SOME TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES VIA A LOG-REVERSE OF YOUNG'S RESULT

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if A, $B>0$ in the operator order and $\nu\in[0,1]$, then

$$\begin{aligned}0&\leq(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\\&\leq\nu(1-\nu)[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B].\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}0&\leq[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\&\leq\nu(1-\nu)[(A\ln A+B\ln B)\circ 1-A\circ\ln B-(\ln A)\circ B].\end{aligned}$$

for all $\nu\in[0,1]$.

1. Integration

The famous *Young's inequality* for scalars says that if $a,b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [13]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\[2ex] 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [14] while the first one is due to Furuichi [5].

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It is an open question for the author if in the right hand side of (1.3) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R=\max\{1-\nu,\nu\}$.

Kittaneh and Manasrah [9], [10] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \le R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.4}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's constant* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ \ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \le (1-\nu)a+\nu b \le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [16] while the second by Liao et al. [12].

In the recent paper [4] we obtained the following reverses of Young's inequality as well:

$$0\le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

It has been shown in [4] that there is no ordering for the upper bounds of the quantity $(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$ as provided by the inequalities (1.4) and (1.7). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (1.3), (1.6) and (1.8).

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k} f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k) \tag{1.9}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if* f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [7, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.10}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.11}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.12}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle Ae_j, e_j \rangle \langle Be_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.13}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [7, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.14}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1 \text{ for } A>0. \tag{1.15}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2} \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^{2}\circ B^{2}\right)^{1/2} \text{ for } A,\ B\geq 0.$$

It has been shown in [8] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by these results, in this paper we provide among others some upper bounds for the Young differences

$$(1-\nu)\,A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}$$

and

$$[(1-\nu)\,A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}$$

for $\nu\in[0,1]$ and $A,B>0$.

2. Main Results

We start to the following main result:

Theorem 1. *Assume that* A, $B>0$ *and* $\nu\in[0,1]$, *then*

$$\begin{aligned} 0 &\leq (1-\nu)\,A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu} \\ &\leq \nu\,(1-\nu)\,[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B]\,. \end{aligned} \tag{2.1}$$

In particular,

$$\begin{aligned} 0 &\leq \frac{1}{2}\,(A\otimes 1+1\otimes B)-A^{1/2}\otimes B^{1/2} \\ &\leq \frac{1}{4}\,[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B]\,. \end{aligned} \tag{2.2}$$

Proof. From (1.7) we have

$$0\leq(1-\nu)\,t+\nu s-t^{1-\nu}s^{\nu}\leq\nu\,(1-\nu)\,(t-s)\,(\ln t-\ln s) \tag{2.3}$$

for all t, $s>0$ and $\nu\in[0,1]$.

If

$$A=\int_{[0,\infty)}tdE\,(t) \text{ and } B=\int_{[0,\infty)}sdF\,(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_{[0,\infty)}\int_{[0,\infty)}$ over $dE\,(t)\otimes dF\,(s)$ in (2.3) we get

$$\begin{aligned} 0 &\leq \int_{[0,\infty)}\int_{[0,\infty)}\left[(1-\nu)\,t+\nu s-t^{1-\nu}s^{\nu}\right]dE\,(t)\otimes dF\,(s) \\ &\leq \nu\,(1-\nu)\int_{[0,\infty)}\int_{[0,\infty)}(t-s)\,(\ln t-\ln s)\,dE\,(t)\otimes dF\,(s)\,. \end{aligned} \tag{2.4}$$

Observe that, by (1.9)

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)}\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s)\\&=(1-\nu)\int_{[0,\infty)}\int_{[0,\infty)}tdE(t)\otimes dF(s)+\nu\int_{[0,\infty)}\int_{[0,\infty)}sdE(t)\otimes dF(s)\\&-\int_{[0,\infty)}\int_{[0,\infty)}t^{1-\nu}s^{\nu}dE(t)\otimes dF(s)\\&=(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\end{aligned}$$

