TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES IN TERMS OF KANTOROVICH RATIO

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space. In this paper we show among others that, if f,g are continuous on the interval I with

$$0<\gamma\leq\frac{f(t)}{g(t)}\leq\Gamma \text{ for } t\in I$$

and if A and B are selfadjoint operators with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$, then

$$\begin{aligned}&\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\\&\leq(1-\nu)\,f(A)\otimes g(B)+\nu g(A)\otimes f(B)\\&\leq\left[\frac{(\gamma+\Gamma)^{2}}{4\gamma\Gamma}\right]^{R}\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right].\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}&\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\\&\leq(1-\nu)\,f(A)\circ g(B)+\nu g(A)\circ f(B)\\&\leq\left[\frac{(\gamma+\Gamma)^{2}}{4\gamma\Gamma}\right]^{R}\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right].\end{aligned}$$

1. Introduction

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

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This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if* f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [6, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} [(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.5}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [6, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.7}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1 \right)^{1/2} \left(B^2 \circ 1 \right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2 \right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [10] that $\left(A^2 \circ 1 \right)^{1/2} \left(B^2 \circ 1 \right)^{1/2}$ and $\left(A^2 \circ B^2 \right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

The famous *Young inequality* for scalars says that, if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \tag{1.8}$$

with equality if and only if $a = b$. The inequality (1.8) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [8], [9] provided a refinement and an additive reverse for Young inequality as follows:

$$r \left(\sqrt{a} - \sqrt{b} \right)^2 \leq (1-\nu)\, a + \nu b - a^{1-\nu} b^{\nu} \leq R \left(\sqrt{a} - \sqrt{b} \right)^2 \tag{1.9}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.9) to an identity and is of no interest.

We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}} \right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.10}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function S is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's

$$S\left(\left(\frac{a}{b} \right)^r \right) a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \leq S\left(\frac{a}{b} \right) a^{1-\nu} b^{\nu}, \tag{1.11}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.11) is due to Tominaga [14] while the first one is due to Furuichi [7].

It is an open question for the author if in the right hand side of (1.11) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R = \max\{1-\nu, \nu\}$.

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.12}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.13}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.13) was obtained by Zuo et al. in [16] while the second by Liao et al. [12].

In [16] the authors also showed that

$$K^{r}(h)\geq S(h^{r})\ \text{ for } h>0 \text{ and } r\in\left[0,\frac{1}{2}\right]$$

implying that the lower bound in (1.13) is better than the lower bound from (1.11).

We can give a simple direct proof for (1.13) as follows.

Recall the following result obtained by the author in 2006 [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ & \leq\frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \\ & \leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \tag{1.14}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n=\sum_{j=1}^{n}p_j>0$.

For $n=2$, we deduce from (1.14) that

$$\begin{aligned} & 2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \\ & \leq\nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y] \\ & \leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.15}$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$.

Now, if we write the inequality (1.15) for the convex function $\Phi(x)=-\ln x$, and for the positive numbers a and b we get (1.13).

Motivated by the above results, in this paper we show among others that, if f,g are continuous on the interval I with

$$0<\gamma\leq\frac{f(t)}{g(t)}\leq\Gamma \text{ for } t\in I$$

and if A and B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned}
&\left[f^{1-\nu}(A) g^{\nu}(A)\right] \otimes \left[f^{\nu}(B) g^{1-\nu}(B)\right] \\
&\leq (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\
&\leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \left[f^{1-\nu}(A) g^{\nu}(A)\right] \otimes \left[f^{\nu}(B) g^{1-\nu}(B)\right].
\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&\left[f^{1-\nu}(A) g^{\nu}(A)\right] \circ \left[f^{\nu}(B) g^{1-\nu}(B)\right] \\
&\leq (1-\nu) f(A) \circ g(B) + \nu g(A) \circ f(B) \\
&\leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \left[f^{1-\nu}(A) g^{\nu}(A)\right] \circ \left[f^{\nu}(B) g^{1-\nu}(B)\right].
\end{aligned}$$

2. Main Results

We have:

Theorem 1. *Let I and J be two intervals and f, g defined and continuous on an interval containing $I \cup J$. Assume that*

$$0 < \gamma_1 \leq \frac{f(t)}{g(t)} \leq \Gamma_1 \text{ for } t \in I$$

and

$$0 < \gamma_2 \leq \frac{f(s)}{g(s)} \leq \Gamma_2 \text{ for } s \in J.$$

Define

$$U(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) := \begin{cases} K\left(\frac{\Gamma_1}{\gamma_2}\right) \text{ if } 1 \leq \frac{\gamma_1}{\Gamma_2}, \\ \\ \max\left\{K\left(\frac{\Gamma_1}{\gamma_2}\right), K\left(\frac{\gamma_1}{\Gamma_2}\right)\right\} \\ \text{if } \frac{\gamma_1}{\Gamma_2} < 1 < \frac{\Gamma_1}{\gamma_2}, \\ \\ K\left(\frac{\gamma_1}{\Gamma_2}\right) \text{ if } \frac{\Gamma_1}{\gamma_2} \leq 1, \end{cases}$$

and

$$u(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) = \begin{cases} K\left(\frac{\gamma_1}{\Gamma_2}\right) \text{ if } 1 \leq \frac{\gamma_1}{\Gamma_2}, \\ \\ 1 \text{ if } \frac{\gamma_1}{\Gamma_2} < 1 < \frac{\Gamma_1}{\gamma_2}, \\ \\ K\left(\frac{\Gamma_1}{\gamma_2}\right) \text{ if } \frac{\Gamma_1}{\gamma_2} \leq 1. \end{cases}$$

If A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ and $\operatorname{Sp}(B) \subset J$, then

$$\begin{aligned}
&u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1-\nu}(A) g^{\nu}(A)\right] \otimes \left[f^{\nu}(B) g^{1-\nu}(B)\right] \\
&\leq (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\
&\leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1-\nu}(A) g^{\nu}(A)\right] \otimes \left[f^{\nu}(B) g^{1-\nu}(B)\right]
\end{aligned} \tag{2.1}$$

for $\nu \in [0,1]$, where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular,

$$u^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[f^{1/2}(A)\,g^{1/2}(A)\right]\otimes\left[f^{1/2}(B)\,g^{1/2}(B)\right] \tag{2.2}$$
$$\leq \frac{1}{2}\left[f(A)\otimes g(B)+g(A)\otimes f(B)\right]$$
$$\leq U^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[f^{1/2}(A)\,g^{1/2}(A)\right]\otimes\left[f^{1/2}(B)\,g^{1/2}(B)\right].$$

Proof. If $a\in[\gamma_1,\Gamma_1]\subset(0,\infty)$ and $b\in[\gamma_2,\Gamma_2]\subset(0,\infty)$, then

$$\frac{a}{b}\in\left[\frac{\gamma_1}{\Gamma_2},\frac{\Gamma_1}{\gamma_2}\right]\subset(0,\infty).$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, then we observe that

$$\max_{\tau\in\left[\frac{\gamma_1}{\Gamma_2},\frac{\Gamma_1}{\gamma_2}\right]}K(\tau)=U(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)$$

and

$$\min_{\tau\in\left[\frac{\gamma_1}{\Gamma_2},\frac{\Gamma_1}{\gamma_2}\right]}K(\tau)=u(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2).$$

By (1.13) we then get

$$u^{r}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,a^{1-\nu}b^{\nu} \tag{2.3}$$
$$\leq K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b$$
$$\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq U^{R}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,a^{1-\nu}b^{\nu},$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

Now, if we take

$$a=\frac{f(t)}{g(t)},\ t\in I \text{ and } b=\frac{f(s)}{g(s)},\ s\in J$$

in (2.3), then we get

$$u^{r}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left(\frac{f(t)}{g(t)}\right)^{1-\nu}\left(\frac{f(s)}{g(s)}\right)^{\nu} \tag{2.4}$$
$$\leq(1-\nu)\frac{f(t)}{g(t)}+\nu\frac{f(s)}{g(s)}$$
$$\leq U^{R}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left(\frac{f(t)}{g(t)}\right)^{1-\nu}\left(\frac{f(s)}{g(s)}\right)^{\nu},$$

for $t\in I$ and $s\in J$.

This is equivalent to

$$u^{r}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,f^{1-\nu}(t)\,g^{\nu}(t)\,f^{\nu}(s)\,g^{1-\nu}(s) \tag{2.5}$$
$$\leq(1-\nu)\,f(t)\,g(s)+\nu g(t)\,f(s)$$
$$\leq U^{R}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,f^{1-\nu}(t)\,g^{\nu}(t)\,f^{\nu}(s)\,g^{1-\nu}(s),$$

for $t\in I$ and $s\in J$.

If

$$A=\int_I t\,dE(t) \text{ and } B=\int_J s\,dF(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$ in (2.5), we derive that

$$\begin{aligned}(2.6)\qquad & u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \int_I \int_J f^{1-\nu}(t) g^\nu(t) f^\nu(s) g^{1-\nu}(s) \, dE(t) \otimes dF(s) \\ & \le \int_I \int_J [(1-\nu) f(t) g(s) + \nu g(t) f(s)] \, dE(t) \otimes dF(s) \\ & \le U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \int_I \int_J f^{1-\nu}(t) g^\nu(t) f^\nu(s) g^{1-\nu}(s) \, dE(t) \otimes dF(s).\end{aligned}$$

By utilizing (1.1) we get

$$\begin{aligned} & \int_I \int_J f^{1-\nu}(t) g^\nu(t) f^\nu(s) g^{1-\nu}(s) \, dE(t) \otimes dF(s) \\ & = \left[f^{1-\nu}(A) g^\nu(A)\right] \otimes \left[f^\nu(B) g^{1-\nu}(B)\right]\end{aligned}$$

and

$$\begin{aligned} & \int_I \int_J [(1-\nu) f(t) g(s) + \nu g(t) f(s)] \, dE(t) \otimes dF(s) \\ & = (1-\nu) \int_I \int_J f(t) g(s) \, dE(t) \otimes dF(s) + \nu \int_I \int_J g(t) f(s) \, dE(t) \otimes dF(s) \\ & = (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B).\end{aligned}$$

Therefore, by (2.6) we obtain the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 1,*

$$\begin{aligned}(2.7)\qquad & u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1-\nu}(A) g^\nu(A)\right] \circ \left[f^\nu(B) g^{1-\nu}(B)\right] \\ & \le (1-\nu) f(A) \circ g(B) + \nu g(A) \circ f(B) \\ & \le U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1-\nu}(A) g^\nu(A)\right] \circ \left[f^\nu(B) g^{1-\nu}(B)\right]\end{aligned}$$

for $\nu \in [0, 1]$.

In particular,

$$\begin{aligned}(2.8)\qquad & u^{1/2}(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1/2}(A) g^{1/2}(A)\right] \circ \left[f^{1/2}(B) g^{1/2}(B)\right] \\ & \le \frac{1}{2} [f(A) \circ g(B) + g(A) \circ f(B)] \\ & \le U^{1/2}(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1/2}(A) g^{1/2}(A)\right] \circ \left[f^{1/2}(B) g^{1/2}(B)\right].\end{aligned}$$

Proof. We have the representation

$$X \circ Y = \mathcal{U}^* (X \otimes Y) \mathcal{U}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take $\mathcal{U}^*$ at the left and $\mathcal{U}$ at the right in (2.1), then we get

$$\begin{aligned} & u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu}(A) g^\nu(A)\right] \otimes \left[f^\nu(B) g^{1-\nu}(B)\right]\right) \mathcal{U} \\ & \le \mathcal{U}^* [(1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B)] \mathcal{U} \\ & \le U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu}(A) g^\nu(A)\right] \otimes \left[f^\nu(B) g^{1-\nu}(B)\right]\right) \mathcal{U},\end{aligned}$$

namely

$$\begin{aligned}
&u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\mathcal{U}^*\left(\left[f^{1-\nu}(A)g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)g^{1-\nu}(B)\right]\right)\mathcal{U}\\
&\le(1-\nu)\mathcal{U}^*\left[f(A)\otimes g(B)\right]\mathcal{U}+\nu\mathcal{U}^*\left[g(A)\otimes f(B)\right]\mathcal{U}\\
&\le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\mathcal{U}^*\left(\left[f^{1-\nu}(A)g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)g^{1-\nu}(B)\right]\right)\mathcal{U},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\mathcal{U}^*\left(\left[f^{1-\nu}(A)g^{\nu}(A)\right]\circ\left[f^{\nu}(B)g^{1-\nu}(B)\right]\right)\mathcal{U}\\
&\le(1-\nu)\mathcal{U}^*\left[f(A)\circ g(B)\right]\mathcal{U}+\nu\mathcal{U}^*\left[g(A)\circ f(B)\right]\mathcal{U}\\
&\le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\mathcal{U}^*\left(\left[f^{1-\nu}(A)g^{\nu}(A)\right]\circ\left[f^{\nu}(B)g^{1-\nu}(B)\right]\right)\mathcal{U}
\end{aligned}$$

and the inequality (2.7) is obtained. □

Corollary 2. *Assume that f,g are continuous on I and*

$$0<\gamma\le\frac{f(t)}{g(t)}\le\Gamma \text{ for } t\in I.$$

If A and B are selfadjoint operators with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$, then

$$\begin{aligned}
&\left[f^{1-\nu}(A)g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)g^{1-\nu}(B)\right]\\
&\le(1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B)\\
&\le\left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R\left[f^{1-\nu}(A)g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)g^{1-\nu}(B)\right].
\end{aligned}\tag{2.9}$$

In particular,

$$\begin{aligned}
&\left[f^{1/2}(A)g^{1/2}(A)\right]\otimes\left[f^{1/2}(B)g^{1/2}(B)\right]\\
&\le\frac{1}{2}\left[f(A)\otimes g(B)+g(A)\otimes f(B)\right]\\
&\le\frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)g^{1/2}(A)\right]\otimes\left[f^{1/2}(B)g^{1/2}(B)\right].
\end{aligned}\tag{2.10}$$

We also have for $B=A$ that

$$\begin{aligned}
&\left[f^{1-\nu}(A)g^{\nu}(A)\right]\otimes\left[f^{\nu}(A)g^{1-\nu}(A)\right]\\
&\le(1-\nu)f(A)\otimes g(A)+\nu g(A)\otimes f(A)\\
&\le\left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R\left[f^{1-\nu}(A)g^{\nu}(A)\right]\otimes\left[f^{\nu}(A)g^{1-\nu}(A)\right].
\end{aligned}\tag{2.11}$$

In particular,

$$\begin{aligned}
&\left[f^{1/2}(A)g^{1/2}(A)\right]\otimes\left[f^{1/2}(A)g^{1/2}(A)\right]\\
&\le\frac{1}{2}\left[f(A)\otimes g(A)+g(A)\otimes f(A)\right]\\
&\le\frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)g^{1/2}(A)\right]\otimes\left[f^{1/2}(A)g^{1/2}(A)\right].
\end{aligned}\tag{2.12}$$

The proof follows by taking $\gamma_1=\gamma_2=\gamma$ and $\Gamma_1=\Gamma_2=\Gamma$ in Theorem 1.

Remark 1. *With the assumptions of Corollary 2 we have the following inequalities for the Hadamard product*

$$\begin{aligned}(2.13)\qquad &\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\\ &\le(1-\nu)\,f(A)\circ g(B)+\nu g(A)\circ f(B)\\ &\le\left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^{R}\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right].\end{aligned}$$

In particular,

$$\begin{aligned}(2.14)\qquad &\left[f^{1/2}(A)\,g^{1/2}(A)\right]\circ\left[f^{1/2}(B)\,g^{1/2}(B)\right]\\ &\le\frac{1}{2}\left[f(A)\circ g(B)+g(A)\circ f(B)\right]\\ &\le\frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)\,g^{1/2}(A)\right]\otimes\left[f^{1/2}(B)\,g^{1/2}(B)\right].\end{aligned}$$

We also have for $B=A$ that

$$\begin{aligned}(2.15)\qquad &\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(A)\,g^{1-\nu}(A)\right]\\ &\le f(A)\circ g(A)\\ &\le\left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^{R}\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(A)\,g^{1-\nu}(A)\right].\end{aligned}$$

In particular,

$$\begin{aligned}(2.16)\qquad &\left[f^{1/2}(A)\,g^{1/2}(A)\right]\circ\left[f^{1/2}(A)\,g^{1/2}(A)\right]\\ &\le f(A)\circ g(A)\\ &\le\frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)\,g^{1/2}(A)\right]\circ\left[f^{1/2}(A)\,g^{1/2}(A)\right].\end{aligned}$$

We also have:

Theorem 2. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}(2.17)\qquad u^{r}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)&\le(1-\nu)\left[f^{\nu}(A)\,g^{-\nu}(A)\right]\otimes\left[f^{-\nu}(B)\,g^{\nu}(B)\right]\\ &\quad+\nu\left[g^{1-\nu}(A)\,f^{-1+\nu}(A)\right]\otimes\left[g^{-1+\nu}(B)\,f^{1-\nu}(B)\right]\\ &\le U^{R}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2),\end{aligned}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}(2.18)\qquad &u^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\\ &\le\frac{1}{2}\left[f^{1/2}(A)\,g^{-1/2}(A)\right]\otimes\left[f^{-1/2}(B)\,g^{1/2}(B)\right]\\ &+\frac{1}{2}\left[g^{1/2}(A)\,f^{-1/2}(A)\right]\otimes\left[g^{-1/2}(B)\,f^{1/2}(B)\right]\\ &\le U^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2).\end{aligned}$$

Proof. From (2.5) we also have

$$u^{r}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\leq\frac{\left(1-\nu\right)f\left(t\right)g\left(s\right)+\nu g\left(t\right)f\left(s\right)}{f^{1-\nu}\left(t\right)g^{\nu}\left(t\right)f^{\nu}\left(s\right)g^{1-\nu}\left(s\right)}\leq U^{R}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right),$$

namely

$$\begin{aligned}u^{r}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)&\leq\left(1-\nu\right)f^{\nu}\left(t\right)g^{-\nu}\left(t\right)f^{-\nu}\left(s\right)g^{\nu}\left(s\right)\\&\quad+\nu g^{1-\nu}\left(t\right)f^{-1+\nu}\left(t\right)g^{-1+\nu}\left(s\right)f^{1-\nu}\left(s\right)\\&\leq U^{R}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right),\end{aligned}\tag{2.19}$$

for $t\in I$ and $s\in J$.

By taking the integral $\int_I\int_J$ over $dE\left(t\right)\otimes dF\left(s\right)$ in (2.19), we derive the desired inequality (2.17). □

Corollary 3. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}u^{r}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)&\leq\left(1-\nu\right)\left[f^{\nu}\left(A\right)g^{-\nu}\left(A\right)\right]\circ\left[f^{-\nu}\left(B\right)g^{\nu}\left(B\right)\right]\\&\quad+\nu\left[g^{1-\nu}\left(A\right)f^{-1+\nu}\left(A\right)\right]\circ\left[g^{-1+\nu}\left(B\right)f^{1-\nu}\left(B\right)\right]\\&\leq U^{R}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right),\end{aligned}\tag{2.20}$$

for all $\nu\in\left[0,1\right].$

In particular,

$$\begin{aligned}u^{1/2}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)&\leq\frac{1}{2}\left[f^{1/2}\left(A\right)g^{-1/2}\left(A\right)\right]\circ\left[f^{-1/2}\left(B\right)g^{1/2}\left(B\right)\right]\\&\quad+\frac{1}{2}\left[g^{1/2}\left(A\right)f^{-1/2}\left(A\right)\right]\circ\left[g^{-1/2}\left(B\right)f^{1/2}\left(B\right)\right]\\&\leq U^{1/2}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right).\end{aligned}\tag{2.21}$$

If we assume that f and g satisfy the conditions of Corollary 2 and A has the spectrum $\operatorname{Sp}\left(A\right)\subset I$, then by (2.21) we get the following inequality of interest

$$1\leq\left[f^{1/2}\left(A\right)g^{-1/2}\left(A\right)\right]\circ\left[f^{-1/2}\left(A\right)g^{1/2}\left(A\right)\right]\leq\frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}.\tag{2.22}$$

3. Inequalities for Sums

We can state the following result:

Proposition 1. *With the assumptions of Theorem 1 and if* A_i *and* B_i *are self-adjoint operators with* $\operatorname{Sp}\left(A_i\right)\subset I$ *and* $\operatorname{Sp}\left(B_i\right)\subset J$, $p_i\geq0$, $i\in\{1,...,n\}$ *with*

$\sum_{i=1}^{n} p_i = 1$, *then*

$$u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[\sum_{i=1}^{n} p_i f^{1-\nu}(A_i) g^{\nu}(A_i)\right]\otimes\left[\sum_{i=1}^{n} p_i f^{\nu}(B_i) g^{1-\nu}(B_i)\right] \tag{3.1}$$
$$\leq (1-\nu)\left(\sum_{i=1}^{n} p_i f(A_i)\right)\otimes\left(\sum_{i=1}^{n} p_i g(B_i)\right)$$
$$+\nu\left(\sum_{i=1}^{n} p_i g(A_i)\right)\otimes\left(\sum_{i=1}^{n} p_i f(B_i)\right)$$
$$\leq U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[\sum_{i=1}^{n} p_i f^{1-\nu}(A_i) g^{\nu}(A_i)\right]\otimes\left[\sum_{i=1}^{n} p_i f^{\nu}(B_i) g^{1-\nu}(B_i)\right]$$

for $\nu\in[0,1]$, *where* $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

In particular, for $\nu = 1/2$, *we get*

$$u^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) \tag{3.2}$$
$$\times\left[\sum_{i=1}^{n} p_i f^{1/2}(A_i) g^{1/2}(A_i)\right]\otimes\left[\sum_{i=1}^{n} p_i f^{1/2}(B_i) g^{1/2}(B_i)\right]$$
$$\leq \frac{1}{2}\left\{\left(\sum_{i=1}^{n} p_i f(A_i)\right)\otimes\left(\sum_{i=1}^{n} p_i g(B_i)\right)\right.$$
$$\left.+\left(\sum_{i=1}^{n} p_i g(A_i)\right)\otimes\left(\sum_{i=1}^{n} p_i f(B_i)\right)\right\}$$
$$\leq U^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)$$
$$\times\times\left[\sum_{i=1}^{n} p_i f^{1/2}(A_i) g^{1/2}(A_i)\right]\otimes\left[\sum_{i=1}^{n} p_i f^{1/2}(B_i) g^{1/2}(B_i)\right].$$

Proof. From (2.1) we get

$$u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[f^{1-\nu}(A_i) g^{\nu}(A_i)\right]\otimes\left[f^{\nu}(B_j) g^{1-\nu}(B_j)\right] \tag{3.3}$$
$$\leq (1-\nu) f(A_i)\otimes g(B_j)+\nu g(A_i)\otimes f(B_j)$$
$$\leq U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[f^{1-\nu}(A_i) g^{\nu}(A_i)\right]\otimes\left[f^{\nu}(B_j) g^{1-\nu}(B_j)\right]$$

for $i,j\in\{1,...,n\}$.

If we multiply (3.3) by $p_i p_j \geq 0$ and sum, then we get

$$u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\sum_{i,j=1}^{n} p_i p_j\left[f^{1-\nu}(A_i) g^{\nu}(A_i)\right]\otimes\left[f^{\nu}(B_j) g^{1-\nu}(B_j)\right] \tag{3.4}$$
$$\leq (1-\nu)\sum_{i,j=1}^{n} p_i p_j f(A_i)\otimes g(B_j)+\nu\sum_{i,j=1}^{n} p_i p_j g(A_i)\otimes f(B_j)$$
$$\leq U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\sum_{i,j=1}^{n} p_i p_j\left[f^{1-\nu}(A_i) g^{\nu}(A_i)\right]\otimes\left[f^{\nu}(B_j) g^{1-\nu}(B_j)\right],$$

which is equivalent to (3.1). □

Remark 2. *Assume that f,g are continuous on I and*

$$0<\gamma\leq\frac{f\left(t\right)}{g\left(t\right)}\leq\Gamma \text{ for } t\in I.$$

For $B_i=A_i$, $i\in\{1,...,n\}$ we get from (3.1) that

$$\begin{aligned}(3.5)\quad&\left[\sum_{i=1}^{n}p_i f^{1-\nu}\left(A_i\right)g^{\nu}\left(A_i\right)\right]\otimes\left[\sum_{i=1}^{n}p_i f^{\nu}\left(A_i\right)g^{1-\nu}\left(A_i\right)\right]\\&\leq\left(1-\nu\right)\left(\sum_{i=1}^{n}p_i f\left(A_i\right)\right)\otimes\left(\sum_{i=1}^{n}p_i g\left(A_i\right)\right)\\&+\nu\left(\sum_{i=1}^{n}p_i g\left(A_i\right)\right)\otimes\left(\sum_{i=1}^{n}p_i f\left(A_i\right)\right)\\&\leq\left[\frac{\left(\gamma+\Gamma\right)^2}{4\gamma\Gamma}\right]^{R}\left[\sum_{i=1}^{n}p_i f^{1-\nu}\left(A_i\right)g^{\nu}\left(A_i\right)\right]\otimes\left[\sum_{i=1}^{n}p_i f^{\nu}\left(A_i\right)g^{1-\nu}\left(A_i\right)\right].\end{aligned}$$

In particular, for $\nu=1/2$

$$\begin{aligned}(3.6)\quad&\left[\sum_{i=1}^{n}p_i f^{1/2}\left(A_i\right)g^{1/2}\left(A_i\right)\right]\otimes\left[\sum_{i=1}^{n}p_i f^{1/2}\left(A_i\right)g^{1/2}\left(A_i\right)\right]\\&\leq\frac{1}{2}\left[\left(\sum_{i=1}^{n}p_i f\left(A_i\right)\right)\otimes\left(\sum_{i=1}^{n}p_i g\left(A_i\right)\right)\right.\\&\left.+\left(\sum_{i=1}^{n}p_i g\left(A_i\right)\right)\otimes\left(\sum_{i=1}^{n}p_i f\left(A_i\right)\right)\right]\\&\leq\frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[\sum_{i=1}^{n}p_i f^{1-\nu}\left(A_i\right)g^{\nu}\left(A_i\right)\right]\otimes\left[\sum_{i=1}^{n}p_i f^{\nu}\left(A_i\right)g^{1-\nu}\left(A_i\right)\right].\end{aligned}$$

From (3.7) we get the following inequality for the Hadamard product

$$\begin{aligned}(3.7)\quad&\left[\sum_{i=1}^{n}p_i f^{1-\nu}\left(A_i\right)g^{\nu}\left(A_i\right)\right]\circ\left[\sum_{i=1}^{n}p_i f^{\nu}\left(A_i\right)g^{1-\nu}\left(A_i\right)\right]\\&\leq\left(\sum_{i=1}^{n}p_i f\left(A_i\right)\right)\circ\left(\sum_{i=1}^{n}p_i g\left(A_i\right)\right)\\&\leq\left[\frac{\left(\gamma+\Gamma\right)^2}{4\gamma\Gamma}\right]^{R}\left[\sum_{i=1}^{n}p_i f^{1-\nu}\left(A_i\right)g^{\nu}\left(A_i\right)\right]\circ\left[\sum_{i=1}^{n}p_i f^{\nu}\left(A_i\right)g^{1-\nu}\left(A_i\right)\right].\end{aligned}$$

In particular, we have

$$\left[\sum_{i=1}^{n} p_i f^{1/2}(A_i)\, g^{1/2}(A_i)\right] \circ \left[\sum_{i=1}^{n} p_i f^{1/2}(A_i)\, g^{1/2}(A_i)\right] \tag{3.8}$$
$$\leq \left(\sum_{i=1}^{n} p_i f(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i g(A_i)\right)$$
$$\leq \frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}} \left[\sum_{i=1}^{n} p_i f^{1/2}(A_i)\, g^{1/2}(A_i)\right] \circ \left[\sum_{i=1}^{n} p_i f^{1/2}(A_i)\, g^{1/2}(A_i)\right].$$

4. Examples

Assume that the operators A and B satisfy the conditions

$$0 < m \leq A,\ B \leq M$$

for some constants m and M.

Consider the functions $f(t) = t^p$, $g(t) = t^q$ for $t > 0$ and $p \neq q$ are real numbers. We have $\frac{t^p}{t^q} = t^{p-q}$ and

$$m^{p-q} \leq \frac{f(t)}{g(t)} \leq M^{p-q} \text{ for } p > q$$

and

$$M^{p-q} \leq \frac{f(t)}{g(t)} \leq m^{p-q} \text{ for } p < q$$

for all $t \in [m, M]$.

For $p > q$ we get by Corollary 2

$$A^{(1-\nu)p+\nu q} \otimes B^{\nu p+(1-\nu)q} \tag{4.1}$$
$$\leq (1-\nu) A^p \otimes B^q + \nu A^q \otimes B^p$$
$$\leq \left[\frac{(m^{p-q} + M^{p-q})^2}{4m^{p-q}M^{p-q}}\right]^R A^{(1-\nu)p+\nu q} \otimes B^{\nu p+(1-\nu)q}$$

where $\nu \in [0,1]$ and $R = \max\{1-\nu, \nu\}$.

In particular,

$$A^{\frac{p+q}{2}} \otimes B^{\frac{p+q}{2}} \leq \frac{1}{2}\left[A^p \otimes B^q + A^q \otimes B^p\right] \tag{4.2}$$
$$\leq \frac{m^{p-q} + M^{p-q}}{2m^{\frac{p-q}{2}} M^{\frac{p-q}{2}}} A^{\frac{p+q}{2}} \otimes B^{\frac{p+q}{2}}.$$

We also have for $B = A$ that

$$A^{(1-\nu)p+\nu q} \otimes A^{\nu p+(1-\nu)q} \tag{4.3}$$
$$\leq (1-\nu) A^p \otimes A^q + \nu A^q \otimes A^p$$
$$\leq \left[\frac{(m^{p-q} + M^{p-q})^2}{4m^{p-q}M^{p-q}}\right]^R A^{(1-\nu)p+\nu q} \otimes A^{\nu p+(1-\nu)q}.$$

In particular,

$$A^{\frac{p+q}{2}}\otimes A^{\frac{p+q}{2}}\leq\frac{1}{2}\left[A^{p}\otimes A^{q}+A^{q}\otimes A^{p}\right]\leq\frac{m^{p-q}+M^{p-q}}{2m^{\frac{p-q}{2}}M^{\frac{p-q}{2}}}A^{\frac{p+q}{2}}\otimes A^{\frac{p+q}{2}}. \tag{4.4}$$

For $p>q$ we get by Remark 1 the following inequalities for Hadamard product

$$\begin{aligned}&A^{(1-\nu)p+\nu q}\circ B^{\nu p+(1-\nu)q}\\&\leq(1-\nu)A^{p}\circ B^{q}+\nu A^{q}\circ B^{p}\\&\leq\left[\frac{\left(m^{p-q}+M^{p-q}\right)^{2}}{4m^{p-q}M^{p-q}}\right]^{R}A^{(1-\nu)p+\nu q}\circ B^{\nu p+(1-\nu)q}.\end{aligned} \tag{4.5}$$

In particular,

$$A^{\frac{p+q}{2}}\circ B^{\frac{p+q}{2}}\leq\frac{1}{2}\left[A^{p}\circ B^{q}+A^{q}\circ B^{p}\right]\leq\frac{m^{p-q}+M^{p-q}}{2m^{\frac{p-q}{2}}M^{\frac{p-q}{2}}}A^{\frac{p+q}{2}}\otimes B^{\frac{p+q}{2}}. \tag{4.6}$$

We also have for $B=A$ that

$$A^{(1-\nu)p+\nu q}\circ A^{\nu p+(1-\nu)q}\leq A^{p}\circ A^{q}\leq\left[\frac{\left(m^{p-q}+M^{p-q}\right)^{2}}{4m^{p-q}M^{p-q}}\right]^{R}A^{(1-\nu)p+\nu q}\circ A^{\nu p+(1-\nu)q}. \tag{4.7}$$

In particular,

$$A^{\frac{p+q}{2}}\circ A^{\frac{p+q}{2}}\leq A^{p}\circ A^{q}\leq\frac{m^{p-q}+M^{p-q}}{2m^{\frac{p-q}{2}}M^{\frac{p-q}{2}}}A^{\frac{p+q}{2}}\circ A^{\frac{p+q}{2}}. \tag{4.8}$$

Similar inequalities may be stated if one consider the functions $f(t)=\exp(\alpha t)$, $g(t)=\exp(\beta t)$ with $\alpha\neq\beta$ and $t\in\mathbb{R}$.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[5] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[7] S. Furuichi, Refined Young inequalities with Specht's ratio, *Journal of the Egyptian Mathematical Society* **20**(2012) , 46–49.

[8] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269

[9] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.

[10] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[11] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[12] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[13] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.

[14] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[15] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[16] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa