_i A_i\right) \otimes \left(\sum_{i=1}^n p_i B_i^{-1}\right) + \left(\sum_{i=1}^n p_i A_i^{-1}\right) \otimes \left(\sum_{i=1}^n p_i B_i\right)}{2} - 1\right)\end{aligned}$$

for all $\nu \in [0, 1]$.

Proof. From Theorem 1 we have

$$\begin{aligned}(3.3)\qquad & 0 \le \frac{1}{2M}\nu(1-\nu)\left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\ & \le (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \\ & \le \frac{1}{2m}\nu(1-\nu)\left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right)\end{aligned}$$

and

$$0 \le \frac{1}{2} m\nu (1-\nu) \left(A_i \otimes B_j^{-1} + A_i^{-1} \otimes B_j - 2 \right) \tag{3.4}$$
$$\le (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}$$
$$\le \frac{1}{2} M\nu (1-\nu) \left(A_i \otimes B_j^{-1} + A_i^{-1} \otimes B - 2 \right)$$

for all $i, j \in \{1, ..., n\}$.

If we multiply the inequalities (3.3) and (3.4) by $p_i p_j$ and sum, then we get

$$0 \le \frac{1}{2M} \nu (1-\nu) \sum_{i,j=1}^{n} p_i p_j \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j \right)$$
$$\le \sum_{i,j=1}^{n} p_i p_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \right]$$
$$\le \frac{1}{2m} \nu (1-\nu) \sum_{i,j=1}^{n} p_i p_j \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j \right)$$

and

$$0 \le \frac{1}{2} m\nu (1-\nu) \sum_{i,j=1}^{n} p_i p_j \left(A_i \otimes B_j^{-1} + A_i^{-1} \otimes B_j - 2 \right)$$
$$\le \sum_{i,j=1}^{n} p_i p_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \right]$$
$$\le \frac{1}{2} M\nu (1-\nu) \sum_{i,j=1}^{n} p_i p_j \left(A_i \otimes B_j^{-1} + A_i^{-1} \otimes B - 2 \right),$$

which gives (3.1) and (3.2). □

Corollary 3. *With the assumptions of Proposition 1 we have the following inequalities for the Hadamard product*

$$0 \le \frac{1}{M} \nu (1-\nu) \tag{3.5}$$
$$\times \left(\left(\sum_{i=1}^{n} p_i \left(\frac{A_i^2 + B_i^2}{2} \right) \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{i=1}^{n} p_i B_i \right) \right)$$
$$\le \left(\sum_{i=1}^{n} p_i \left[(1-\nu) A_i + \nu B_i \right] \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^{n} p_i B_i^{\nu} \right)$$
$$\le \frac{1}{m} \nu (1-\nu)$$
$$\times \left(\left(\sum_{i=1}^{n} p_i \left(\frac{A_i^2 + B_i^2}{2} \right) \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{i=1}^{n} p_i B_i \right) \right)$$

and

$$0 \le m\nu(1-\nu) \times \left(\frac{\left(\sum_{i=1}^n p_i A_i\right) \circ \left(\sum_{i=1}^n p_i B_i^{-1}\right) + \left(\sum_{i=1}^n p_i A_i^{-1}\right) \circ \left(\sum_{i=1}^n p_i B_i\right)}{2} - 1 \right) \le \left(\sum_{i=1}^n p_i \left[(1-\nu) A_i + \nu B_i\right] \right) \circ 1 - \left(\sum_{i=1}^n p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^n p_i B_i^{\nu} \right) \le M\nu(1-\nu) \times \left(\frac{\left(\sum_{i=1}^n p_i A_i\right) \circ \left(\sum_{i=1}^n p_i B_i^{-1}\right) + \left(\sum_{i=1}^n p_i A_i^{-1}\right) \circ \left(\sum_{i=1}^n p_i B_i\right)}{2} - 1 \right). \tag{3.6}$$

Remark 4. *We observe that for $B_i = A_i$, $i \in \{1, ..., n\}$ in Corollary 3, then*

$$0 \le \frac{1}{M}\nu(1-\nu) \left(\left(\sum_{i=1}^n p_i A_i^2 \right) \circ 1 - \left(\sum_{i=1}^n p_i A_i \right) \circ \left(\sum_{i=1}^n p_i A_i \right) \right) \le \left(\sum_{i=1}^n p_i A_i \right) \circ 1 - \left(\sum_{i=1}^n p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^n p_i A_i^{\nu} \right) \le \frac{1}{m}\nu(1-\nu) \left(\left(\sum_{i=1}^n p_i A_i^2 \right) \circ 1 - \left(\sum_{i=1}^n p_i A_i \right) \circ \left(\sum_{i=1}^n p_i A_i \right) \right) \tag{3.7}$$

and

$$0 \le m\nu(1-\nu) \left(\left(\sum_{i=1}^n p_i A_i \right) \circ \left(\sum_{i=1}^n p_i A_i^{-1} \right) - 1 \right) \le \left(\sum_{i=1}^n p_i A_i \right) \circ 1 - \left(\sum_{i=1}^n p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^n p_i A_i^{\nu} \right) \le M\nu(1-\nu) \left(\left(\sum_{i=1}^n p_i A_i \right) \circ \left(\sum_{i=1}^n p_i A_i^{-1} \right) - 1 \right). \tag{3.8}$$

