TENSORIAL AND HADAMARD PRODUCT INEQUALITIES OF SCHWARZ TYPE FOR SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space. In this paper we show among others that, if the functions $f,\ g : I \subset \mathbb{R} \to [0,\infty)$ are continuous and $A,\ B$ are selfadjoint operators with spectra $\operatorname{Sp}(A),\ \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned}
&f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B) \\
&\geq \left[f^{2(1-\lambda)}(A)\, g^{2\lambda}(A)\right] \otimes \left[f^{2\lambda}(B)\, g^{2(1-\lambda)}(B)\right] \\
&+ \left[f^{2\lambda}(A)\, g^{2(1-\lambda)}(A)\right] \otimes \left[f^{2(1-\lambda)}(B)\, g^{2\lambda}(B)\right] \\
&\geq 2\left[f(A)\, g(A)\right] \otimes \left[f(B)\, g(B)\right],
\end{aligned}$$

for all $\lambda \in [0,1]$. We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&f^2(A) \circ g^2(B) + g^2(A) \circ f^2(B) \\
&\geq \left[f^{2(1-\lambda)}(A)\, g^{2\lambda}(A)\right] \circ \left[f^{2\lambda}(B)\, g^{2(1-\lambda)}(B)\right] \\
&+ \left[f^{2\lambda}(A)\, g^{2(1-\lambda)}(A)\right] \circ \left[f^{2(1-\lambda)}(B)\, g^{2\lambda}(B)\right] \\
&\geq 2\left[f(A)\, g(A)\right] \circ \left[f(B)\, g(B)\right],
\end{aligned}$$

for all $\lambda \in [0,1]$.

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

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This construction [2] extends the definition of Korányi [4] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [6, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle Ae_j, e_j \rangle \langle Be_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.5}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [6, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.7}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [7] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if the functions $f,\ g : I \subset \mathbb{R} \to [0, \infty)$ are continuous and A, B are selfadjoint operators with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned} & f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B) \\ & \geq \left[f^{2(1-\lambda)}(A)\, g^{2\lambda}(A)\right] \otimes \left[f^{2\lambda}(B)\, g^{2(1-\lambda)}(B)\right] \\ & + \left[f^{2\lambda}(A)\, g^{2(1-\lambda)}(A)\right] \otimes \left[f^{2(1-\lambda)}(B)\, g^{2\lambda}(B)\right] \\ & \geq 2\left[f(A)\, g(A)\right] \otimes \left[f(B)\, g(B)\right], \end{aligned}$$

for all $\lambda \in [0, 1]$. We also have the following inequalities for the Hadamard product

$$\begin{aligned} & f^2(A) \circ g^2(B) + g^2(A) \circ f^2(B) \\ & \geq \left[f^{2(1-\lambda)}(A)\, g^{2\lambda}(A)\right] \circ \left[f^{2\lambda}(B)\, g^{2(1-\lambda)}(B)\right] \\ & + \left[f^{2\lambda}(A)\, g^{2(1-\lambda)}(A)\right] \circ \left[f^{2(1-\lambda)}(B)\, g^{2\lambda}(B)\right] \\ & \geq 2\left[f(A)\, g(A)\right] \circ \left[f(B)\, g(B)\right], \end{aligned}$$

for all $\lambda \in [0, 1]$.

2. Main Results

The following result providing a refinement of Cauchy-Schwarz inequality holds:

Theorem 1. *Assume that the functions* $f,\ g : I \subset \mathbb{R} \to [0, \infty)$ *are continuous and* A, B *are selfadjoint operators with spectra* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$\begin{aligned} & f^2(A) \otimes g^2(B) + g^2(A) \otimes f^2(B) \\ & \geq \left[f^{2(1-\lambda)}(A)\, g^{2\lambda}(A)\right] \otimes \left[f^{2\lambda}(B)\, g^{2(1-\lambda)}(B)\right] \\ & + \left[f^{2\lambda}(A)\, g^{2(1-\lambda)}(A)\right] \otimes \left[f^{2(1-\lambda)}(B)\, g^{2\lambda}(B)\right] \\ & \geq 2\left[f(A)\, g(A)\right] \otimes \left[f(B)\, g(B)\right], \end{aligned} \tag{2.1}$$

for all $\lambda \in [0, 1]$.

