TENSORIAL AND HADAMARD PRODUCT REVERSE INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES RELATED TO YOUNG'S RESULT

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0 < m \leq A$, $B \leq M$, for some constants m, M, then

$$0 \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)}$$

for all $\nu \in [0,1]$. We also have the following inequalities for the Hadamard product

$$0 \leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)}$$

for all $\nu \in [0,1]$.

1. Integration

The famous *Young's inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.3}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

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The second inequality in (1.3) is due to Tominaga [14] while the first one is due to Furuichi [5].

It is an open question for the author if in the right hand side of (1.3) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R=\max\{1-\nu,\nu\}$.

Kittaneh and Manasrah [9], [10] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)a+\nu b-a^{1-\nu}b^\nu \le R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.4}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's constant* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \le (1-\nu)a+\nu b \le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [16] while the second by Liao et al. [12].

In the recent paper [4] we obtained the following reverses of Young's inequality as well:

$$0\le (1-\nu)a+\nu b-a^{1-\nu}b^\nu \le \nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

It has been shown in [4] that there is no ordering for the upper bounds of the quantity $(1-\nu)a+\nu b-a^{1-\nu}b^\nu$ as provided by the inequalities (1.4) and (1.7). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu}$ incorporated in the inequalities (1.3), (1.6) and (1.8).

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k} f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k) \tag{1.9}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [7, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.10}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) \, dE(t) \otimes dF(s) \tag{1.11}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} \left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.12}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.13}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [7, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.14}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.15}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [8] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by these results, in this paper we provide among others some upper bounds for the Young differences

$$(1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}$$

and

$$[(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu}$$

for $\nu \in [0,1]$ and A, B such that $0 < m_1 \leq A \leq M_1$, $0 < m_2 \leq B \leq M_2$ for some constants m_1, m_2, M_1 and M_2.

2. Main Results

The first main result is as follows:

Theorem 1. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \leq A,\ B \leq M$, then*

$$0 \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} \tag{2.1}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \leq \ln \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}. \tag{2.2}$$

Proof. If $a,\ b \in [m, M] \subset (0, \infty)$, then

$$\begin{aligned} 0 &\leq (a-b)(\ln a - \ln b) = |(a-b)(\ln a - \ln b)| \\ &= |a-b| |\ln a - \ln b| \leq (M-m)(\ln M - \ln m). \end{aligned}$$

By (1.7) we then get

$$0 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq \nu (1-\nu)(M-m)(\ln M - \ln m) \tag{2.3}$$

for all $a,\ b \in [m, M]$.

If

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.3) we get

$$\begin{aligned} 0 &\le \int_m^M \int_m^M \left[(1-\nu)t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &\le \nu(1-\nu)(M-m)(\ln M - \ln m) \int_m^M \int_m^M dE(t) \otimes dF(s) \\ &= \nu(1-\nu)(M-m)(\ln M - \ln m). \end{aligned} \tag{2.4}$$

Observe that, by (1.9),

$$\begin{aligned} &\int_{[0,\infty)} \int_{[0,\infty)} \left[(1-\nu)t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &= (1-\nu) \int_{[0,\infty)} \int_{[0,\infty)} t dE(t) \otimes dF(s) + \nu \int_{[0,\infty)} \int_{[0,\infty)} s dE(t) \otimes dF(s) \\ &- \int_{[0,\infty)} \int_{[0,\infty)} t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) \\ &= (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}, \end{aligned}$$

which gives, by (2.4), the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 1, we have the following inequalities for the Hadamard product*

$$0 \le [(1-\nu)A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \le \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} \tag{2.5}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \le \frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2} \le \ln \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}. \tag{2.6}$$

Proof. For the operators X and Y we have the representation

$$X \circ Y = \mathcal{U}^* (X \otimes Y) \mathcal{U},$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.1), we get

$$\begin{aligned} 0 &\le \mathcal{U}^* \left[(1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}\right] \mathcal{U} \\ &\le \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} \mathcal{U}^* \mathcal{U}. \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} &\mathcal{U}^* \left[(1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}\right] \mathcal{U} \\ &= (1-\nu) \mathcal{U}^* (A \otimes 1) \mathcal{U} + \nu \mathcal{U}^* (1 \otimes B) \mathcal{U} - \mathcal{U}^* \left(A^{1-\nu} \otimes B^{\nu}\right) \mathcal{U} \\ &= (1-\nu)(A \circ 1) + \nu (1 \circ B) - A^{1-\nu} \circ B^{\nu} \\ &= (1-\nu)(A \circ 1) + \nu B \circ 1 - A^{1-\nu} \circ B^{\nu} \\ &= [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu}, \end{aligned}$$