and

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)}(t-s)(\ln t-\ln s)dE(t)\otimes dF(s)\\&=\int_{[0,\infty)}\int_{[0,\infty)}(t\ln t+s\ln s-t\ln s-s\ln t)dE(t)\otimes dF(s)\\&=\int_{[0,\infty)}\int_{[0,\infty)}t\ln tdE(t)\otimes dF(s)+\int_{[0,\infty)}\int_{[0,\infty)}s\ln sdE(t)\otimes dF(s)\\&-\int_{[0,\infty)}\int_{[0,\infty)}t\ln sdE(t)\otimes dF(s)-\int_{[0,\infty)}\int_{[0,\infty)}s\ln tdE(t)\otimes dF(s)\\&=(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B\end{aligned}$$

and by (2.4) we get (2.1). □

Corollary 1. *With the assumptions of Theorem 1 we have the following inequalities for the Hadamard product*

$$\begin{aligned}0&\le[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\&\le\nu(1-\nu)[(A\ln A+B\ln B)\circ 1-A\circ\ln B-(\ln A)\circ B]\end{aligned}\tag{2.5}$$

for $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0&\le\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\\&\le\frac{1}{4}[(A\ln A+B\ln B)\circ 1-A\circ\ln B-(\ln A)\circ B].\end{aligned}\tag{2.6}$$

Proof. For the operators X and Y we have the representation

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\mathcal{U},$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.1), we get

$$\begin{aligned}0&\le\mathcal{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\right]\mathcal{U}\\&\le\nu(1-\nu)\mathcal{U}^*[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B]\mathcal{U}.\end{aligned}\tag{2.7}$$

Observe that

$$\begin{aligned}&\mathcal{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\right]\mathcal{U}\\&=(1-\nu)\mathcal{U}^*(A\otimes 1)\mathcal{U}+\nu\mathcal{U}^*(1\otimes B)\mathcal{U}-\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U}\\&=(1-\nu)(A\circ 1)+\nu(1\circ B)-\left(A^{1-\nu}\circ B^{\nu}\right)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{U}^*\left[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B\right]\mathcal{U}\\&=\mathcal{U}^*\left((A\ln A)\otimes 1\right)\mathcal{U}+\mathcal{U}^*\left(1\otimes(B\ln B)\right)\mathcal{U}\\&-\mathcal{U}^*\left(A\otimes\ln B\right)\mathcal{U}-\mathcal{U}^*\left((\ln A)\otimes B\right)\mathcal{U}\\&=(A\ln A)\circ 1+1\circ(B\ln B)-A\circ\ln B-(\ln A)\circ B\end{aligned}$$

and by (2.7) we derive (2.5). □

Remark 1. *If we take $B=A$ in Corollary 1, then we get*

$$0\le A\circ 1-A^{1-\nu}\circ A^{\nu}\le 2\nu(1-\nu)\left[\left((A\ln A)\circ 1-A\circ\ln A\right)\right] \tag{2.8}$$

for all $\nu\in[0,1]$.

In particular,

$$0\le A\circ 1-A^{1/2}\circ A^{1/2}\le\frac{1}{2}\left[\left((A\ln A)\circ 1-A\circ\ln A\right)\right]. \tag{2.9}$$

Corollary 2. *Assume that $A_i,\ B_j>0$ and $p_i,\ q_j\ge 0$ for $i\in\{1,...,n\}$, $j\in\{1,...,m\}$, and put $P_n:=\sum_{i=1}^n p_i$, $Q_m:=\sum_{j=1}^m q_j$, then*

$$\begin{aligned}0&\le(1-\nu)Q_m\left(\sum_{j=1}^m p_iA_i\right)\otimes 1+\nu P_n 1\otimes\left(\sum_{j=1}^m q_jB_j\right)\\&-\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^m q_jB_j^{\nu}\right)\\&\le\nu(1-\nu)\left[Q_m\left(\sum_{i=1}^n p_iA_i\ln A_i\right)\otimes 1+P_n 1\otimes\left(\sum_{j=1}^m q_jB_j\ln B_j\right)\right.\\&\left.-\left(\sum_{i=1}^n p_iA_i\right)\otimes\left(\sum_{j=1}^m q_j\ln B_j\right)-\left(\sum_{i=1}^n p_i\ln A_i\right)\otimes\left(\sum_{j=1}^m q_jB_j\right)\right]\end{aligned} \tag{2.10}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0&\le\frac{1}{2}\left[Q_m\left(\sum_{j=1}^m p_iA_i\right)\otimes 1+P_n 1\otimes\left(\sum_{j=1}^m q_jB_j\right)\right]\\&-\left(\sum_{i=1}^n p_iA_i^{1/2}\right)\otimes\left(\sum_{j=1}^m q_jB_j^{1/2}\right)\\&\le\frac{1}{4}\left[Q_m\left(\sum_{i=1}^n p_iA_i\ln A_i\right)\otimes 1+P_n 1\otimes\left(\sum_{j=1}^m q_jB_j\ln B_j\right)\right.\\&\left.-\left(\sum_{i=1}^n p_iA_i\right)\otimes\left(\sum_{j=1}^m q_j\ln B_j\right)-\left(\sum_{i=1}^n p_i\ln A_i\right)\otimes\left(\sum_{j=1}^m q_jB_j\right)\right].\end{aligned} \tag{2.11}$$

Proof. From (2.1) we have

$$\begin{aligned}0 &\leq (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \\ &\leq \nu (1-\nu) \left[(A_i \ln A_i) \otimes 1 + 1 \otimes (B_j \ln B_j) - A_i \otimes \ln B_j - (\ln A_i) \otimes B_j \right].\end{aligned}$$

If we multiply this inequality by $p_i q_j$ and sum, then we get

$$\begin{aligned}0 &\leq (1-\nu) \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j A_i \otimes 1 + \nu \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j 1 \otimes B_j \\ &- \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j A_i^{1-\nu} \otimes B_j^{\nu} \\ &\leq \nu (1-\nu) \left[\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j (A_i \ln A_i) \otimes 1 + \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j 1 \otimes (B_j \ln B_j) \right. \\ &\left. - \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j A_i \otimes \ln B_j - \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j (\ln A_i) \otimes B_j \right]. \end{aligned} \tag{2.12}$$

Observe that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j A_i \otimes 1 = Q_m \left(\sum_{j=1}^{m} p_i A_i \right) \otimes 1,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j 1 \otimes B_j = P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j \right),$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j A_i^{1-\nu} \otimes B_j^{\nu} = \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{j=1}^{m} q_j B_j^{\nu} \right),$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j (A_i \ln A_i) \otimes 1 = Q_m \left(\sum_{i=1}^{n} p_i A_i \ln A_i \right) \otimes 1,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j 1 \otimes (B_j \ln B_j) = P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j \ln B_j \right),$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j A_i \otimes \ln B_j = \left(\sum_{i=1}^{n} p_i A_i \right) \otimes \left(\sum_{j=1}^{m} q_j \ln B_j \right)$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j (\ln A_i) \otimes B_j = \left(\sum_{i=1}^{n} p_i \ln A_i \right) \otimes \left(\sum_{j=1}^{m} q_j B_j \right).$$

By (2.12) we derive (2.10). □

Corollary 3. *With the assumptions of Corollary 2, we have the inequalities for the Hadamard product*

$$\begin{aligned} 0 &\le \left((1-\nu) Q_m \sum_{i=1}^{n} p_i A_i + \nu P_n \sum_{j=1}^{m} q_j B_j \right) \circ 1 \\ &- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{j=1}^{m} q_j B_j^{\nu} \right) \\ &\le \nu (1-\nu) \left[\left(Q_m \sum_{i=1}^{n} p_i A_i \ln A_i + P_n \sum_{j=1}^{m} q_j B_j \ln B_j \right) \circ 1 \right. \\ &\left. - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{j=1}^{m} q_j \ln B_j \right) - \left(\sum_{i=1}^{n} p_i \ln A_i \right) \circ \left(\sum_{j=1}^{m} q_j B_j \right) \right] \end{aligned} \tag{2.13}$$

for all $\nu \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\le \frac{1}{2} \left[\left(Q_m \sum_{i=1}^{n} p_i A_i + P_n \sum_{j=1}^{m} q_j B_j \right) \circ 1 \right] \\ &- \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \circ \left(\sum_{j=1}^{m} q_j B_j^{1/2} \right) \\ &\le \frac{1}{4} \left[\left(Q_m \sum_{i=1}^{n} p_i A_i \ln A_i + P_n \sum_{j=1}^{m} q_j B_j \ln B_j \right) \circ 1 \right. \\ &\left. - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{j=1}^{m} q_j \ln B_j \right) - \left(\sum_{i=1}^{n} p_i \ln A_i \right) \circ \left(\sum_{j=1}^{m} q_j B_j \right) \right]. \end{aligned} \tag{2.14}$$