4. Inequalities for Power Series

We also have the following result for power series

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 \le A$, $B \le 1$ and $0 \le \alpha, \beta < R$, then*

$$0 \le \nu(1-\nu) \left[\frac{f(\beta) f\left(\alpha A^2\right) \otimes 1 + f(\alpha) 1 \otimes f\left(\beta B^2\right)}{2} - f(\alpha A) \otimes f(\beta B) \right] \le (1-\nu) f(\beta) f(\alpha A) \otimes 1 + \nu f(\alpha) 1 \otimes f(\beta B) - f\left(\alpha A^{1-\nu}\right) \otimes f\left(\beta B^{\nu}\right) \tag{4.1}$$

for all $\nu \in [0,1]$.

In particular, for $\nu = 1/2$ we get

$$0 \le \frac{1}{4}\left[\frac{f(\beta) f\left(\alpha A^2\right) \otimes 1 + f(\alpha) 1 \otimes f\left(\beta B^2\right)}{2} - f(\alpha A) \otimes f(\beta B)\right] \tag{4.2}$$
$$\le \frac{1}{2}\left[f(\beta) f(\alpha A) \otimes 1 + f(\alpha) 1 \otimes f(\beta B)\right] - f\left(\alpha A^{1/2}\right) \otimes f\left(\beta B^{1/2}\right).$$

If $R = \infty$, then for $0 \le A,\ B \le 1$ and $0 \le \alpha, \beta$,

$$0 \le \nu(1-\nu)\left[\frac{f(\beta) f\left(\alpha A^2\right) \otimes 1 + f(\alpha) 1 \otimes f\left(\beta B^2\right)}{2} - f(\alpha A) \otimes f(\beta B)\right] \tag{4.3}$$
$$\le (1-\nu) f(\beta) f(\alpha A) \otimes 1 + \nu f(\alpha) 1 \otimes f(\beta B) - f\left(\alpha A^{1-\nu}\right) \otimes f\left(\beta B^{\nu}\right)$$
$$\le \nu(1-\nu)\left[\frac{f(\alpha A) \otimes f\left(\beta B^{-1}\right) + f\left(\alpha A^{-1}\right) \otimes f(\beta B)}{2} - f(\alpha) f(\beta) 1\right]$$

for all $\nu \in [0,1]$.

In particular, for $\nu = 1/2$ we get

$$0 \le \frac{1}{2}\left[\frac{f(\beta) f\left(\alpha A^2\right) \otimes 1 + f(\alpha) 1 \otimes f\left(\beta B^2\right)}{2} - f(\alpha A) \otimes f(\beta B)\right] \tag{4.4}$$
$$\le \frac{1}{2}\left[f(\beta) f(\alpha A) \otimes 1 + f(\alpha) 1 \otimes f(\beta B)\right] - f\left(\alpha A^{1/2}\right) \otimes f\left(\beta B^{1/2}\right)$$
$$\le \frac{1}{4}\left[\frac{f(\alpha A) \otimes f\left(\beta B^{-1}\right) + f\left(\alpha A^{-1}\right) \otimes f(\beta B)}{2} - f(\alpha) f(\beta) 1\right].$$

Proof. From Theorem 1 we have for $0 \le A,\ B \le 1$ that

$$0 \le \nu(1-\nu)\left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A \otimes B\right) \tag{4.5}$$
$$\le (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}$$
$$\le \nu(1-\nu)\left(\frac{A \otimes B^{-1} + A^{-1} \otimes B}{2} - 1\right)$$

for all $\nu \in [0,1]$.

Since $0 \le A,\ B \le 1$, then $0 \le A^i,\ B^j \le 1$ for $i, j = 0, 1, ...$ and by (4.5) we get

$$0 \le \nu(1-\nu)\left(\frac{A^{2i} \otimes 1 + 1 \otimes B^{2j}}{2} - A^i \otimes B^j\right)$$
$$\le (1-\nu) A^i \otimes 1 + \nu 1 \otimes B^j - A^{(1-\nu)i} \otimes B^{\nu j}$$
$$\le \nu(1-\nu)\left(\frac{A^i \otimes B^{-j} + A^{-i} \otimes B^j}{2} - 1\right).$$