hence by (2.7) we get (2.5). □

Remark 1. *If we take $B = A$ in Corollary 1, then we get*

$$0 \le A \circ 1 - A^{1-\nu} \circ A^{\nu} \le \ln \left(\frac{M}{m} \right)^{\nu(1-\nu)(M-m)} \tag{2.8}$$

for all $\nu \in [0,1]$.
In particular,

$$0 \le A \circ 1 - A^{1/2} \circ A^{1/2} \le \ln \left(\frac{M}{m} \right)^{\frac{1}{4}(M-m)}. \tag{2.9}$$

We also have:

Theorem 2. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m_1 \le A \le M_1$, $0 < m_2 \le B \le M_2$, then*

$$0 \le (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \le \Gamma_{\nu} (m_1, M_1, m_2, M_2) A \otimes 1, \tag{2.10}$$

where

$$\Gamma_{\nu}(m_1, M_1, m_2, M_2) := \begin{cases} \ln \left(\frac{M_1}{m_2} \right)^{\nu(1-\nu)\left(\frac{M_1}{m_2} - 1 \right)} & \text{if } 1 \le \frac{m_1}{M_2} \\ \max \left\{ \ln \left(\frac{m_1}{M_2} \right)^{\nu(1-\nu)\left(\frac{m_1}{M_2} - 1 \right)}, \ln \left(\frac{M_1}{m_2} \right)^{\nu(1-\nu)\left(\frac{M_1}{m_2} - 1 \right)} \right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \ln \left(\frac{m_1}{M_2} \right)^{\nu(1-\nu)\left(\frac{m_1}{M_2} - 1 \right)} & \text{if } \frac{M_1}{m_2} \le 1 \end{cases}$$

for all $\nu \in [0,1]$.
In particular,

$$0 \le \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \le \Gamma_{1/2} (m_1, M_1, m_2, M_2) A \otimes 1. \tag{2.11}$$

Proof. We consider the function $f(t) = (t-1) \ln t$, $t > 0$. We observe that

$$f'(t) = \ln t + 1 - \frac{1}{t}$$

and

$$f''(t) = \frac{1}{t} + \frac{1}{t^2} = \frac{t+1}{t^2}$$

for $t > 0$.

This shows that the function f is strictly convex on $(0, \infty)$, decreasing on $(0,1)$, increasing on $(1, \infty)$ with

$$\min_{t \in (0,\infty)} f(t) = f(1) = 0.$$

If $0 < m_1 \le t \le M_1$, $0 < m_2 \le s \le M_2$, then $u := \frac{t}{s} \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]$, which shows that

$$\max_{u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]} f(u) = \begin{cases} f\left(\frac{M_1}{m_2}\right) \text{ if } 1 \le \frac{m_1}{M_2} \\ \\ \max\left\{f\left(\frac{m_1}{M_2}\right), f\left(\frac{M_1}{m_2}\right)\right\} \text{ if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ f\left(\frac{m_1}{M_2}\right) \text{ if } \frac{M_1}{m_2} \le 1 \end{cases}$$

$$= \begin{cases} \ln\left(\frac{M_1}{m_2}\right)^{\frac{M_1}{m_2}-1} \text{ if } 1 \le \frac{m_1}{M_2} \\ \\ \max\left\{\ln\left(\frac{m_1}{M_2}\right)^{\frac{m_1}{M_2}-1}, \ln\left(\frac{M_1}{m_2}\right)^{\frac{M_1}{m_2}-1}\right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ \ln\left(\frac{m_1}{M_2}\right)^{\frac{m_1}{M_2}-1} \text{ if } \frac{M_1}{m_2} \le 1. \end{cases}.$$

By (1.7) we get

$$\begin{aligned} 0 &\le (1-\nu)t + \nu s - t^{1-\nu}s^{\nu} \le \nu(1-\nu)t\left(\frac{s}{t} - 1\right)\ln\frac{s}{t} \\ &\le \nu(1-\nu)t\Gamma(m_1, M_1, m_2, M_2) \end{aligned} \tag{2.12}$$

for $0 < m_1 \le t \le M_1$, $0 < m_2 \le s \le M_2$.