Remark 2. *We observe that for* $m = n$, $B_i = A_i$ *and* $q_i = p_i$ *in Corollary 3, we get*

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu} \right) \\ &\le 2\nu (1-\nu) \left[\left(P_n \sum_{i=1}^{n} p_i A_i \ln A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{i=1}^{n} p_i \ln A_i \right) \right] \end{aligned} \tag{2.15}$$

for all $\nu \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \\ &\le \frac{1}{2} \left[\left(P_n \sum_{i=1}^{n} p_i A_i \ln A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{i=1}^{n} p_i \ln A_i \right) \right]. \end{aligned} \tag{2.16}$$

3. Inequalities for Power Series

We also have the following result for power series:

Theorem 2. *Let* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)\subset\mathbb{C}$, $R>0$ *and let* $\nu\in[0,1]$. *Assume that* $0<\alpha<R$, $0<\beta<R$, $0<\alpha A<R$, $0<\beta B<R$, $0<\alpha A^{1-\nu}<R$, $0<\beta B^{\nu}<R$ *and* $0\leq\alpha,\ \beta<R$, *then*

$$\begin{aligned}(3.1)\quad & 0\leq(1-\nu)f(\beta)f(\alpha A)\otimes 1+\nu f(\alpha)1\otimes f(\beta B)-f\left(\alpha A^{1-\nu}\right)\otimes f(\beta B^{\nu})\\ &\leq\nu(1-\nu)\left[\alpha f(\beta)\left(f'(\alpha A)A\ln A\right)\otimes 1+\beta f(\alpha)1\otimes\left(f'(\beta B)B\ln B\right)\right.\\ &\left.-\beta f'(\beta)f(\alpha A)\otimes\ln B-\alpha f'(\alpha)\ln A\otimes f(\beta B)\right],\end{aligned}$$

and, in particular,

$$\begin{aligned}(3.2)\quad & 0\leq\frac{f(\beta)f(\alpha A)\otimes 1+f(\alpha)1\otimes f(\beta B)}{2}-f\left(\alpha A^{1/2}\right)\otimes f\left(\beta B^{1/2}\right)\\ &\leq\frac{1}{4}\left[\alpha f(\beta)\left(f'(\alpha A)A\ln A\right)\otimes 1+\beta f(\alpha)1\otimes\left(f'(\beta B)B\ln B\right)\right.\\ &\left.-\beta f'(\beta)f(\alpha A)\otimes\ln B-\alpha f'(\alpha)\ln A\otimes f(\beta B)\right].\end{aligned}$$

Proof. From (2.1) we get

$$\begin{aligned}& 0\leq(1-\nu)A^i\otimes 1+\nu 1\otimes B^j-A^{i(1-\nu)}\otimes B^{j\nu}\\ &\leq\nu(1-\nu)\left[\left(iA^i\ln A\right)\otimes 1+1\otimes\left(jB^j\ln B\right)-jA^i\otimes\ln B-(i\ln A)\otimes B^j\right],\end{aligned}$$

for all $i\in\{0,...,n\}$, $j\in\{0,...,m\}$.

Now, if we multiply by $a_i\alpha^i$ and by $a_j\beta^j$, then we get

$$\begin{aligned}& 0\leq(1-\nu)a_j\beta^ja_i\alpha^iA^i\otimes 1+\nu a_i\alpha^i1\otimes a_j\beta^jB^j\\ &-a_i\alpha^iA^{i(1-\nu)}\otimes a_j\beta^jB^{j\nu}\\ &\leq\nu(1-\nu)\left[\left(a_j\beta^ja_i\alpha^iiA^i\ln A\right)\otimes 1+a_i\alpha^i1\otimes\left(a_j\beta^jjB^j\ln B\right)\right.\\ &\left.-ja_j\beta^ja_i\alpha^iA^i\otimes\ln B-\left(ia_i\alpha^i\ln A\right)\otimes a_j\beta^jB^j\right],\end{aligned}$$

for all $i\in\{0,...,n\}$, $j\in\{0,...,m\}$.

Now if we sum over i from 0 to n and from j from 0 to m, then we get

$$\begin{aligned}(3.3)\qquad 0 &\leq (1-\nu)\left(\sum_{j=0}^{m} a_j\beta^j\right)\left(\sum_{i=0}^{n} a_i\alpha^i A^i\right)\otimes 1 \\ &+ \nu\left(\sum_{i=0}^{n} a_i\alpha^i\right) 1\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^j\right) \\ &- \left(\sum_{i=0}^{n} a_i\alpha^i A^{i(1-\nu)}\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^{j\nu}\right) \\ &\leq \nu(1-\nu)\left[\left(\left(\sum_{j=0}^{m} a_j\beta^j\right)\left(\sum_{i=0}^{n} ia_i\alpha^i A^i\right)\ln A\right)\otimes 1\right. \\ &+ \left(\sum_{i=0}^{n} a_i\alpha^i\right) 1\otimes\left(\left(\sum_{j=0}^{m} ja_j\beta^j B^j\right)\ln B\right) \\ &- \left(\sum_{j=0}^{m} ja_j\beta^j\right)\left(\sum_{i=0}^{n} a_i\alpha^i A^i\right)\otimes \ln B \\ &\left. - \left(\left(\sum_{i=0}^{n} ia_i\alpha^i\right)\ln A\right)\otimes\left(\sum_{j=0}^{m} a_j\beta^j B^j\right)\right],\end{aligned}$$

for all $\nu \in [0,1]$.

Since $0<\alpha<R$, $0<\beta<R$, $0<\alpha A<R$, $0<\beta B<R$, $0<\alpha A^{1-\nu}<R$, $0<\beta B^{\nu}<R$, then the following series are convergent and

$$\sum_{i=0}^{\infty} a_i\alpha^i = f(\alpha)\quad \sum_{j=0}^{\infty} a_j\beta^j = f(\beta),\quad \sum_{i=0}^{\infty} a_i\alpha^i A^i = f(\alpha A),$$

$$\sum_{i=0}^{\infty} a_i\alpha^i A^{i(1-\nu)} = f\left(\alpha A^{1-\nu}\right),\quad \sum_{j=0}^{\infty} a_j\beta^j B^{j\nu} = f(\beta B^{\nu}).$$

Also

$$\sum_{i=0}^{\infty} ia_i\alpha^i = \sum_{i=1}^{\infty} ia_i\alpha^i = \alpha\left(a_1 + 2a_2\alpha + 3a_3\alpha^2 + ...\right) = \alpha f'(\alpha)$$

$$\sum_{j=0}^{\infty} ja_j\beta^j = \beta f'(\beta),\quad \sum_{i=0}^{\infty} ia_i\alpha^i A^i = \alpha A f'(\alpha A),\quad \sum_{j=0}^{\infty} ja_j\beta^j B^j = \beta B f'(\beta B).$$

By taking the limit over $n,m\to\infty$ in (3.3), we deduce the desired result (3.1). □

Corollary 4. *With the assumptions of Theorem 2, we have the following inequalities for the Hadamard product*

$$\begin{aligned}(3.4)\qquad 0 &\leq [(1-\nu) f(\beta) f(\alpha A) + \nu f(\alpha) f(\beta B)]\circ 1 - f\left(\alpha A^{1-\nu}\right)\circ f(\beta B^{\nu}) \\ &\leq \nu(1-\nu)\left[[\alpha f(\beta) f'(\alpha A) A\ln A + \beta f(\alpha) f'(\beta B) B\ln B]\circ 1\right. \\ &\left. - \beta f'(\beta) f(\alpha A)\circ \ln B - \alpha f'(\alpha)\ln A\circ f(\beta B)\right],\end{aligned}$$

and, in particular,

$$0 \le \frac{f(\beta) f(\alpha A) + f(\alpha) f(\beta B)}{2} \circ 1 - f\left(\alpha A^{1/2}\right) \circ f\left(\beta B^{1/2}\right) \tag{3.5}$$
$$\le \frac{1}{4} [(\alpha f(\beta) f'(\alpha A) A \ln A + \beta f(\alpha) f'(\beta B) B \ln B) \circ 1$$
$$-\beta f'(\beta) f(\alpha A) \circ \ln B - \alpha f'(\alpha) \ln A \circ f(\beta B)],$$

for $\nu \in [0,1]$.

If we take $f(t) = \exp t$, then from Corollary 4 we get for $\alpha \in [0,1]$ that

$$0 \le [(1-\nu) \exp(\beta + \alpha A) + \nu \exp(\alpha + \beta B)] \circ 1 \tag{3.6}$$
$$-\exp\left(\alpha A^{1-\nu}\right) \circ \exp(\beta B^{\nu})$$
$$\le \nu(1-\nu) [[\alpha \exp(\beta + \alpha A) A \ln A + \beta \exp(\alpha + \beta B) B \ln B] \circ 1$$
$$-\beta \exp(\beta + \alpha A) \circ \ln B - \alpha \ln A \circ \exp(\alpha + \beta B)],$$

and, in particular,

$$0 \le \frac{\exp(\beta + \alpha A) + \exp(\alpha + \beta B)}{2} \circ 1 \tag{3.7}$$
$$-\exp\left(\alpha A^{1/2}\right) \circ \exp\left(\beta B^{1/2}\right)$$
$$\le \frac{1}{4} [[\alpha \exp(\beta + \alpha A) A \ln A + \beta \exp(\alpha + \beta B) B \ln B] \circ 1$$
$$-\beta \exp(\beta + \alpha A) \circ \ln B - \alpha \ln A \circ \exp(\alpha + \beta B)],$$

for all $\alpha, \beta \ge 0$ and A, $B > 0$.

4. Related Results

Recall that if a, $b > 0$ and

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } a \ne b, \\ \\ b \text{ if } a = b \end{cases}$$

is the *logarithmic mean* and $G(a,b) := \sqrt{ab}$ is the *geometric mean*, then $L(a,b) \ge G(a,b)$ for all a, $b > 0$.

Then from (1.7) we have for $a \ne b$ that

$$0 \le (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b)$$
$$= \nu(1-\nu)(a-b)^2 \frac{\ln a - \ln b}{a-b} \le \nu(1-\nu) \frac{(a-b)^2}{\sqrt{ab}},$$

which implies that

$$0 \le (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \le \nu(1-\nu)\left(a^{3/2} b^{-1/2} - 2a^{1/2} b^{1/2} + a^{-1/2} b^{3/2}\right) \tag{4.1}$$

for all $a, b > 0$.

The following result provides a simpler upper bound than the one from Theorem 1 that involves the logarithmic function.

Theorem 3. *Assume that* A, $B>0$ *and* $\nu\in[0,1]$, *then*

$$0\leq(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu} \tag{4.2}$$
$$\leq\nu(1-\nu)\left(A^{3/2}\otimes B^{-1/2}-2A^{1/2}\otimes B^{1/2}+A^{-1/2}\otimes B^{3/2}\right).$$

In particular,

$$0\leq\frac{1}{2}(A\otimes 1+1\otimes B)-A^{1/2}\otimes B^{1/2} \tag{4.3}$$
$$\leq\frac{1}{4}\left(A^{3/2}\otimes B^{-1/2}-2A^{1/2}\otimes B^{1/2}+A^{-1/2}\otimes B^{3/2}\right).$$

The argument follows in a similar way to the one from the proof of Theorem 1 by employing the inequality (4.2) above.

Corollary 5. *With the assumption of Theorem 1, we have the following inequality for the Hadamard product*

$$0\leq[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu} \tag{4.4}$$
$$\leq\nu(1-\nu)\left(A^{3/2}\circ B^{-1/2}-2A^{1/2}\circ B^{1/2}+A^{-1/2}\circ B^{3/2}\right)$$

for all $\nu\in[0,1]$.

In particular,

$$0\leq\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2} \tag{4.5}$$
$$\leq\frac{1}{4}\left(A^{3/2}\circ B^{-1/2}-2A^{1/2}\circ B^{1/2}+A^{-1/2}\circ B^{3/2}\right).$$

If we take $B=A$ in Corollary 5, we get

$$0\leq A\circ 1-A^{1-\nu}\circ A^{\nu}\leq 2\nu(1-\nu)\left(A^{3/2}\circ A^{-1/2}-A^{1/2}\circ A^{1/2}\right) \tag{4.6}$$

for all $\nu\in[0,1]$.

In particular,

$$0\leq A\circ 1-A^{1/2}\circ A^{1/2}\leq\frac{1}{2}\left(A^{3/2}\circ A^{-1/2}-A^{1/2}\circ A^{1/2}\right). \tag{4.7}$$

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