If we multiply this inequality by $a_i\alpha^i$ and $a_j\beta^j$, then we get

$$\begin{aligned}0 &\leq \nu(1-\nu)\left(\frac{a_i\alpha^i A^{2i}\otimes a_j\beta^j 1 + a_i\alpha^i 1\otimes a_j\beta^j B^{2j}}{2} - a_i\alpha^i A^i\otimes a_j\beta^j B^j\right)\\ &\leq (1-\nu)\, a_i\alpha^i A^i\otimes a_j\beta^j 1 + \nu a_i\alpha^i 1\otimes a_j\beta^j B^j - a_i\alpha^i A^{(1-\nu)i}\otimes a_j\beta^j B^{\nu j}\\ &\leq \nu(1-\nu)\left(\frac{a_i\alpha^i A^i\otimes a_j\beta^j B^{-j} + a_i\alpha^i A^{-i}\otimes a_j\beta^j B^j}{2} - a_i\alpha^i a_j\beta^j 1\right)\end{aligned}$$

for $i, j = 0, 1, ...$

If we sum over i from 0 to n and over j from 0 to m, then we get

$$\begin{aligned}0 &\leq \nu(1-\nu)\\ &\times\left[\frac{1}{2}\left(\sum_{i=0}^n a_i\alpha^i A^{2i}\right)\otimes\left(\sum_{j=0}^m a_j\beta^j\right)1\right.\\ &+\frac{1}{2}\left(\sum_{i=0}^n a_i\alpha^i\right)1\otimes\left(\sum_{j=0}^m a_j\beta^j B^{2j}\right)\\ &\left.-\left(\sum_{i=0}^n a_i\alpha^i A^i\right)\otimes\left(\sum_{j=0}^m a_j\beta^j B^j\right)\right]\\ &\leq (1-\nu)\left(\sum_{i=0}^n a_i\alpha^i A^i\right)\otimes\left[\sum_{j=0}^m a_j\beta^j\right]1\\ &+\nu\left(\sum_{i=0}^n a_i\alpha^i\right)1\otimes\left(\sum_{j=0}^m a_j\beta^j B^j\right)\\ &-\left(\sum_{i=0}^n a_i\alpha^i A^{(1-\nu)i}\right)\otimes\left(\sum_{j=0}^m a_j\beta^j B^{\nu j}\right)\\ &\leq \nu(1-\nu)\\ &\times\left[\frac{1}{2}\left(\sum_{i=0}^n a_i\alpha^i A^i\right)\otimes\left(\sum_{j=0}^m a_j\beta^j B^{-j}\right)\right.\\ &+\frac{1}{2}\left(\sum_{i=0}^n a_i\alpha^i A^{-i}\right)\otimes\left(\sum_{j=0}^m a_j\beta^j B^j\right)\\ &\left.-\left(\sum_{i=0}^n a_i\alpha^i\right)\left(\sum_{j=0}^m a_j\beta^j\right)1\right]\end{aligned} \tag{4.6}$$

for all $m, n > 0$.

If $0 \leq A,\ B \leq 1$ and $0 \leq \alpha, \beta < R$, then $0 \leq \alpha A^2, \beta B^2, \alpha A, \beta B, \alpha A^{(1-\nu)}, \beta B^{\nu} < R$, which shows that the series

$$\sum_{i=0}^{\infty} a_i \alpha^i, \ \sum_{j=0}^{\infty} a_j \beta^j, \ \sum_{i=0}^{\infty} a_i \alpha^i A^i, \ \sum_{j=0}^{\infty} a_j \beta^j B^j, \ \sum_{i=0}^{\infty} a_i \alpha^i A^{2i}, \ \sum_{j=0}^{m} a_j \beta^j B^{2j},$$

$$\sum_{i=0}^{n} a_i \alpha^i A^{(1-\nu)i} \text{ and } \sum_{j=0}^{\infty} a_j \beta^j B^{\nu j}$$

are convergent, and by taking $m, n \to \infty$ in the first two inequalities in (4.6) we deduce (4.1).

If $R = \infty$, then the series $\sum_{i=0}^{\infty} a_i \alpha^i A^{-i}$ and $\sum_{j=0}^{\infty} a_j \beta^j B^{-j}$ are also convergent, and by taking $m, n \to \infty$ in all inequalities in (4.6), we derive (4.3). □

Corollary 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 \leq A$, $B \leq 1$ and $0 \leq \alpha, \beta < R$, then*

$$0 \leq \nu(1-\nu)\left[\frac{f(\beta) f\left(\alpha A^2\right) + f(\alpha) f\left(\beta B^2\right)}{2} \circ 1 - f(\alpha A) \circ f(\beta B)\right] \leq [(1-\nu) f(\beta) f(\alpha A) + \nu f(\alpha) f(\beta B)] \circ 1 - f\left(\alpha A^{1-\nu}\right) \circ f\left(\beta B^{\nu}\right) \tag{4.7}$$

for all $\nu \in [0,1]$.