If

$$A = \int_{m_1}^{M_1} t dE(t) \text{ and } B = \int_{m_2}^{M_2} s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_{m_1}^{M_1}\int_{m_2}^{M_2}$ over $dE(t) \otimes dF(s)$ in (2.12) we get

$$\begin{aligned} 0 &\le \int_{m_1}^{M_1}\int_{m_2}^{M_2} \left[(1-\nu)t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &\le \nu(1-\nu)\Gamma(m_1, M_1, m_2, M_2)\int_{m_1}^{M_1}\int_{m_2}^{M_2} t dE(t) \otimes dF(s). \end{aligned} \tag{2.13}$$

Since

$$\begin{aligned} &\int_{m_1}^{M_1}\int_{m_2}^{M_2} \left[(1-\nu)t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &= (1-\nu)A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \end{aligned}$$

and

$$\int_{m_1}^{M_1}\int_{m_2}^{M_2} t dE(t) \otimes dF(s) = A \otimes 1,$$

hence by (2.13) we get (2.10). $\square$

Corollary 2. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \leq A,\ B \leq M,$ then*

$$0 \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)\left(\frac{M-m}{M}\right)} A \otimes 1, \tag{2.14}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \leq \ln \left(\frac{M}{m}\right)^{\frac{M-m}{4M}} A \otimes 1. \tag{2.15}$$

The proof follows by Theorem 2 for $m_1 = m_2 = m$ and $M_1 = M_2 = M$.

Corollary 3. *With the assumptions of Theorem 2, we have the following inequalities for the Hadamard product*

$$0 \leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq \Gamma_{\nu} (m_1, M_1, m_2, M_2) A \circ 1 \tag{2.16}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2} \leq \Gamma_{1/2} (m_1, M_1, m_2, M_2) A \circ 1. \tag{2.17}$$

If $0 < m \leq A,\ B \leq M,$ then

$$0 \leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)\left(\frac{M-m}{M}\right)} A \circ 1, \tag{2.18}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2} \leq \ln \left(\frac{M}{m}\right)^{\frac{M-m}{4M}} A \circ 1. \tag{2.19}$$

If we take $B = A$ in Corollary 3, then we get

$$0 \leq A \circ 1 - A^{1-\nu} \circ A^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)\left(\frac{M-m}{M}\right)} A \circ 1, \tag{2.20}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq A \circ 1 - A^{1/2} \circ A^{1/2} \leq \ln \left(\frac{M}{m}\right)^{\frac{M-m}{4M}} A \circ 1. \tag{2.21}$$

Further, we can also state the following multiplicative reverse of Young's inequality:

Theorem 3. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m_1 \leq A \leq M_1,\ 0 < m_2 \leq B \leq M_2,$ then*

$$\begin{aligned} A^{1-\nu} \otimes B^{\nu} &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \\ &\leq \exp\left[\Delta_{\nu} (m_1, M_1, m_2, M_2)\right] A^{1-\nu} \otimes B^{\nu}, \end{aligned} \tag{2.22}$$

where

$$\Delta_{\nu}(m_1, M_1, m_2, M_2) := \nu(1-\nu) \times \begin{cases} \frac{(M_1-m_2)^2}{m_2 M_1} \text{ if } 1 \leq \frac{m_1}{M_2} \\ \\ \max\left\{\frac{(M_2-m_1)^2}{m_1 M_2}, \frac{(M_1-m_2)^2}{m_2 M_1}\right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ \frac{(M_2-m_1)^2}{m_1 M_2} \text{ if } \frac{M_1}{m_2} \leq 1 \end{cases}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \otimes B^{1/2} \leq \frac{A \otimes 1 + 1 \otimes B}{2} \leq \exp\left[\Delta_{1/2}(m_1, M_1, m_2, M_2)\right] A^{1/2} \otimes B^{1/2}. \tag{2.23}$$

Proof. If $0 < m_1 \leq t \leq M_1$, $0 < m_2 \leq s \leq M_2$, then $u := \frac{t}{s} \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]$, which shows that

$$\max_{u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]} K(u) = \begin{cases} K\left(\frac{M_1}{m_2}\right) \text{ if } 1 \leq \frac{m_1}{M_2} \\ \\ \max\left\{K\left(\frac{m_1}{M_2}\right), K\left(\frac{M_1}{m_2}\right)\right\} \text{ if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ K\left(\frac{m_1}{M_2}\right) \text{ if } \frac{M_1}{m_2} \leq 1, \end{cases}$$