In particular, for $B = A$

$$f^2(A) \otimes g^2(A) + g^2(A) \otimes f^2(A) \tag{2.2}$$
$$\geq \left[f^{2(1-\lambda)}(A) g^{2\lambda}(A)\right] \otimes \left[f^{2\lambda}(A) g^{2(1-\lambda)}(A)\right]$$
$$+ \left[f^{2\lambda}(A) g^{2(1-\lambda)}(A)\right] \otimes \left[f^{2(1-\lambda)}(A) g^{2\lambda}(A)\right]$$
$$\geq 2[f(A) g(A)] \otimes [f(A) g(A)].$$

Proof. Using the weighted arithmetic mean-geometric mean inequality for positive numbers, we have for $a, b, c, d \geq 0$ and $\lambda \in [0,1]$ that

$$(1-\lambda) a^2 b^2 + \lambda c^2 d^2 \geq a^{2(1-\lambda)} b^{2(1-\lambda)} c^{2\lambda} d^{2\lambda}$$

and

$$\lambda a^2 b^2 + (1-\lambda) c^2 d^2 \geq a^{2\lambda} b^{2\lambda} c^{2(1-\lambda)} d^{2(1-\lambda)}.$$

If we add these two inequalities, then we get

$$a^2 b^2 + c^2 d^2 \geq a^{2(1-\lambda)} b^{2(1-\lambda)} c^{2\lambda} d^{2\lambda} + a^{2\lambda} b^{2\lambda} c^{2(1-\lambda)} d^{2(1-\lambda)}.$$

By arithmetic mean-geometric mean inequality we also have

$$a^{2(1-\lambda)} b^{2(1-\lambda)} c^{2\lambda} d^{2\lambda} + a^{2\lambda} b^{2\lambda} c^{2(1-\lambda)} d^{2(1-\lambda)}$$
$$= \left(a^{1-\lambda} b^{1-\lambda} c^{\lambda} d^{\lambda}\right)^2 + \left(a^{\lambda} b^{\lambda} c^{1-\lambda} d^{1-\lambda}\right)^2$$
$$\geq 2a^{1-\lambda} b^{1-\lambda} c^{\lambda} d^{\lambda} a^{\lambda} b^{\lambda} c^{1-\lambda} d^{1-\lambda} = 2abcd$$

for $a, b, c, d \geq 0$ and $\lambda \in [0,1]$.

Therefore we have

$$a^2 b^2 + c^2 d^2 \geq a^{2(1-\lambda)} b^{2(1-\lambda)} c^{2\lambda} d^{2\lambda} + a^{2\lambda} b^{2\lambda} c^{2(1-\lambda)} d^{2(1-\lambda)} \geq 2abcd \tag{2.3}$$

for $a, b, c, d \geq 0$ and $\lambda \in [0,1]$.

If we take $a = f(t)$, $b = g(s)$, $c = g(t)$ and $d = f(s)$ for $t, s \in I$ in (2.3) to get

$$f^2(t) g^2(s) + g^2(t) f^2(s) \tag{2.4}$$
$$\geq f^{2(1-\lambda)}(t) g^{2(1-\lambda)}(s) g^{2\lambda}(t) f^{2\lambda}(s)$$
$$+ f^{2\lambda}(t) g^{2\lambda}(s) g^{2(1-\lambda)}(t) f^{2(1-\lambda)}(s)$$
$$\geq 2f(t) g(s) g(t) f(s)$$

for all $t, s \in I$ and $\lambda \in [0,1]$.

Now, if we take the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$, then we get

$$\int_I \int_I \left[f^2(t) g^2(s) + g^2(t) f^2(s)\right] dE(t) \otimes dF(s) \tag{2.5}$$
$$\geq \int_I \int_I f^{2(1-\lambda)}(t) g^{2(1-\lambda)}(s) g^{2\lambda}(t) f^{2\lambda}(s) \, dE(t) \otimes dF(s)$$
$$+ \int_I \int_I f^{2\lambda}(t) g^{2\lambda}(s) g^{2(1-\lambda)}(t) f^{2(1-\lambda)}(s) \, dE(t) \otimes dF(s)$$
$$\geq 2 \int_I \int_I f(t) g(s) g(t) f(s) \, dE(t) \otimes dF(s),$$

namely, by (1.1),

$$\begin{aligned}
&\int_I f^2(t)\,dE(t)\otimes\int_I g^2(s)\,dF(s)+\int_I g^2(t)\,dE(t)\otimes\int_I f^2(s)\,dF(s)\\
&\geq\int_I f^{2(1-\lambda)}(t)\,g^{2\lambda}(t)\,dE(t)\otimes\int_I g^{2(1-\lambda)}(s)\,f^{2\lambda}(s)\,dF(s)\\
&+\int_I f^{2\lambda}(t)\,g^{2(1-\lambda)}(t)\,dE(t)\otimes\int_I g^{2\lambda}(s)\,f^{2(1-\lambda)}(s)\,dF(s)\\
&\geq 2\int_I f(t)\,g(t)\,dE(t)\otimes\int_I g(s)\,f(s)\,dF(s)
\end{aligned}$$

and the inequality (2.1) is thus proved. □

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}
&f^2(A)\circ g^2(B)+g^2(A)\circ f^2(B)\\
&\geq\left[f^{2(1-\lambda)}(A)\,g^{2\lambda}(A)\right]\circ\left[f^{2\lambda}(B)\,g^{2(1-\lambda)}(B)\right]\\
&+\left[f^{2\lambda}(A)\,g^{2(1-\lambda)}(A)\right]\circ\left[f^{2(1-\lambda)}(B)\,g^{2\lambda}(B)\right]\\
&\geq 2\left[f(A)\,g(A)\right]\circ\left[f(B)\,g(B)\right],
\end{aligned}\tag{2.6}$$

for all $\lambda\in[0,1]$.

In particular, for $B=A$

$$\begin{aligned}
f^2(A)\circ g^2(A)&\geq\left[f^{2(1-\lambda)}(A)\,g^{2\lambda}(A)\right]\circ\left[f^{2\lambda}(A)\,g^{2(1-\lambda)}(A)\right]\\
&\geq\left[f(A)\,g(A)\right]\circ\left[f(A)\,g(A)\right].
\end{aligned}\tag{2.7}$$

Proof. By the inequality (2.1), we derive

$$\begin{aligned}
&\mathcal{U}^*\left[f^2(A)\otimes g^2(B)+g^2(A)\otimes f^2(B)\right]\mathcal{U}\\
&\geq\mathcal{U}^*\left\{\left[f^{2(1-\lambda)}(A)\,g^{2\lambda}(A)\right]\otimes\left[f^{2\lambda}(B)\,g^{2(1-\lambda)}(B)\right]\right.\\
&\left.+\left[f^{2\lambda}(A)\,g^{2(1-\lambda)}(A)\right]\otimes\left[f^{2(1-\lambda)}(B)\,g^{2\lambda}(B)\right]\right\}\\
&\geq 2\mathcal{U}^*\left\{\left[f(A)\,g(A)\right]\otimes\left[f(B)\,g(B)\right]\right\}\mathcal{U},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\mathcal{U}^*\left[f^2(A)\otimes g^2(B)\right]\mathcal{U}+\mathcal{U}^*\left[g^2(A)\otimes f^2(B)\right]\mathcal{U}\\
&\geq\mathcal{U}^*\left\{\left[f^{2(1-\lambda)}(A)\,g^{2\lambda}(A)\right]\otimes\left[f^{2\lambda}(B)\,g^{2(1-\lambda)}(B)\right]\right\}\mathcal{U}\\
&+\mathcal{U}^*\left\{\left[f^{2\lambda}(A)\,g^{2(1-\lambda)}(A)\right]\otimes\left[f^{2(1-\lambda)}(B)\,g^{2\lambda}(B)\right]\right\}\mathcal{U}\\
&\geq 2\mathcal{U}^*\left\{\left[f(A)\,g(A)\right]\otimes\left[f(B)\,g(B)\right]\right\}\mathcal{U}.
\end{aligned}$$

By utilizing the representation (1.5) we deduce the desired result (2.6). □

Corollary 2. *Assume that the functions* $f,\ g: I\subset\mathbb{R}\to[0,\infty)$ *are continuous and* $A_i,\ B_i,\ i\in\{1,...,n\}$ *are selfadjoint operators with spectra* $\operatorname{Sp}(A_i),\ \operatorname{Sp}(B_i)\subset I$, *and* $p_i,\ q_i \geq 0$ *for* $i \in \{1,...,n\}$*, then*

$$\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{j=1}^{n} q_j g^2(B_j)\right) + \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{j=1}^{n} q_j f^2(B_j)\right)$$
$$\geq \left[\sum_{i=1}^{n} p_i f^{2(1-\lambda)}(A_i)\, g^{2\lambda}(A_i)\right] \otimes \left[\sum_{j=1}^{n} q_j f^{2\lambda}(B_j)\, g^{2(1-\lambda)}(B_j)\right] + \left[\sum_{i=1}^{n} p_i f^{2\lambda}(A_i)\, g^{2(1-\lambda)}(A_i)\right] \otimes \left[\sum_{j=1}^{n} q_j f^{2(1-\lambda)}(B_j)\, g^{2\lambda}(B_j)\right]$$
$$\geq 2\left[\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right] \otimes \left[\sum_{j=1}^{n} q_j f(B_j)\, g(B_j)\right] \tag{2.8}$$

and

$$\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \circ \left(\sum_{j=1}^{n} q_j g^2(B_j)\right) + \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \circ \left(\sum_{j=1}^{n} q_j f^2(B_j)\right)$$
$$\geq \left[\sum_{i=1}^{n} p_i f^{2(1-\lambda)}(A_i)\, g^{2\lambda}(A_i)\right] \circ \left[\sum_{j=1}^{n} q_j f^{2\lambda}(B_j)\, g^{2(1-\lambda)}(B_j)\right] + \left[\sum_{i=1}^{n} p_i f^{2\lambda}(A_i)\, g^{2(1-\lambda)}(A_i)\right] \circ \left[\sum_{j=1}^{n} q_j f^{2(1-\lambda)}(B_j)\, g^{2\lambda}(B_j)\right]$$
$$\geq 2\left[\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right] \circ \left[\sum_{j=1}^{n} q_j f(B_j)\, g(B_j)\right] \tag{2.9}$$

for all $\lambda \in [0,1]$.

Proof. From (2.1) we get

$$\sum_{i,j=1}^{n} p_i q_j f^2(A_i) \otimes g^2(B_j) + \sum_{i,j=1}^{n} p_i q_j g^2(A_i) \otimes f^2(B_j)$$
$$\geq \sum_{i,j=1}^{n} p_i q_j \left[f^{2(1-\lambda)}(A_i)\, g^{2\lambda}(A_i)\right] \otimes \left[f^{2\lambda}(B_j)\, g^{2(1-\lambda)}(B_j)\right]$$

$$+\sum_{i,j=1}^{n} p_i q_j \left[f^{2\lambda}(A_i)\, g^{2(1-\lambda)}(A_i)\right] \otimes \left[f^{2(1-\lambda)}(B_j)\, g^{2\lambda}(B_j)\right]$$
$$\geq 2 \sum_{i,j=1}^{n} p_i q_j \left[f(A_i)\, g(A_i)\right] \otimes \left[f(B_j)\, g(B_j)\right],$$

which gives (2.8). □

Remark 1. *If we take in Corollary 2 $q_i = p_i$ and $B_i = A_i$, $i \in \{1, ..., n\}$, then*

$$\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) + \left(\sum_{i=1}^{n} p_i g^2(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f^2(A_i)\right)$$
$$\geq \left[\sum_{i=1}^{n} p_i f^{2(1-\lambda)}(A_i)\, g^{2\lambda}(A_i)\right] \otimes \left[\sum_{i=1}^{n} p_i f^{2\lambda}(A_i)\, g^{2(1-\lambda)}(A_i)\right]$$
$$+ \left[\sum_{i=1}^{n} p_i f^{2\lambda}(A_i)\, g^{2(1-\lambda)}(A_i)\right] \otimes \left[\sum_{i=1}^{n} p_i f^{2(1-\lambda)}(A_i)\, g^{2\lambda}(A_i)\right]$$
$$\geq 2 \left[\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right] \otimes \left[\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right] \tag{2.10}$$

and

$$\left(\sum_{i=1}^{n} p_i f^2(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i g^2(A_i)\right)$$
$$\geq \left[\sum_{i=1}^{n} p_i f^{2(1-\lambda)}(A_i)\, g^{2\lambda}(A_i)\right] \circ \left[\sum_{i=1}^{n} p_i f^{2\lambda}(A_i)\, g^{2(1-\lambda)}(A_i)\right]$$
$$\geq \left[\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right] \circ \left[\sum_{i=1}^{n} p_i f(A_i)\, g(A_i)\right] \tag{2.11}$$

for all $\lambda \in [0, 1]$.

Theorem 2. *Assume that f is monotonic nondecreasing and convex on $[0, \infty)$ and A, $B \geq 0$. If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$f(A \otimes B) \leq \frac{1}{p} f(A^p) \otimes 1 + \frac{1}{q} 1 \otimes f(B^q). \tag{2.12}$$

In particular,

$$f(A \otimes B) \leq \frac{1}{2}\left[f\left(A^2\right) \otimes 1 + 1 \otimes f\left(B^2\right)\right]. \tag{2.13}$$

If f is monotonic nonincreasing and concave on $[0, \infty)$, then the reverse inequality holds in (2.12) and (2.13).

Proof. Using Young's inequality and the monotonicity and convexity of f we have

$$f(ts) \leq f\left(\frac{1}{p} t^p + \frac{1}{q} s^q\right) \leq \frac{1}{p} f(t^p) + \frac{1}{q} f(s^q)$$

for all $t, s \geq 0$.

Now, if we take the double integral $\int_{[0,\infty)} \int_{[0,\infty)}$ over $dE(t) \otimes dF(s)$, then we get

$$\begin{aligned} \text{(2.14)} \qquad & \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \\ & \leq \int_{[0,\infty)} \int_{[0,\infty)} \left[\frac{1}{p} f(t^p) + \frac{1}{q} f(s^q) \right] dE(t) \otimes dF(s). \end{aligned}$$

Since

$$\begin{aligned} & \int_{[0,\infty)} \int_{[0,\infty)} \left[\frac{1}{p} f(t^p) + \frac{1}{q} f(s^q) \right] dE(t) \otimes dF(s) \\ & = \frac{1}{p} \int_{[0,\infty)} \int_{[0,\infty)} f(t^p)\, dE(t) \otimes dF(s) + \frac{1}{q} \int_{[0,\infty)} \int_{[0,\infty)} f(s^q)\, dE(t) \otimes dF(s) \\ & = \frac{1}{p} f(A^p) \otimes 1 + \frac{1}{q} 1 \otimes f(B^q), \end{aligned}$$

hence by (1.3) and (2.14), we derive (2.12). □

Corollary 3. *Assume that f is monotonic nondecreasing and operator convex on $[0,\infty)$ and A, $B \geq 0$. If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$f(A \circ B) \leq \left[\frac{1}{p} f(A^p) + \frac{1}{q} f(B^q) \right] \circ 1. \tag{2.15}$$

In particular,

$$f(A \circ B) \leq \frac{1}{2} \left[f\left(A^2\right) + f\left(B^2\right) \right] \circ 1. \tag{2.16}$$

Proof. By (1.5) and Davis-Choi-Jensen's inequality, we have

$$f(A \circ B) = f(\mathcal{U}^* (A \otimes B) \mathcal{U}) \leq \mathcal{U}^* f(A \otimes B) \mathcal{U}. \tag{2.17}$$

By (2.12) we get

$$\begin{aligned} \text{(2.18)} \qquad \mathcal{U}^* f(A \otimes B) \mathcal{U} & \leq \mathcal{U}^* \left[\frac{1}{p} f(A^p) \otimes 1 + \frac{1}{q} 1 \otimes f(B^q) \right] \mathcal{U} \\ & = \frac{1}{p} \mathcal{U}^* (f(A^p) \otimes 1) \mathcal{U} + \frac{1}{q} \mathcal{U}^* (1 \otimes f(B^q)) \mathcal{U} \\ & = \frac{1}{p} f(A^p) \circ 1 + \frac{1}{q} 1 \circ f(B^q). \end{aligned}$$

By making use of (2.17) and (2.18) we derive (2.15). □

Theorem 3. *Assume that f is convex on $\mathbb{R}$ and A, $B > 0$. If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$f(\ln(A \otimes B)) \leq \frac{1}{p} f(p \ln A) \otimes 1 + \frac{1}{q} 1 \otimes f(q \ln B). \tag{2.19}$$

In particular,

$$f(\ln(A \otimes B)) \leq \frac{1}{2} \left[f(2 \ln A) \otimes 1 + 1 \otimes f(2 \ln B) \right]. \tag{2.20}$$

Proof. We observe that for $t,s>0$ that

$$\ln(ts)=\ln t+\ln s=\frac{1}{p}\ln(t^p)+\frac{1}{q}\ln(s^q).$$

Now, if we take the function f and use its convexity, then we get

$$f(\ln(ts))=f\left(\frac{1}{p}\ln(t^p)+\frac{1}{q}\ln(s^q)\right)\le\frac{1}{p}f(\ln(t^p))+\frac{1}{q}f(\ln(s^q))$$

for $t,s>0$.

Now, if we take the double integral $\int_{[0,\infty)}\int_{[0,\infty)}$ over $dE(t)\otimes dF(s)$, then we get

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)}f(\ln(ts))\,dE(t)\otimes dF(s)\\&\le\int_{[0,\infty)}\int_{[0,\infty)}\left[\frac{1}{p}f(\ln(t^p))+\frac{1}{q}f(\ln(s^q))\right]dE(t)\otimes dF(s).\end{aligned}\tag{2.21}$$

Observe that

$$\int_{[0,\infty)}\int_{[0,\infty)}f(\ln(ts))\,dE(t)\otimes dF(s)=f(\ln(A\otimes B))$$

and

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)}\left[\frac{1}{p}f(\ln(t^p))+\frac{1}{q}f(\ln(s^q))\right]dE(t)\otimes dF(s)\\&=\frac{1}{p}\int_{[0,\infty)}\int_{[0,\infty)}f(\ln(t^p))\,dE(t)\otimes dF(s)\\&+\frac{1}{q}\int_{[0,\infty)}\int_{[0,\infty)}f(\ln(s^q))\,dE(t)\otimes dF(s)\\&=\frac{1}{p}f(\ln A^p)\otimes 1+\frac{1}{q}1\otimes f(\ln B^q)=\frac{1}{p}f(p\ln A)\otimes 1+\frac{1}{q}1\otimes f(q\ln B),\end{aligned}$$

then by (2.21) we derive (2.19). □

Corollary 4. *Assume that f is convex on $\mathbb{R}$ with $f\circ\ln$ is operator convex on $(0,\infty)$. If $A,\ B>0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$f(\ln(A\circ B))\le\left[\frac{1}{p}f(p\ln A)+\frac{1}{q}f(q\ln B)\right]\circ 1.\tag{2.22}$$

In particular,

$$f(\ln(A\circ B))\le\frac{1}{2}[f(2\ln A)+f(2\ln B)]\circ 1.\tag{2.23}$$

The proof is similar to the one from Corollary 3 for the operator convex function $f\circ\ln$.

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Assume that $0\le A,\ B<R^{1/2}$, then*

$$f^2(A\otimes B)\le f\left(A^2\right)\otimes f\left(B^2\right).\tag{2.24}$$

If $R=\infty$, then the inequality (2.24) holds for $A,\ B\ge 0$.

Proof. For $0 \leq s,\ t < \sqrt{R}$ we get that $0 \leq st,\ s^2,\ t^2 < R$. By Cauchy-Schwarz inequality, we have

$$\left(\sum_{k=0}^{n} a_k (ts)^k\right)^2 = \left(\sum_{k=0}^{n} a_k t^k s^k\right)^2 \leq \left(\sum_{k=0}^{n} a_k t^{2k}\right)\left(\sum_{k=0}^{n} a_k s^{2k}\right). \tag{2.25}$$

Since the series

$$\sum_{k=0}^{\infty} a_k (ts)^k,\ \sum_{k=0}^{\infty} a_k t^{2k} \text{ and } \sum_{k=0}^{\infty} a_k s^{2k}$$

are convergent for $0 \leq s,\ t < \sqrt{R}$, then by taking the limit over $n \to \infty$ in (2.25) we deduce that

$$f^2(ts) \leq f\left(t^2\right) f\left(s^2\right) \text{ for all } s,\ t \in [0, \sqrt{R}).$$

Now, if we take the double integral $\int_{[0,\sqrt{R})} \int_{[0,\sqrt{R})}$ over $dE(t) \otimes dF(s)$, then we get

$$\begin{aligned}
&\int_{[0,\sqrt{R})} \int_{[0,\sqrt{R})} f^2(ts)\, dE(t) \otimes dF(s) \\
&\leq \int_{[0,\sqrt{R})} \int_{[0,\sqrt{R})} f\left(t^2\right) f\left(s^2\right) dE(t) \otimes dF(s) \\
&= \left(\int_{[0,\sqrt{R})} f\left(t^2\right) dE(t)\right) \otimes \left(\int_{[0,\sqrt{R})} f\left(s^2\right) dF(s)\right),
\end{aligned}$$

which gives (2.24). □

3. Some Examples

Assume that $A,\ B > 0$. If we take $f(t) = t^p$ and $g(t) = t^q$, $p, q \neq 0$, in (2.1), then we get