In particular, for $\nu = 1/2$ we get

$$0 \leq \frac{1}{4}\left[\frac{f(\beta) f\left(\alpha A^2\right) + f(\alpha) f\left(\beta B^2\right)}{2} \circ 1 - f(\alpha A) \circ f(\beta B)\right] \leq \frac{1}{2}[f(\beta) f(\alpha A) + f(\alpha) f(\beta B)] \circ 1 - f\left(\alpha A^{1/2}\right) \circ f\left(\beta B^{1/2}\right). \tag{4.8}$$

If $R = \infty$, then for $0 \leq A$, $B \leq 1$ and $0 \leq \alpha, \beta$,

$$\begin{aligned} 0 &\leq \nu(1-\nu)\left[\frac{f(\beta) f\left(\alpha A^2\right) + f(\alpha) f\left(\beta B^2\right)}{2} \circ 1 - f(\alpha A) \circ f(\beta B)\right] \\ &\leq [(1-\nu) f(\beta) f(\alpha A) + \nu f(\alpha) f(\beta B)] \circ 1 - f\left(\alpha A^{1-\nu}\right) \circ f\left(\beta B^{\nu}\right) \\ &\leq \nu(1-\nu)\left[\frac{f(\alpha A) \circ f\left(\beta B^{-1}\right) + f\left(\alpha A^{-1}\right) \circ f(\beta B)}{2} - f(\alpha) f(\beta) 1\right] \end{aligned} \tag{4.9}$$

for all $\nu \in [0,1]$.

In particular, for $\nu = 1/2$ we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\frac{f(\beta) f\left(\alpha A^2\right) + f(\alpha) f\left(\beta B^2\right)}{2} \circ 1 - f(\alpha A) \circ f(\beta B)\right] \\ &\leq \frac{1}{2}[f(\beta) f(\alpha A) + f(\alpha) f(\beta B)] \circ 1 - f\left(\alpha A^{1/2}\right) \circ f\left(\beta B^{1/2}\right) \\ &\leq \frac{1}{4}\left[\frac{f(\alpha A) \circ f\left(\beta B^{-1}\right) + f\left(\alpha A^{-1}\right) \circ f(\beta B)}{2} - f(\alpha) f(\beta) 1\right]. \end{aligned} \tag{4.10}$$

Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. We have the following examples

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^{n}=\ln\frac{1}{1-z},\ z\in D(0,1); \tag{4.11}$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=0}^{\infty}z^{n}=\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^{n}=\exp(z)\qquad z\in\mathbb{C}, \tag{4.12}$$

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1);$$

$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$

and

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1) \tag{4.13}$$

$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^{n},\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

Assume that $0\le A,\ B\le 1$ and $0\le\alpha,\beta<1$, then by writing the inequality (4.7) for the function $f(z)=(1-z)^{-1}$, we get

$$\begin{aligned}
0&\le\nu(1-\nu)\left[\frac{(1-\beta)^{-1}\left(1-\alpha A^{2}\right)^{-1}+(1-\alpha)^{-1}\left(1-\beta B^{2}\right)^{-1}}{2}\circ 1\right.\\
&\quad\left.-(1-\alpha A)^{-1}\circ(1-\beta B)^{-1}\right]\\
&\le\left[(1-\nu)(1-\beta)^{-1}(1-\alpha A)^{-1}+\nu(1-\alpha)^{-1}(1-\beta B)^{-1}\right]\circ 1\\
&\quad-\left(1-\alpha A^{1-\nu}\right)^{-1}\circ(1-\beta B^{\nu})^{-1}
\end{aligned} \tag{4.14}$$

for all $\nu\in[0,1]$.

If $0 \leq A$, $B \leq 1$ and $0 \leq \alpha, \beta$, then by (2.19) for $f(z) = \exp z$, we get

$$
\begin{aligned}
(4.15)\qquad 0 &\leq \nu (1-\nu) \\
&\times \left[\frac{\exp\left(\beta + \alpha A^2\right) + \exp\left(\alpha + \beta B^2\right)}{2} \circ 1 - \exp(\alpha A) \circ \exp(\beta B) \right] \\
&\leq \left[(1-\nu) \exp(\beta + \alpha A) + \nu \exp(\alpha + \beta B) \right] \circ 1 \\
&- \exp\left(\alpha A^{1-\nu}\right) \circ \exp\left(\beta B^{\nu}\right) \\
&\leq \nu (1-\nu) \\
&\times \left[\frac{\exp(\alpha A) \circ \exp\left(\beta B^{-1}\right) + \exp\left(\alpha A^{-1}\right) \circ \exp(\beta B)}{2} - \exp(\alpha + \beta) 1 \right]
\end{aligned}
$$

for all $\nu \in [0, 1]$.

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