which gives that

$$\begin{aligned} &\max_{u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]} K(u) - 1 \\ &= \begin{cases} \frac{(M_1-m_2)^2}{4m_2 M_1} \text{ if } 1 \leq \frac{m_1}{M_2} \\ \\ \max\left\{\frac{(M_2-m_1)^2}{4m_1 M_2}, \frac{(M_1-m_2)^2}{4m_2 M_1}\right\} \text{ if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ \frac{(M_2-m_1)^2}{4m_1 M_2} \text{ if } \frac{M_1}{m_2} \leq 1, \end{cases} \\ &= \frac{1}{4} \times \begin{cases} \frac{(M_1-m_2)^2}{m_2 M_1} \text{ if } 1 \leq \frac{m_1}{M_2} \\ \\ \max\left\{\frac{(M_2-m_1)^2}{m_1 M_2}, \frac{(M_1-m_2)^2}{m_2 M_1}\right\} \text{ if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ \frac{(M_2-m_1)^2}{m_1 M_2} \text{ if } \frac{M_1}{m_2} \leq 1. \end{cases} \end{aligned}$$

From (1.8) we derive

$$\begin{aligned} 1 &\leq \frac{(1-\nu)t + \nu s}{t^{1-\nu} s^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{t}{s}\right) - 1\right)\right] \\ &\leq \exp\left[\Delta_{\nu}(m_1, M_1, m_2, M_2)\right], \end{aligned}$$

which gives

$$1 \leq (1-\nu)t + \nu s \leq \exp\left[\Delta_{\nu}(m_1, M_1, m_2, M_2)\right] t^{1-\nu} s^{\nu}, \tag{2.24}$$

for $0 < m_1 \leq t \leq M_1$, $0 < m_2 \leq s \leq M_2$.

Now, by making a similar argument to the one in the proof of Theorem 2 and utilizing inequality (2.24), we deduce (2.22). □

Corollary 4. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \leq A,\ B \leq M$, then*

$$A^{1-\nu} \otimes B^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq \exp\left[\nu(1-\nu)\frac{(M-m)^2}{mM}\right] A^{1-\nu} \otimes B^{\nu}. \tag{2.25}$$

In particular,

$$A^{1/2} \otimes B^{1/2} \leq \frac{A \otimes 1 + 1 \otimes B}{2} \leq \exp\left[\frac{(M-m)^2}{4mM}\right] A^{1/2} \otimes B^{1/2}. \tag{2.26}$$

We also have the following inequalities for the Hadamard product:

Corollary 5. *With the assumptions of Theorem 2, we have the following inequalities*

$$A^{1-\nu} \circ B^{\nu} \leq [(1-\nu) A + \nu B] \circ 1 \leq \exp[\Delta_{\nu}(m_1, M_1, m_2, M_2)] A^{1-\nu} \circ B^{\nu}, \tag{2.27}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \circ B^{1/2} \leq \frac{A+B}{2} \circ 1 \leq \exp\left[\Delta_{1/2}(m_1, M_1, m_2, M_2)\right] A^{1/2} \circ B^{1/2}. \tag{2.28}$$

With the assumptions of Corollary 4,

$$A^{1-\nu} \circ B^{\nu} \leq [(1-\nu) A + \nu B] \circ 1 \leq \exp\left[\nu(1-\nu)\frac{(M-m)^2}{mM}\right] A^{1-\nu} \circ B^{\nu}. \tag{2.29}$$

In particular,

$$A^{1/2} \circ B^{1/2} \leq \frac{A+B}{2} \circ 1 \leq \exp\left[\frac{(M-m)^2}{4mM}\right] A^{1/2} \circ B^{1/2}. \tag{2.30}$$

If we take $B = A$ in (2.29) and (2.30), then we get for $\nu \in [0,1]$ that

$$A^{1-\nu} \circ A^{\nu} \leq A \circ 1 \leq \exp\left[\nu(1-\nu)\frac{(M-m)^2}{mM}\right] A^{1-\nu} \circ A^{\nu}. \tag{2.31}$$

In particular,

$$A^{1/2} \circ A^{1/2} \leq A \circ 1 \leq \exp\left[\frac{(M-m)^2}{4mM}\right] A^{1/2} \circ A^{1/2}. \tag{2.32}$$

3. Some Inequalities for Sums

We have:

Proposition 1. *Assume that the operators* A_i, B_j *satisfy the conditions* $0 < m \leq A_i,\ B_j \leq M,\ p_i,\ q_j \geq 0$ *for* $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, *and put* $P_n := \sum_{i=1}^{n} p_i$, $Q_m := \sum_{j=1}^{m} q_j$, *then for* $\nu \in [0, 1]$,

$$0 \leq (1-\nu) Q_m \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j\right) - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{m} q_j B_j^{\nu}\right) \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} P_n Q_m. \tag{3.1}$$

In particular,

$$0 \leq \frac{1}{2} \left[Q_m \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j\right)\right] - \left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \otimes \left(\sum_{j=1}^{m} q_j B_j^{1/2}\right) \leq \ln \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} P_n Q_m. \tag{3.2}$$

Proof. From (2.1) we have

$$0 \leq (1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu} \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} \tag{3.3}$$

for $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$.

Now, if we multiply by $p_i q_j \geq 0$ and sum, then we get

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right] \leq \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j = \ln \left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} P_n Q_m \tag{3.4}$$

Observe that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right]$$

$$
\begin{aligned}
&= (1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j A_i \otimes 1 + \nu \sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j 1 \otimes B_j \\
&- \sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j A_i^{1-\nu} \otimes B_j^{\nu} \\
&= (1-\nu) Q_m \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j\right) \\
&- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{m} q_j B_j^{\nu}\right)
\end{aligned}
$$

and by (3.4) we get (3.1). □

Corollary 6. *With the assumptions of Proposition 1, we have the following inequalities for the Hadamard product*

$$
\begin{aligned}
0 &\le \left[(1-\nu) Q_m \left(\sum_{i=1}^{n} p_i A_i\right) + \nu P_n \left(\sum_{j=1}^{m} q_j B_j\right)\right] \circ 1 \\
&- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{m} q_j B_j^{\nu}\right) \\
&\le \ln\left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} P_n Q_m .
\end{aligned}
\tag{3.5}
$$

In particular,

$$
\begin{aligned}
0 &\le \frac{Q_m \left(\sum_{i=1}^{n} p_i A_i\right) + P_n \left(\sum_{j=1}^{m} q_j B_j\right)}{2} \circ 1 \\
&- \left(\sum_{i=1}^{n} p_i A_i^{1/2}\right) \circ \left(\sum_{j=1}^{m} q_j B_j^{1/2}\right) \\
&\le \ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} P_n Q_m .
\end{aligned}
\tag{3.6}
$$

We observe that if $m=n$, $q_i = p_i$ and $B_i = A_i$ then we get from (3.5) that

$$
\begin{aligned}
0 &\le P_n \left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) \\
&\le \ln\left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} P_n^2 ,
\end{aligned}
\tag{3.7}
$$

where $0 < m \le A_i \le M$, $p_i \ge 0$ for $i \in \{1, ..., n\}$ and $\nu \in [0,1]$.

In particular,

$$0 \le P_n \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \le \ln \left(\frac{M}{m} \right)^{\frac{1}{4}(M-m)} P_n^2. \tag{3.8}$$

We also have the following inequalities:

Proposition 2. *Assume that the operators A_i, B_j satisfy the conditions $0 < m_1 \le A_i \le M_1$, $0 < m_2 \le B_j \le M_2$, p_i, $q_j \ge 0$ for $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, and put $P_n := \sum_{i=1}^{n} p_i$, $Q_m := \sum_{j=1}^{m} q_j$, then for $\nu \in [0,1]$,*

$$0 \le (1-\nu) Q_m \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j \right) - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{j=1}^{m} q_j B_j^{\nu} \right) \le \Gamma_\nu (m_1, M_1, m_2, M_2) Q_m \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1. \tag{3.9}$$

In particular,

$$0 \le \frac{1}{2} \left[Q_m \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^{m} q_j B_j \right) \right] - \left(\sum_{i=1}^{n} p_i A_i^{1/2} \right) \otimes \left(\sum_{j=1}^{m} q_j B_j^{1/2} \right) \le \Gamma_{1/2} (m_1, M_1, m_2, M_2) Q_m \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1. \tag{3.10}$$

Corollary 7. *With the assumptions of Proposition 2, we have the following inequalities for the Hadamard product*

$$0 \le \left[(1-\nu) Q_m \left(\sum_{i=1}^{n} p_i A_i \right) + \nu P_n \left(\sum_{j=1}^{m} q_j B_j \right) \right] \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{j=1}^{m} q_j B_j^{\nu} \right) \le \Gamma_\nu (m_1, M_1, m_2, M_2) Q_m \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1. \tag{3.11}$$

In particular,

$$0 \le \frac{Q_m\left(\sum_{i=1}^n p_i A_i\right) + P_n\left(\sum_{j=1}^m q_j B_j\right)}{2} \circ 1 - \left(\sum_{i=1}^n p_i A_i^{1/2}\right) \circ \left(\sum_{j=1}^m q_j B_j^{1/2}\right) \le \Gamma_{1/2}\left(m_1, M_1, m_2, M_2\right) Q_m\left(\sum_{i=1}^n p_i A_i\right) \otimes 1. \tag{3.12}$$

We observe that if $m = n$, $q_i = p_i$ and $B_i = A_i$, then we get from (3.5) that

$$0 \le P_n\left(\sum_{i=1}^n p_i A_i\right) \circ 1 - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^n p_i A_i^{\nu}\right) \le \ln\left(\frac{M}{m}\right)^{\nu(1-\nu)(M-m)} P_n\left(\sum_{i=1}^n p_i A_i\right) \circ 1, \tag{3.13}$$

where $0 < m \le A_i \le M$, $p_i \ge 0$ for $i \in \{1, ..., n\}$ and $\nu \in [0, 1]$.

In particular,

$$0 \le P_n\left(\sum_{i=1}^n p_i A_i\right) \circ 1 - \left(\sum_{i=1}^n p_i A_i^{1/2}\right) \circ \left(\sum_{i=1}^n p_i A_i^{1/2}\right) \le \ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} P_n\left(\sum_{i=1}^n p_i A_i\right) \circ 1. \tag{3.14}$$

Proposition 3. *With the assumptions of Proposition 2, we have the multiplicative reverse of Young's inequality*

$$\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^m q_j B_j^{\nu}\right) \le (1-\nu) Q_m\left(\sum_{i=1}^n p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^m q_j B_j\right) \le \exp\left[\Delta_\nu\left(m_1, M_1, m_2, M_2\right)\right]\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^m q_j B_j^{\nu}\right), \tag{3.15}$$

for all $\nu \in [0, 1]$.

In particular,

$$\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\otimes\left(\sum_{j=1}^{m}q_jB_j^{1/2}\right) \tag{3.16}$$
$$\leq\frac{1}{2}\left[Q_m\left(\sum_{i=1}^{n}p_iA_i\right)\otimes 1+P_n1\otimes\left(\sum_{j=1}^{m}q_jB_j\right)\right]$$
$$\leq\exp\left[\Delta_{1/2}\left(m_1,M_1,m_2,M_2\right)\right]\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\otimes\left(\sum_{j=1}^{m}q_jB_j^{1/2}\right).$$

Corollary 8. *With the assumptions of Proposition 2, we have the following inequalities for the Hadamard product*

$$\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^{m}q_jB_j^{\nu}\right) \tag{3.17}$$
$$\leq\left[\left(1-\nu\right)Q_m\left(\sum_{i=1}^{n}p_iA_i\right)+\nu P_n\left(\sum_{j=1}^{m}q_jB_j\right)\right]\circ 1$$
$$\leq\exp\left[\Delta_{\nu}\left(m_1,M_1,m_2,M_2\right)\right]\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^{m}q_jB_j^{\nu}\right),$$

for all $\nu\in[0,1]$.

Proposition 4. *In particular,*

$$\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\circ\left(\sum_{j=1}^{m}q_jB_j^{1/2}\right) \tag{3.18}$$
$$\leq\frac{Q_m\left(\sum_{i=1}^{n}p_iA_i\right)+P_n\left(\sum_{j=1}^{m}q_jB_j\right)}{2}\circ 1$$
$$\leq\exp\left[\Delta_{1/2}\left(m_1,M_1,m_2,M_2\right)\right]\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\circ\left(\sum_{j=1}^{m}q_jB_j^{1/2}\right).$$

We observe that if $m=n$, $q_i=p_i$ and $B_i=A_i$, then we get from (3.17) that

$$\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right) \tag{3.19}$$
$$\leq P_n\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1$$
$$\leq\exp\left[\nu\left(1-\nu\right)\frac{\left(M-m\right)^2}{mM}\right]\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right)$$

for all $\nu\in[0,1]$, where $0<m\leq A_i\leq M$, $p_i\geq 0$ for $i\in\{1,...,n\}$.

In particular,

$$\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right) \leq P_n\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1 \leq \exp\left[\frac{(M-m)^2}{4mM}\right]\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{1/2}\right), \tag{3.20}$$

where $0<m\leq A_i\leq M$, $p_i\geq 0$ for $i\in\{1,...,n\}$.

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