TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES VIA A CARTWRIGHT-FIELD RESULT

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if the functions f and g are continous and positive on the interval I and such that there exists the positive numbers $m < M$ with

$$0 < m \leq \frac{f(t)}{g(t)} \leq M \text{ for all } t \in I,$$

then, for the selfadjoint operators A, B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A) \subset I$, we have the tensorial inequalities

$$\begin{aligned}
0 &\leq \frac{1}{M}\nu(1-\nu) \\
&\times \left[\frac{\left(f^2(A)\, g^{-1}(A)\right) \otimes g(B) + g(A) \otimes \left(f^2(B)\, g^{-1}(B)\right)}{2} - f(A) \otimes f(B)\right] \\
&\leq (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) - \left(f^{1-\nu}(A)\, g^{\nu}(A)\right) \otimes \left(f^{\nu}(B)\, g^{1-\nu}(B)\right) \\
&\leq \frac{1}{m}\nu(1-\nu) \\
&\times \left[\frac{\left(f^2(A)\, g^{-1}(A)\right) \otimes g(B) + g(A) \otimes \left(f^2(B)\, g^{-1}(B)\right)}{2} - f(A) \otimes f(B)\right].
\end{aligned}$$

Some similar inequalities for Hadamard product are also given.

1. Introduction

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.1}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [4] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Since $\max\{a,b\}\min\{a,b\} = ab$ for $a, b > 0$, then by (1.1) we get

$$\begin{aligned}
\frac{1}{2}\nu(1-\nu)\min\{a,b\}\frac{(b-a)^2}{ab} &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\
&\leq \frac{1}{2}\nu(1-\nu)\max\{a,b\}\frac{(b-a)^2}{ab},
\end{aligned}$$

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namely

$$0 \leq \frac{1}{2}\nu (1-\nu) \min \{a,b\} \left(\frac{a}{b} + \frac{b}{a} - 2 \right) \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \tag{1.2}$$
$$\leq \frac{1}{2}\nu (1-\nu) \max \{a,b\} \left(\frac{a}{b} + \frac{b}{a} - 2 \right),$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_1) \, dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.3}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [7, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.4}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE (t) \text{ and } B = \int_{[0,\infty)} s dF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) \, dE (t) \otimes dF (s) \tag{1.5}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [9] obtained the following *Callebaut type inequalities* for tensorial product

$$(1.6)\quad \begin{aligned}(A\#B)\otimes(A\#B) &\leq \frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right]\\ &\leq \frac{1}{2}\left(A\otimes B+B\otimes A\right)\end{aligned}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)\,e_j,e_j\rangle=\langle Ae_j,e_j\rangle\,\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$(1.7)\qquad A\circ B=\mathcal{U}^{*}\left(A\otimes B\right)\mathcal{U}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [7, p. 173]

$$(1.8)\qquad f\left(A\circ B\right)\geq(\leq)\,f\left(A\right)\circ f\left(B\right)\text{ for all }A,\ B\geq 0.$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1\text{ for }A,\ B\geq 0$$

and *Fiedler inequality*

$$(1.9)\qquad A\circ A^{-1}\geq 1\text{ for }A>0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}\text{ for }A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^{2}\circ B^{2}\right)^{1/2}\text{ for }A,\ B\geq 0.$$

It has been shown in [8] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we obtain some lower and upper bounds for the quantities

$$(1-\nu)\,f\left(A\right)\otimes g\left(B\right)+\nu g\left(A\right)\otimes f\left(B\right)-\left(f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right)\otimes\left(f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right)$$

and

$$(1-\nu)\,f\left(A\right)\circ g\left(B\right)+\nu g\left(A\right)\circ f\left(B\right)-\left(f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right)\circ\left(f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right)$$

with $\nu\in[0,1]$,under the assumptions that the functions f and g are continuous and positive on the interval I and such that there exists the positive numbers $m<M$ such that

$$0<m\leq\frac{f\left(t\right)}{g\left(t\right)}\leq M\text{ for all }t\in I,$$

while the selfadjoint operators $A,\ B$ are with spectra $\operatorname{Sp}\left(A\right),\ \operatorname{Sp}\left(A\right)\subset I$.

2. Main Results

We have the following main result:

Theorem 1. *Assume that the functions f and g are continuous and positive on the interval I and such that there exists the positive numbers $m < M$ such that*

$$0 < m \leq \frac{f(t)}{g(t)} \leq M \text{ for all } t \in I,$$

then for the selfadjoint operators A, B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A) \subset I$, *we have the tensorial inequalities*

$$\begin{aligned}(2.1)\quad 0 &\leq \frac{1}{M}\nu(1-\nu) \\ &\times \left[\frac{\left(f^2(A)g^{-1}(A)\right)\otimes g(B)+g(A)\otimes\left(f^2(B)g^{-1}(B)\right)}{2} - f(A)\otimes f(B)\right] \\ &\leq (1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B) \\ &- \left(f^{1-\nu}(A)g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)g^{1-\nu}(B)\right) \\ &\leq \frac{1}{m}\nu(1-\nu) \\ &\times \left[\frac{\left(f^2(A)g^{-1}(A)\right)\otimes g(B)+g(A)\otimes\left(f^2(B)g^{-1}(B)\right)}{2} - f(A)\otimes f(B)\right]\end{aligned}$$

and

$$\begin{aligned}(2.2)\quad 0 &\leq m\nu(1-\nu) \\ &\times \left[\frac{f(A)\otimes\left(f^{-1}(B)g(B)\right)+\left(f^{-1}(A)g(A)\right)\otimes f(B)}{2} - g(A)\otimes g(B)\right] \\ &\leq (1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B) \\ &- \left(f^{1-\nu}(A)g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)g^{1-\nu}(B)\right) \\ &\leq M\nu(1-\nu) \\ &\times \left[\frac{f(A)\otimes\left(f^{-1}(B)g(B)\right)+\left(f^{-1}(A)g(A)\right)\otimes f(B)}{2} - g(A)\otimes g(B)\right]\end{aligned}$$

for $\nu \in [0,1]$.

Proof. Now if a, $b \in [m, M] \subset (0, \infty)$, then we have from (1.1) and (1.2) the following two inequalities

$$\begin{aligned}(2.3)\qquad 0 &\leq \frac{1}{2M}\nu(1-\nu)\left(a^2-2ab+b^2\right) \leq (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \\ &\leq \frac{1}{2m}\nu(1-\nu)\left(a^2-2ab+b^2\right)\end{aligned}$$

and

$$\begin{aligned}(2.4)\qquad 0 &\leq \frac{1}{2}m\nu(1-\nu)\left(\frac{a}{b}+\frac{b}{a}-2\right) \leq (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \\ &\leq \frac{1}{2}M\nu(1-\nu)\left(\frac{a}{b}+\frac{b}{a}-2\right)\end{aligned}$$

for $\nu \in [0,1]$.

Since

$$a = \frac{f(t)}{g(t)},\ b = \frac{f(s)}{g(s)} \in [m, M] \text{ for all } t, s \in I,$$

then by (2.3) and (2.4) we get

$$0 \leq \frac{1}{2M}\nu(1-\nu)\left(\left(\frac{f(t)}{g(t)}\right)^2 - 2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)} + \left(\frac{f(s)}{g(s)}\right)^2\right) \tag{2.5}$$
$$\leq (1-\nu)\frac{f(t)}{g(t)} + \nu\frac{f(s)}{g(s)} - \left(\frac{f(t)}{g(t)}\right)^{1-\nu}\left(\frac{f(s)}{g(s)}\right)^{\nu}$$
$$\leq \frac{1}{2m}\nu(1-\nu)\left(\left(\frac{f(t)}{g(t)}\right)^2 - 2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)} + \left(\frac{f(s)}{g(s)}\right)^2\right)$$

and

$$0 \leq \frac{1}{2}m\nu(1-\nu)\left(\frac{f(t)}{g(t)}\frac{g(s)}{f(s)} + \frac{g(t)}{f(t)}\frac{f(s)}{g(s)} - 2\right) \tag{2.6}$$
$$\leq (1-\nu)\frac{f(t)}{g(t)} + \nu\frac{f(s)}{g(s)} - \left(\frac{f(t)}{g(t)}\right)^{1-\nu}\left(\frac{f(s)}{g(s)}\right)^{\nu}$$
$$\leq \frac{1}{2}M\nu(1-\nu)\left(\frac{f(t)}{g(t)}\frac{g(s)}{f(s)} + \frac{g(t)}{f(t)}\frac{f(s)}{g(s)} - 2\right)$$

for all $t, s \in I$ and $\nu \in [0,1]$.

If we multiply the inequalities (2.5) and (2.6) by $g(t)\,g(s)$, then we get

$$0 \leq \frac{1}{2M}\nu(1-\nu)\left(\frac{f^2(t)}{g(t)}g(s) - 2f(t)\,f(s) + \frac{f^2(s)}{g(s)}g(t)\right) \tag{2.7}$$
$$\leq (1-\nu)\,f(t)\,g(s) + \nu g(t)\,f(s) - f^{1-\nu}(t)\,g^{\nu}(t)\,f^{\nu}(s)\,g^{1-\nu}(s)$$
$$\leq \frac{1}{2m}\nu(1-\nu)\left(\frac{f^2(t)}{g(t)}g(s) - 2f(t)\,f(s) + \frac{f^2(s)}{g(s)}g(t)\right)$$

and

$$0 \leq \frac{1}{2}m\nu(1-\nu)\left(f(t)\frac{g(s)}{f(s)} + \frac{g(t)}{f(t)}f(s) - 2g(t)\,g(s)\right) \tag{2.8}$$
$$\leq (1-\nu)\,f(t)\,g(s) + \nu g(t)\,f(s) - f^{1-\nu}(t)\,g^{\nu}(t)\,f^{\nu}(s)\,g^{1-\nu}(s)$$
$$\leq \frac{1}{2}M\nu(1-\nu)\left(f(t)\frac{g(s)}{f(s)} + \frac{g(t)}{f(t)}f(s) - 2g(t)\,g(s)\right)$$

for all $t, s \in I$ and $\nu \in [0,1]$.

If

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (2.7) and (2.8) we get

$$
\begin{aligned}
(2.9)\quad 0 &\le \frac{1}{2M}\nu(1-\nu) \\
&\times \int_I \int_I \left(\frac{f^2(t)}{g(t)} g(s) - 2f(t) f(s) + \frac{f^2(s)}{g(s)} g(t) \right) dE(t) \otimes dF(s) \\
&\le \int_I \int_I \left[(1-\nu) f(t) g(s) + \nu g(t) f(s) - f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) \right] \\
&\times dE(t) \otimes dF(s) \\
&\le \frac{1}{2m}\nu(1-\nu) \\
&\times \int_I \int_I \left(\frac{f^2(t)}{g(t)} g(s) - 2f(t) f(s) + \frac{f^2(s)}{g(s)} g(t) \right) dE(t) \otimes dF(s)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.10)\quad 0 &\le \frac{1}{2} m\nu(1-\nu) \\
&\times \int_I \int_I \left(f(t) \frac{g(s)}{f(s)} + \frac{g(t)}{f(t)} f(s) - 2g(t) g(s) \right) dE(t) \otimes dF(s) \\
&\le \int_I \int_I \left[(1-\nu) f(t) g(s) + \nu g(t) f(s) - f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) \right] \\
&\times dE(t) \otimes dF(s) \\
&\le \frac{1}{2} M\nu(1-\nu) \\
&\times \int_I \int_I \left(f(t) \frac{g(s)}{f(s)} + \frac{g(t)}{f(t)} f(s) - 2g(t) g(s) \right) dE(t) \otimes dF(s)
\end{aligned}
$$

for all $\nu \in [0,1]$.

Now, by (1.3) we get

$$
\begin{aligned}
&\int_I \int_I \left(\frac{f^2(t)}{g(t)} g(s) - 2f(t) f(s) + \frac{f^2(s)}{g(s)} g(t) \right) dE(t) \otimes dF(s) \\
&= \int_I \int_I \frac{f^2(t)}{g(t)} g(s)\, dE(t) \otimes dF(s) + \int_I \int_I g(t) \frac{f^2(s)}{g(s)} dE(t) \otimes dF(s) \\
&- 2 \int_I \int_I f(t) f(s)\, dE(t) \otimes dF(s) \\
&= \left(f^2(A) g^{-1}(A) \right) \otimes g(B) + g(A) \otimes \left(f^2(B) g^{-1}(B) \right) \\
&- 2f(A) \otimes f(B),
\end{aligned}
$$

$$\begin{aligned}
&\int_I \int_I \left[(1-\nu) f(t) g(s) + \nu g(t) f(s) - f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s)\right] \\
&\times dE(t) \otimes dF(s) \\
&= (1-\nu) \int_I \int_I f(t) g(s) \, dE(t) \otimes dF(s) + \nu \int_I \int_I g(t) f(s) \, dE(t) \otimes dF(s) \\
&- \int_I \int_I f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) \, dE(t) \otimes dF(s) \\
&= (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\
&- \left(f^{1-\nu}(A) g^{\nu}(A)\right) \otimes \left(f^{\nu}(B) g^{1-\nu}(B)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_I \left(f(t) \frac{g(s)}{f(s)} + \frac{g(t)}{f(t)} f(s) - 2g(t) g(s)\right) dE(t) \otimes dF(s) \\
&= \int_I \int_I f(t) \frac{g(s)}{f(s)} dE(t) \otimes dF(s) + \int_I \int_I \frac{g(t)}{f(t)} f(s) \, dE(t) \otimes dF(s) \\
&- 2 \int_I \int_I g(t) g(s) \, dE(t) \otimes dF(s) \\
&= f(A) \otimes \left(f^{-1}(B) g(B)\right) + \left(f^{-1}(A) g(A)\right) \otimes f(B) \\
&- 2g(A) \otimes g(B).
\end{aligned}$$

Then by (2.9) and (2.10) we get (2.1) and (2.2). □

Remark 1. *We observe that for $\nu = 1/2$ we obtain the following inequalities*

$$\begin{aligned}
0 &\le \frac{1}{4M} \left[\frac{1}{2} \left[\left(f^2(A) g^{-1}(A)\right) \otimes g(B) + g(A) \otimes \left(f^2(B) g^{-1}(B)\right)\right]\right. \\
&\left.-f(A) \otimes f(B)\right] \\
&\le \frac{f(A) \otimes g(B) + g(A) \otimes f(B)}{2} \\
&- \left(f^{1/2}(A) g^{1/2}(A)\right) \otimes \left(f^{1/2}(B) g^{1/2}(B)\right) \\
&\le \frac{1}{4M} \left[\frac{1}{2} \left[\left(f^2(A) g^{-1}(A)\right) \otimes g(B) + g(A) \otimes \left(f^2(B) g^{-1}(B)\right)\right]\right. \\
&\left.-f(A) \otimes f(B)\right]
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
0 &\le \frac{1}{4} m \left[\frac{1}{2} \left[f(A) \otimes \left(f^{-1}(B) g(B)\right) + \left(f^{-1}(A) g(A)\right) \otimes f(B)\right]\right. \\
&\left.-g(A) \otimes g(B)\right] \\
&\le \frac{f(A) \otimes g(B) + g(A) \otimes f(B)}{2} \\
&- \left(f^{1/2}(A) g^{1/2}(A)\right) \otimes \left(f^{1/2}(B) g^{1/2}(B)\right) \\
&\le \frac{1}{4} M \left[\frac{1}{2} \left[f(A) \otimes \left(f^{-1}(B) g(B)\right) + \left(f^{-1}(A) g(A)\right) \otimes f(B)\right]\right. \\
&\left.-g(A) \otimes g(B)\right].
\end{aligned} \tag{2.12}$$

Corollary 1. *With the assumptions of Theorem 1 we have*

$$
\begin{aligned}
(2.13)\quad 0 &\leq \frac{1}{M}\nu\left(1-\nu\right)\\
&\times\left[\frac{\left(f^{2}\left(A\right)g^{-1}\left(A\right)\right)\circ g\left(B\right)+g\left(A\right)\circ\left(f^{2}\left(B\right)g^{-1}\left(B\right)\right)}{2}-f\left(A\right)\circ f\left(B\right)\right]\\
&\leq\left(1-\nu\right)f\left(A\right)\circ g\left(B\right)+\nu g\left(A\right)\circ f\left(B\right)\\
&-\left(f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right)\circ\left(f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right)\\
&\leq\frac{1}{m}\nu\left(1-\nu\right)\\
&\times\left[\frac{\left(f^{2}\left(A\right)g^{-1}\left(A\right)\right)\circ g\left(B\right)+g\left(A\right)\circ\left(f^{2}\left(B\right)g^{-1}\left(B\right)\right)}{2}-f\left(A\right)\circ f\left(B\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(2.14)\quad 0 &\leq m\nu\left(1-\nu\right)\\
&\times\left[\frac{f\left(A\right)\circ\left(f^{-1}\left(B\right)g\left(B\right)\right)+\left(f^{-1}\left(A\right)g\left(A\right)\right)\circ f\left(B\right)}{2}-g\left(A\right)\circ g\left(B\right)\right]\\
&\leq\left(1-\nu\right)f\left(A\right)\circ g\left(B\right)+\nu g\left(A\right)\circ f\left(B\right)\\
&-\left(f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right)\circ\left(f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right)\\
&\leq M\nu\left(1-\nu\right)\\
&\times\left[\frac{f\left(A\right)\circ\left(f^{-1}\left(B\right)g\left(B\right)\right)+\left(f^{-1}\left(A\right)g\left(A\right)\right)\circ f\left(B\right)}{2}-g\left(A\right)\circ g\left(B\right)\right]
\end{aligned}
$$

for all $\nu\in[0,1]$.

Proof. For $X,Y\in B\left(H\right)$, we have the representation

$$X\circ Y=\mathcal{U}^{*}\left(X\otimes Y\right)\mathcal{U}$$

where $\mathcal{U}:H\rightarrow H\otimes H$ is the isometry defined by $\mathcal{U}e_{j}=e_{j}\otimes e_{j}$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^{*}$ at the left and $\mathcal{U}$ at the right in the inequality (2.1), then we get

$$
\begin{aligned}
0 &\leq \frac{1}{M}\nu\left(1-\nu\right)\\
&\times\mathcal{U}^{*}\left[\frac{\left(f^{2}\left(A\right)g^{-1}\left(A\right)\right)\otimes g\left(B\right)+g\left(A\right)\otimes\left(f^{2}\left(B\right)g^{-1}\left(B\right)\right)}{2}-f\left(A\right)\otimes f\left(B\right)\right]\mathcal{U}\\
&\leq\mathcal{U}^{*}\left[\left(1-\nu\right)f\left(A\right)\otimes g\left(B\right)+\nu g\left(A\right)\otimes f\left(B\right)\right.\\
&\left.-\left(f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right)\otimes\left(f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right)\right]\mathcal{U}\\
&\leq\frac{1}{m}\nu\left(1-\nu\right)\\
&\times\mathcal{U}^{*}\left[\frac{\left(f^{2}\left(A\right)g^{-1}\left(A\right)\right)\otimes g\left(B\right)+g\left(A\right)\otimes\left(f^{2}\left(B\right)g^{-1}\left(B\right)\right)}{2}-f\left(A\right)\otimes f\left(B\right)\right]\mathcal{U},
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\leq \frac{1}{M}\nu(1-\nu) \\
&\times \left[\frac{\mathcal{U}^*\left[\left(f^2(A)\,g^{-1}(A)\right)\otimes g(B)\right]\mathcal{U}+\mathcal{U}^*\left[g(A)\otimes\left(f^2(B)\,g^{-1}(B)\right)\right]\mathcal{U}}{2}\right. \\
&\left.-\mathcal{U}^*\left(f(A)\otimes f(B)\right)\mathcal{U}\right] \\
&\leq (1-\nu)\,\mathcal{U}^*\left[f(A)\otimes g(B)\right]\mathcal{U}+\nu\,\mathcal{U}^*\left(g(A)\otimes f(B)\right)\mathcal{U} \\
&-\mathcal{U}^*\left[\left(f^{1-\nu}(A)\,g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)\,g^{1-\nu}(B)\right)\right]\mathcal{U} \\
&\leq \frac{1}{m}\nu(1-\nu) \\
&\times \left[\frac{\mathcal{U}^*\left[\left(f^2(A)\,g^{-1}(A)\right)\otimes g(B)\right]\mathcal{U}+\mathcal{U}^*\left[g(A)\otimes\left(f^2(B)\,g^{-1}(B)\right)\right]\mathcal{U}}{2}\right. \\
&\left.-\mathcal{U}^*\left(f(A)\otimes f(B)\right)\mathcal{U}\right],
\end{aligned}
$$

which is equivalent to (2.13). □

Remark 2. *We observe that for $\nu = 1/2$ we obtain the following inequalities*

$$
\begin{aligned}
0 &\leq \frac{1}{4M}\left[\frac{1}{2}\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(B)+g(A)\circ\left(f^2(B)\,g^{-1}(B)\right)\right]\right. \\
&\left.-f(A)\circ f(B)\right] \\
&\leq \frac{f(A)\circ g(B)+g(A)\circ f(B)}{2} \\
&-\left(f^{1/2}(A)\,g^{1/2}(A)\right)\circ\left(f^{1/2}(B)\,g^{1/2}(B)\right) \\
&\leq \frac{1}{4M}\left[\frac{1}{2}\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(B)+g(A)\circ\left(f^2(B)\,g^{-1}(B)\right)\right]\right. \\
&\left.-f(A)\circ f(B)\right]
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{4}m\left[\frac{1}{2}\left[f(A)\circ\left(f^{-1}(B)\,g(B)\right)+\left(f^{-1}(A)\,g(A)\right)\circ f(B)\right]\right. \\
&\left.-g(A)\circ g(B)\right] \\
&\leq \frac{f(A)\circ g(B)+g(A)\circ f(B)}{2} \\
&-\left(f^{1/2}(A)\,g^{1/2}(A)\right)\circ\left(f^{1/2}(B)\,g^{1/2}(B)\right) \\
&\leq \frac{1}{4}M\left[\frac{1}{2}\left[f(A)\circ\left(f^{-1}(B)\,g(B)\right)+\left(f^{-1}(A)\,g(A)\right)\circ f(B)\right]\right. \\
&\left.-g(A)\circ g(B)\right].
\end{aligned}
\tag{2.16}
$$

Now, if we take $B=A$ in Corollary 1, then we get

$$0\leq\frac{1}{M}\nu(1-\nu)\left[\left(f^{2}(A)g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right]$$
$$\leq f(A)\circ g(A)-\left(f^{1-\nu}(A)g^{\nu}(A)\right)\circ\left(f^{\nu}(A)g^{1-\nu}(A)\right)$$
$$\leq\frac{1}{m}\nu(1-\nu)\left[\left(f^{2}(A)g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right] \tag{2.17}$$

and

$$0\leq m\nu(1-\nu)\left[f(A)\circ\left(f^{-1}(A)g(A)\right)-g(A)\circ g(A)\right]$$
$$\leq f(A)\circ g(A)-\left(f^{1-\nu}(A)g^{\nu}(A)\right)\circ\left(f^{\nu}(A)g^{1-\nu}(A)\right)$$
$$\leq M\nu(1-\nu)\left[f(A)\circ\left(f^{-1}(A)g(A)\right)-g(A)\circ g(A)\right] \tag{2.18}$$

for all $\nu\in[0,1]$.

In particular, for $\nu=1/2$ we get

$$0\leq\frac{1}{4M}\left[\left(f^{2}(A)g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right]$$
$$\leq f(A)\circ g(A)-\left(f^{1/2}(A)g^{1/2}(A)\right)\circ\left(f^{1/2}(A)g^{1/2}(A)\right)$$
$$\leq\frac{1}{4m}\left[\left(f^{2}(A)g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right] \tag{2.19}$$

and

$$0\leq\frac{1}{4}m\left[f(A)\circ\left(f^{-1}(A)g(A)\right)-g(A)\circ g(A)\right]$$
$$\leq f(A)\circ g(A)-\left(f^{1/2}(A)g^{1/2}(A)\right)\circ\left(f^{1/2}(A)g^{1/2}(A)\right)$$
$$\leq\frac{1}{4}M\left[f(A)\circ\left(f^{-1}(A)g(A)\right)-g(A)\circ g(A)\right]. \tag{2.20}$$

3. Inequalities for Power Series

Assume that the operators A and B satisfy the conditions

$$0<m\leq A,\ B\leq M$$

for some constants m and M.

Consider the functions $f(t)=t^{p}$, $g(t)=t^{q}$ for $t>0$ and $p\neq q$ are real numbers. We have $\frac{t^{p}}{t^{q}}=t^{p-q}$ and

$$m^{p-q}\leq\frac{f(t)}{g(t)}\leq M^{p-q}\text{ for }p>q$$

and

$$M^{p-q}\leq\frac{f(t)}{g(t)}\leq m^{p-q}\text{ for }p<q$$

for all $t\in[m,M]$.

For $p>q$ we get by Theorem 1 that

$$0\leq \frac{1}{M^{p-q}}\nu\left(1-\nu\right)\left[\frac{A^{2p-q}\otimes B^{q}+A^{q}\otimes B^{2p-q}}{2}-A^{p}\otimes B^{q}\right] \tag{3.1}$$
$$\leq\left(1-\nu\right)A^{p}\otimes B^{q}+\nu A^{q}\otimes B^{p}-A^{(1-\nu)p+\nu q}\otimes B^{\nu q+(1-\nu)p}$$
$$\leq \frac{1}{m^{p-q}}\nu\left(1-\nu\right)\left[\frac{A^{2p-q}\otimes B^{q}+A^{q}\otimes B^{2p-q}}{2}-A^{p}\otimes B^{q}\right]$$

and

$$0\leq m^{p-q}\nu\left(1-\nu\right)\left[\frac{A^{p}\otimes B^{q-p}+A^{q-p}\otimes B^{p}}{2}-A^{p}\otimes B^{q}\right] \tag{3.2}$$
$$\leq\left(1-\nu\right)A^{p}\otimes B^{q}+\nu A^{q}\otimes B^{p}-A^{(1-\nu)p+\nu q}\otimes B^{\nu q+(1-\nu)p}$$
$$\leq M^{p-q}\nu\left(1-\nu\right)\left[\frac{A^{p}\otimes B^{q-p}+A^{q-p}\otimes B^{p}}{2}-A^{p}\otimes B^{q}\right],$$

for all $\nu\in[0,1]$.

We also have the following result for power series

Theorem 2. *Let $f(z)=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Assume that $0\leq A,\ B\leq 1$, $p>q$ and $0\leq\alpha,\beta<R$, then*

$$0\leq\nu\left(1-\nu\right) \tag{3.3}$$
$$\times\left[\frac{f\left(\alpha A^{2p-q}\right)\otimes f\left(\beta B^{q}\right)+f\left(\alpha A^{q}\right)\otimes f\left(\beta B^{2p-q}\right)}{2}-f\left(\alpha A^{p}\right)\otimes f\left(\beta B^{q}\right)\right]$$
$$\leq\left(1-\nu\right)f\left(\alpha A^{p}\right)\otimes f\left(\beta B^{q}\right)+\nu f\left(\alpha A^{q}\right)\otimes f\left(\beta B^{p}\right)$$
$$-f\left(\alpha A^{(1-\nu)p+\nu q}\right)\otimes f\left(\beta B^{\nu q+(1-\nu)p}\right).$$

If $R=\infty$, then

$$0\leq\nu\left(1-\nu\right) \tag{3.4}$$
$$\times\left[\frac{f\left(\alpha A^{2p-q}\right)\otimes f\left(\beta B^{q}\right)+f\left(\alpha A^{q}\right)\otimes f\left(\beta B^{2p-q}\right)}{2}-f\left(\alpha A^{p}\right)\otimes f\left(\beta B^{q}\right)\right]$$
$$\leq\left(1-\nu\right)f\left(\alpha A^{p}\right)\otimes f\left(\beta B^{q}\right)+\nu f\left(\alpha A^{q}\right)\otimes f\left(\beta B^{p}\right)$$
$$-f\left(\alpha A^{(1-\nu)p+\nu q}\right)\otimes f\left(\beta B^{\nu q+(1-\nu)p}\right)$$
$$\leq\nu\left(1-\nu\right)$$
$$\times\left[\frac{f\left(\alpha A^{p}\right)\otimes f\left(\beta B^{q-p}\right)+f\left(\alpha A^{q-p}\right)\otimes f\left(\beta B^{p}\right)}{2}-f\left(\alpha A^{p}\right)\otimes f\left(\beta B^{q}\right)\right].$$

Proof. From (3.1) and (3.2) we have for $0\leq A,\ B\leq 1$ that

$$0\leq\nu\left(1-\nu\right)\left[\frac{A^{2p-q}\otimes B^{q}+A^{q}\otimes B^{2p-q}}{2}-A^{p}\otimes B^{q}\right] \tag{3.5}$$
$$\leq\left(1-\nu\right)A^{p}\otimes B^{q}+\nu A^{q}\otimes B^{p}-A^{(1-\nu)p+\nu q}\otimes B^{\nu q+(1-\nu)p}$$
$$\leq\nu\left(1-\nu\right)\left[\frac{A^{p}\otimes B^{q-p}+A^{q-p}\otimes B^{p}}{2}-A^{p}\otimes B^{q}\right],$$

for $p > q$ and $\nu \in [0,1]$.

Since $0 \leq A,\ B \leq 1$, then $0 \leq A^i,\ B^j \leq 1$ for $i, j = 0, 1, ...$ and by (3.5) we get

$$0 \leq \nu (1-\nu) \left[\frac{A^{i(2p-q)} \otimes B^{jq} + A^{iq} \otimes B^{j(2p-q)}}{2} - A^{ip} \otimes B^{jq} \right] \tag{3.6}$$
$$\leq (1-\nu) A^{ip} \otimes B^{jq} + \nu A^{iq} \otimes B^{jp} - A^{i[(1-\nu)p+\nu q]} \otimes B^{j[\nu q + (1-\nu)p]}$$
$$\leq \nu (1-\nu) \left[\frac{A^{ip} \otimes B^{j(q-p)} + A^{i(q-p)} \otimes B^{jp}}{2} - A^{ip} \otimes B^{jq} \right],$$

for $i, j = 0, 1, ...$

If we multiply this inequality by $a_i \alpha^i$ and $a_j \beta^j$, then we get

$$0 \leq \nu (1-\nu)$$
$$\times \left[\frac{a_i \alpha^i A^{i(2p-q)} \otimes a_j \beta^j B^{jq} + a_i \alpha^i A^{iq} \otimes a_j \beta^j B^{j(2p-q)}}{2} - a_i \alpha^i A^{ip} \otimes a_j \beta^j B^{jq} \right]$$
$$\leq (1-\nu) a_i \alpha^i A^{ip} \otimes a_j \beta^j B^{jq} + \nu a_i \alpha^i A^{iq} \otimes a_j \beta^j B^{jp}$$
$$- a_i \alpha^i A^{i[(1-\nu)p+\nu q]} \otimes a_j \beta^j B^{j[\nu q + (1-\nu)p]}$$
$$\leq \nu (1-\nu)$$
$$\times \left[\frac{a_i \alpha^i A^{ip} \otimes a_j \beta^j B^{j(q-p)} + a_i \alpha^i A^{i(q-p)} \otimes a_j \beta^j B^{jp}}{2} - a_i \alpha^i A^{ip} \otimes a_j \beta^j B^{jq} \right],$$

for $i, j = 0, 1, ...$

If we sum over i from 0 to n and over j from 0 to m, then we get

$$0 \leq \nu (1-\nu) \tag{3.7}$$
$$\times \left[\frac{\left(\sum_{i=0}^{n} a_i \alpha^i A^{i(2p-q)} \right) \otimes \left(\sum_{j=0}^{m} a_j \beta^j B^{jq} \right)}{2} \right.$$
$$+ \frac{\left(\sum_{i=0}^{n} a_i \alpha^i A^{iq} \right) \otimes \left(\sum_{j=0}^{m} a_j \beta^j B^{j(2p-q)} \right)}{2}$$
$$\left. - \left(\sum_{i=0}^{n} a_i \alpha^i A^{ip} \right) \otimes \left(\sum_{j=0}^{m} a_j \beta^j B^{jq} \right) \right]$$

$$\le (1-\nu)\left(\sum_{i=0}^{n} a_i\alpha^i A^{ip}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{jq}\right)$$
$$+\nu\left(\sum_{i=0}^{n} a_i\alpha^i A^{iq}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{jp}\right)$$
$$-\left(\sum_{i=0}^{n} a_i\alpha^i A^{i[(1-\nu)p+\nu q]}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{j[\nu q+(1-\nu)p]}\right)$$
$$\le \nu(1-\nu)$$
$$\times\left[\frac{\left(\sum_{i=0}^{n} a_i\alpha^i A^{ip}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{j(q-p)}\right)}{2}\right.$$
$$+\frac{\left(\sum_{i=0}^{n} a_i\alpha^i A^{i(q-p)}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{jp}\right)}{2}$$
$$\left.-\left(\sum_{i=0}^{n} a_i\alpha^i A^{ip}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{jq}\right)\right],$$

for $p>q$ and $\nu\in[0,1]$.

If $0\le A$, $B\le 1$ and $0\le\alpha,\beta<R$, then $0\le \alpha A^{2p-q}$, βB^q, αA^q, βB^{2p-q}, αA^p, βB^p, $\alpha A^{[(1-\nu)p+\nu q]}$, $\beta B^{[\nu q+(1-\nu)p]}<R$, which shows that the series

$$\sum_{i=0}^{\infty} a_i\alpha^i A^{i(2p-q)},\ \sum_{j=0}^{\infty} a_j\beta^j B^{jq},\ \sum_{i=0}^{\infty} a_i\alpha^i A^{iq},\ \sum_{j=0}^{\infty} a_j\beta^j B^{j(2p-q)}$$
$$\sum_{i=0}^{\infty} a_i\alpha^i A^{ip},\ \sum_{j=0}^{\infty} a_j\beta^j B^{jq},\ \sum_{i=0}^{\infty} a_i\alpha^i A^{i[(1-\nu)p+\nu q]} \text{ and}$$
$$\sum_{j=0}^{\infty} a_j\beta^j B^{j[\nu q+(1-\nu)p]}$$

are convergent. By taking $m,n\to\infty$ in the first two inequalities in (3.7) we deduce (3.3).

If $R=\infty$, then the series $\sum_{j=0}^{\infty} a_j\beta^j B^{j(q-p)}$ and $\sum_{i=0}^{\infty} a_i\alpha^i A^{i(q-p)}$ are also convergent, and by taking $m,n\to\infty$ in all inequalities in (3.7), we derive (3.4). □

Corollary 2. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Assume that $0\le A$, $B\le 1$, $p>q$ and $0\le\alpha,\beta<R$, then*

$$(3.8)\quad 0\le\nu(1-\nu)$$
$$\times\left[\frac{f\left(\alpha A^{2p-q}\right)\circ f\left(\beta B^q\right)+f\left(\alpha A^q\right)\circ f\left(\beta B^{2p-q}\right)}{2}-f\left(\alpha A^p\right)\circ f\left(\beta B^q\right)\right]$$
$$\le(1-\nu) f\left(\alpha A^p\right)\circ f\left(\beta B^q\right)+\nu f\left(\alpha A^q\right)\circ f\left(\beta B^p\right)$$
$$-f\left(\alpha A^{(1-\nu)p+\nu q}\right)\circ f\left(\beta B^{\nu q+(1-\nu)p}\right).$$

If $R=\infty$, *then*

$$0\leq\nu(1-\nu)\times\left[\frac{f\left(\alpha A^{2p-q}\right)\circ f(\beta B^{q})+f(\alpha A^{q})\circ f\left(\beta B^{2p-q}\right)}{2}-f(\alpha A^{p})\circ f(\beta B^{q})\right]\leq(1-\nu)f(\alpha A^{p})\circ f(\beta B^{q})+\nu f(\alpha A^{q})\circ f(\beta B^{p})-f\left(\alpha A^{(1-\nu)p+\nu q}\right)\circ f\left(\beta B^{\nu q+(1-\nu)p}\right)\leq\nu(1-\nu)\times\left[\frac{f(\alpha A^{p})\circ f(\beta B^{q-p})+f(\alpha A^{q-p})\circ f(\beta B^{p})}{2}-f(\alpha A^{p})\circ f(\beta B^{q})\right]\tag{3.9}$$

for $\nu\in[0,1]$.

Remark 3. *Assume that* $0\leq A\leq 1$, $p>q$ *and* $0\leq\alpha,\beta<R$, *then by taking* $B=A$ *in (3.8), then we get*

$$0\leq\nu(1-\nu)\times\left[\frac{f\left(\alpha A^{2p-q}\right)\circ f(\beta A^{q})+f(\alpha A^{q})\circ f\left(\beta A^{2p-q}\right)}{2}-f(\alpha A^{p})\circ f(\beta A^{q})\right]\leq(1-\nu)f(\alpha A^{p})\circ f(\beta A^{q})+\nu f(\alpha A^{q})\circ f(\beta A^{p})-f\left(\alpha A^{(1-\nu)p+\nu q}\right)\circ f\left(\beta A^{\nu q+(1-\nu)p}\right),\tag{3.10}$$

and for $\alpha=\beta=1$ *we derive*

$$0\leq\nu(1-\nu)\left[f\left(A^{2p-q}\right)-f(A^{p})\right]\circ f(A^{q})\leq f(A^{p})\circ f(A^{q})-f\left(A^{(1-\nu)p+\nu q}\right)\circ f\left(A^{\nu q+(1-\nu)p}\right).\tag{3.11}$$

If $R=\infty$, *then*

$$0\leq\nu(1-\nu)\times\left[\frac{f\left(\alpha A^{2p-q}\right)\circ f(\beta A^{q})+f(\alpha A^{q})\circ f\left(\beta A^{2p-q}\right)}{2}-f(\alpha A^{p})\circ f(\beta A^{q})\right]\leq(1-\nu)f(\alpha A^{p})\circ f(\beta A^{q})+\nu f(\alpha A^{q})\circ f(\beta A^{p})-f\left(\alpha A^{(1-\nu)p+\nu q}\right)\circ f\left(\beta A^{\nu q+(1-\nu)p}\right)\leq\nu(1-\nu)\times\left[\frac{f(\alpha A^{p})\circ f(\beta A^{q-p})+f(\alpha A^{q-p})\circ f(\beta A^{p})}{2}-f(\alpha A^{p})\circ f(\beta A^{q})\right]\tag{3.12}$$

and for $\alpha=\beta=1$ *we derive*

$$0\leq\nu(1-\nu)\left[f\left(A^{2p-q}\right)-f(A^{p})\right]\circ f(A^{q})\leq f(A^{p})\circ f(A^{q})-f\left(A^{(1-\nu)p+\nu q}\right)\circ f\left(A^{\nu q+(1-\nu)p}\right)\leq\nu(1-\nu)\left[f\left(A^{q-p}\right)-f(A^{q})\right]\circ f(A^{p})\tag{3.13}$$

for $\nu\in[0,1]$.

Let $h(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. We have the following examples

$$\begin{aligned}
h(z) &= \sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1); \\
h(z) &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).
\end{aligned}\tag{3.14}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned}
h(z) &= \sum_{n=0}^{\infty}\frac{1}{n!}z^n=\exp(z) \qquad z\in\mathbb{C}, \\
h(z) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad z\in D(0,1); \\
h(z) &= \sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z), \qquad z\in D(0,1);
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
h(z) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z), \qquad z\in D(0,1) \\
h(z) &= {}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0, \\
&\quad z\in D(0,1);
\end{aligned}\tag{3.16}$$

where Γ is *Gamma function.*

Assume that $0\le A,\ B\le 1$, $p>q$ and $0\le\alpha,\beta<1$, then by writing the inequality (3.3) for the function $f(z)=(1-z)^{-1}$, we get

$$\begin{aligned}
0 &\le \nu(1-\nu) \\
&\times\left[\frac{\left(1-\alpha A^{2p-q}\right)^{-1}\otimes(1-\beta B^q)^{-1}+(1-\alpha A^q)^{-1}\otimes\left(1-\beta B^{2p-q}\right)^{-1}}{2}\right. \\
&\left.-(1-\alpha A^p)^{-1}\otimes f(1-\beta B^q)^{-1}\right] \\
&\le (1-\nu)(1-\alpha A^p)^{-1}\otimes(1-\beta B^q)^{-1}+\nu(1-\alpha A^q)^{-1}\otimes(1-\beta B^p)^{-1} \\
&-\left(1-\alpha A^{(1-\nu)p+\nu q}\right)^{-1}\otimes\left(1-\beta B^{\nu q+(1-\nu)p}\right)^{-1}.
\end{aligned}\tag{3.17}$$

From (3.11) we also have the inequalities for the Hadamard product

$$\begin{aligned} &0 \le \nu (1-\nu) \left[\left(1-A^{2p-q}\right)^{-1} - (1-A^p)^{-1} \right] \circ (1-A^q)^{-1} \\ &\le (1-A^p)^{-1} \circ (1-A^q)^{-1} - \left(1-A^{(1-\nu)p+\nu q}\right)^{-1} \circ \left(1-A^{\nu q+(1-\nu)p}\right)^{-1}, \end{aligned} \tag{3.18}$$

for $\nu \in [0,1]$, $0 \le A < 1$, $p > q$.

Assume that $0 \le A,\ B \le 1$, $p > q$ and $0 \le \alpha, \beta$, then by writing the inequality (3.4) for the function $f(z) = \exp z$, we get

$$\begin{aligned} &0 \le \nu (1-\nu) \\ &\times \left[\frac{\exp\left(\alpha A^{2p-q}\right) \otimes \exp(\beta B^q) + \exp(\alpha A^q) \otimes \exp\left(\beta B^{2p-q}\right)}{2} \right. \\ &\left. - \exp(\alpha A^p) \otimes \exp(\beta B^q) \right] \\ &\le (1-\nu) \exp(\alpha A^p) \otimes \exp(\beta B^q) + \nu f(\alpha A^q) \otimes f(\beta B^p) \\ &- \exp\left(\alpha A^{(1-\nu)p+\nu q}\right) \otimes \exp\left(\beta B^{\nu q+(1-\nu)p}\right) \\ &\le \nu (1-\nu) \\ &\times \left[\frac{\exp(\alpha A^p) \otimes \exp(\beta B^{q-p}) + \exp(\alpha A^{q-p}) \otimes \exp(\beta B^p)}{2} \right. \\ &\left. - \exp(\alpha A^p) \otimes \exp(\beta B^q) \right]. \end{aligned} \tag{3.19}$$

Finally, from (3.13) we derive

$$\begin{aligned} &0 \le \nu (1-\nu) \left[\exp\left(A^{2p-q}\right) - \exp(A^p) \right] \circ \exp(A^q) \\ &\le \exp(A^p) \circ \exp(A^q) - \exp\left(A^{(1-\nu)p+\nu q}\right) \circ \exp\left(A^{\nu q+(1-\nu)p}\right) \\ &\le \nu (1-\nu) \left[\exp\left(A^{q-p}\right) - \exp(A^q) \right] \circ \exp(A^p) \end{aligned} \tag{3.20}$$

for $\nu \in [0,1]$, $0 \le A \le 1$, $p > q$.

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