$$\begin{aligned}
A^{2p} \otimes B^{2q} + A^{2q} \otimes B^{2p} &\geq \left(A^{2p(1-\lambda)+2q\lambda}\right) \otimes \left(B^{2p\lambda+2q(1-\lambda)}\right) \\
&\quad + \left(A^{2p\lambda+2q(1-\lambda)}\right) \otimes \left(B^{2p(1-\lambda)+2q\lambda}\right) \\
&\geq 2A^{p+q} \otimes B^{p+q}
\end{aligned} \tag{3.1}$$

and, in particular

$$\begin{aligned}
A^{2p} \otimes A^{2q} + A^{2q} \otimes A^{2p} &\geq \left(A^{2p(1-\lambda)+2q\lambda}\right) \otimes \left(A^{2p\lambda+2q(1-\lambda)}\right) \\
&\quad + \left(A^{2p\lambda+2q(1-\lambda)}\right) \otimes \left(A^{2p(1-\lambda)+2q\lambda}\right) \\
&\geq 2A^{p+q} \otimes A^{p+q}.
\end{aligned} \tag{3.2}$$

where $\lambda \in [0, 1]$.

For $q = 1/2$ and $p = -1/2$ we get

$$A^{-1} \otimes B + A \otimes B^{-1} \geq A^{2\lambda-1} \otimes B^{1-2\lambda} + A^{1-2\lambda} \otimes B^{2\lambda-1} \geq 2 \tag{3.3}$$

and, in particular

$$A^{-1} \otimes A + A \otimes A^{-1} \geq A^{2\lambda-1} \otimes A^{1-2\lambda} + A^{1-2\lambda} \otimes A^{2\lambda-1} \geq 2, \tag{3.4}$$

where $\lambda \in [0, 1]$.

We also have the inequalities for the Hadamard product

$$A^{2p}\circ B^{2q}+A^{2q}\circ B^{2p}\geq\left(A^{2p(1-\lambda)+2q\lambda}\right)\circ\left(B^{2p\lambda+2q(1-\lambda)}\right)+\left(A^{2p\lambda+2q(1-\lambda)}\right)\circ\left(B^{2p(1-\lambda)+2q\lambda}\right)\geq 2A^{p+q}\circ B^{p+q}\tag{3.5}$$

and, in particular

$$A^{2p}\circ A^{2q}+A^{2q}\circ A^{2p}\geq\left(A^{2p(1-\lambda)+2q\lambda}\right)\circ\left(A^{2p\lambda+2q(1-\lambda)}\right)+\left(A^{2p\lambda+2q(1-\lambda)}\right)\circ\left(A^{2p(1-\lambda)+2q\lambda}\right)\geq 2A^{p+q}\circ A^{p+q},\tag{3.6}$$

where $\lambda\in[0,1]$.

For $q=1/2$ and $p=-1/2$ we get

$$A^{-1}\circ B+A\circ B^{-1}\geq A^{2\lambda-1}\circ B^{1-2\lambda}+A^{1-2\lambda}\circ B^{2\lambda-1}\geq 2\tag{3.7}$$

and, in particular

$$A^{-1}\circ A\geq A^{2\lambda-1}\circ A^{1-2\lambda}\geq 1,\tag{3.8}$$

where $\lambda\in[0,1]$.

We notice that the inequality (3.8) is an improvement of Fiedler inequality (1.7).

If $A_i,B_i>0$ and $p_i\geq 0$ for $i\in\{1,...,n\}$, then by (2.8) we get

$$\left(\sum_{i=1}^{n}p_iA_i^{2p}\right)\otimes\left(\sum_{i=1}^{n}p_iB_i^{2q}\right)+\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)\otimes\left(\sum_{i=1}^{n}p_iB_i^{2p}\right)\tag{3.9}$$
$$\geq\left(\sum_{i=1}^{n}p_iA_i^{2p(1-\lambda)+2q\lambda}\right)\otimes\left(\sum_{i=1}^{n}p_iB_i^{2p\lambda+2q(1-\lambda)}\right)+\left(\sum_{i=1}^{n}p_iA_i^{2p\lambda+2q(1-\lambda)}\right)\otimes\left(\sum_{i=1}^{n}p_iB_i^{2p(1-\lambda)+2q\lambda}\right)$$
$$\geq 2\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\otimes\left(\sum_{i=1}^{n}p_iB_i^{p+q}\right)$$

and

$$\left(\sum_{i=1}^{n}p_iA_i^{2p}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{2q}\right)+\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{2p}\right)\tag{3.10}$$
$$\geq\left(\sum_{i=1}^{n}p_iA_i^{2p(1-\lambda)+2q\lambda}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{2p\lambda+2q(1-\lambda)}\right)+\left(\sum_{i=1}^{n}p_iA_i^{2p\lambda+2q(1-\lambda)}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{2p(1-\lambda)+2q\lambda}\right)$$
$$\geq 2\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{p+q}\right).$$

If we take $p=1/2$ and $q=-1/2$ and also assume that $\sum_{i=1}^{n}p_i=1$, then we get

$$\begin{aligned}(3.11)\quad &\left(\sum_{i=1}^{n}p_iA_i\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{-1}\right)+\left(\sum_{i=1}^{n}p_iA_i^{-1}\right)\circ\left(\sum_{i=1}^{n}p_iB_i\right)\\ &\geq\left(\sum_{i=1}^{n}p_iA_i^{1-2\lambda}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{2\lambda-1}\right)+\left(\sum_{i=1}^{n}p_iA_i^{2\lambda-1}\right)\circ\left(\sum_{i=1}^{n}p_iB_i^{1-2\lambda}\right)\\ &\geq 2.\end{aligned}$$

In particular, we have

$$(3.12)\quad \left(\sum_{i=1}^{n}p_iA_i\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{-1}\right)\geq\left(\sum_{i=1}^{n}p_iA_i^{1-2\lambda}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{2\lambda-1}\right)\geq 1,$$

where $\lambda\in[0,1]$.

Consider the function $f(t)=t^r$, $r\geq 2$. This function is monotonic nondecreasing and convex on $[0,\infty)$. If A, $B\geq 0$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (2.12)

$$(3.13)\quad (A\otimes B)^r\leq\frac{1}{p}A^{rp}\otimes 1+\frac{1}{q}1\otimes B^{rq}.$$

In particular,

$$(3.14)\quad (A\otimes B)^r\leq\frac{1}{2}\left(A^{2r}\otimes 1+1\otimes B^{2r}\right).$$

Moreover, if $r\in[1,2]$, then f is operator convex and by (2.15) we get

$$(3.15)\quad (A\circ B)^r\leq\left(\frac{1}{p}A^{rp}+\frac{1}{q}B^{rq}\right)\circ 1.$$

In particular,

$$(3.16)\quad (A\circ B)^r\leq\frac{1}{2}\left(A^{2r}+B^{2r}\right)\circ 1.$$

Now, we consider the function $f(t)=|t|^u$ that is convex on $\mathbb{R}$ for $u\geq 2$. By (2.19) we derive

$$(3.17)\quad |\ln(A\otimes B)|^u\leq p^{u-1}|\ln A|^u\otimes 1+q^{u-1}1\otimes|\ln B|^u,$$

where A, $B>0$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular,

$$(3.18)\quad |\ln(A\otimes B)|^u\leq 2^{u-1}\left(|\ln A|^u\otimes 1+1\otimes|\ln B|^u\right)$$

for A, $B>0$.

Let $h(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. We have the following examples

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^{n}=\ln\frac{1}{1-z},\ z\in D(0,1);\tag{3.19}$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=0}^{\infty}z^{n}=\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^{n}=\exp(z)\qquad z\in\mathbb{C},\tag{3.20}$$

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1);$$

$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\,n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$

and

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1)\tag{3.21}$$

$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^{n},\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function*.

Assume that $0\le A,\ B<1$, then by (2.24)

$$(1-A\otimes B)^{-2}\le\left(1-A^{2}\right)^{-1}\otimes\left(1-B^{2}\right)^{-1},\tag{3.22}$$

$$[\ln(1-A\otimes B)]^{2}\le\ln\left(1-A^{2}\right)\otimes\ln\left(1-B^{2}\right)\tag{3.23}$$

and

$$\left[\sin^{-1}(A\otimes B)\right]^{2}\le\sin^{-1}\left(A^{2}\right)\otimes\sin^{-1}\left(B^{2}\right).\tag{3.24}$$

If $A,\ B\ge 0$, then by (2.24) we get

$$\exp(2A\otimes B)\le\exp\left(A^{2}\right)\otimes\exp\left(B^{2}\right),\tag{3.25}$$

$$[\sinh(A\otimes B)]^{2}\le\sinh\left(A^{2}\right)\otimes\sinh\left(B^{2}\right)\tag{3.26}$$

and

$$[\cosh(A\otimes B)]^{2}\le\cosh\left(A^{2}\right)\otimes\cosh\left(B^{2}\right).\tag{3.27}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa