REFINEMENTS AND REVERSES OF TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES RELATED TO YOUNG'S RESULT

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0 < m \leq A$, $B \leq M$, for some constants m, M, then

$$\begin{aligned} 0 &\leq \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A \otimes B\right) \\ &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\leq \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A \otimes B\right) \end{aligned}$$

for all $\nu \in [0,1]$. We also have the inequalities for Hadamard product

$$\begin{aligned} 0 &\leq \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2 + B^2}{2} \circ 1 - A \circ B\right) \\ &\leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \\ &\leq \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2 + B^2}{2} \circ 1 - A \circ B\right) \end{aligned}$$

for all $\nu \in [0,1]$.

1. Introduction

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [17]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

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$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a,\ b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [18] while the first one is due to Furuichi [8].

Kittaneh and Manasrah [13], [14] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.4}$$

where $a,\ b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^{2}}{4h},\ \ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [20] while the second by Liao et al. [16].

In [20] the authors also showed that $K^{r}(h)\geq S(h^{r})$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.3).

In the recent paper [5] we obtained the following reverses of Young's inequality as well:

$$0\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

In [6] we obtained the following Young related inequalities:

Theorem 1. *For any $a,b>0$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\}&\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\}\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^{2}}{\max^{2}\{a,b\}}\right]&\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\\&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^{2}}{\min^{2}\{a,b\}}\right].\end{aligned} \tag{1.10}$$

For an equivalent form and a different approach in proving the results (1.9) and (1.10) see [1].

The second inequalities in (1.9) and (1.10) are better than the corresponding results obtained by Furuichi and Minculete in [11] where instead of constant $\frac{1}{2}$ they had the constant 1.Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [3], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_1) dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.11}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [3] extends the definition of Korányi [15] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [10, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.12}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} tdE (t) \text{ and } B = \int_{[0,\infty)} sdF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) dE (t) \otimes dF (s) \tag{1.13}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators A, $B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [19] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B)\leq\frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right] \tag{1.14}$$
$$\leq\frac{1}{2}\left(A\otimes B+B\otimes A\right)$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)\,e_j,e_j\rangle=\langle Ae_j,e_j\rangle\,\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [9], we have the representation

$$A\circ B=\mathcal{U}^{*}\left(A\otimes B\right)\mathcal{U} \tag{1.15}$$

where $\mathcal{U}:H\rightarrow H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [10, p. 173]

$$f\left(A\circ B\right)\geq(\leq)\,f\left(A\right)\circ f\left(B\right)\ \text{ for all } A,\ B\geq 0. \tag{1.16}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1\text{ for }A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1\text{ for }A>0. \tag{1.17}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [2] showed that

$$A\circ B\leq\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}\ \text{ for }A,\ B\geq 0$$

and Aujla and Vasudeva [4] gave an alternative upper bound

$$A\circ B\leq\left(A^{2}\circ B^{2}\right)^{1/2}\ \text{ for }A,\ B\geq 0.$$

It has been shown in [12] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by these results, in this paper we provide among others some upper and lower bounds for the Young differences

$$(1-\nu)\,A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}$$

and

$$\left[(1-\nu)\,A+\nu B\right]\circ 1-A^{1-\nu}\circ B^{\nu}$$

for $\nu\in[0,1]$ and $A,\ B>0$.

2. Main Results

The first main result is as follows:

Theorem 2. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \le A,\ B \le M$, then*

$$\begin{aligned} 0 &\le \frac{1}{2} m\nu (1-\nu) \left[\left(\ln^2 A\right) \otimes 1 + 1 \otimes \left(\ln^2 B\right) - 2 \ln A \otimes \ln B \right] \\ &\le (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\le \frac{1}{2} M\nu (1-\nu) \left[\left(\ln^2 A\right) \otimes 1 + 1 \otimes \left(\ln^2 B\right) - 2 \ln A \otimes \ln B \right] \\ &\le \frac{1}{2} \nu (1-\nu) M (\ln M - \ln m)^2 \end{aligned} \tag{2.1}$$

for all $\nu \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\le \frac{1}{8} m \left[\left(\ln^2 A\right) \otimes 1 + 1 \otimes \left(\ln^2 B\right) - 2 \ln A \otimes \ln B \right] \\ &\le \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \\ &\le \frac{1}{8} M \left[\left(\ln^2 A\right) \otimes 1 + 1 \otimes \left(\ln^2 B\right) - 2 \ln A \otimes \ln B \right] \\ &\le \frac{1}{8} M (\ln M - \ln m)^2 . \end{aligned} \tag{2.2}$$

Proof. If $t,\ s \in [m, M] \subset (0, \infty)$, then by (1.9) we get

$$\begin{aligned} 0 &\le \frac{1}{2} m\nu (1-\nu) (\ln t - \ln s)^2 \le (1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \\ &\le \frac{1}{2} M\nu (1-\nu) (\ln t - \ln s)^2 \\ &\le \frac{1}{2} M\nu (1-\nu) (\ln M - \ln m)^2 . \end{aligned} \tag{2.3}$$

If

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B, then by taking in (2.3) the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, we get

$$\begin{aligned} 0 &\le \frac{1}{2} m\nu (1-\nu) \int_m^M \int_m^M (\ln t - \ln s)^2 dE(t) \otimes dF(s) \\ &\le \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) \\ &\le \frac{1}{2} M\nu (1-\nu) \int_m^M \int_m^M (\ln t - \ln s)^2 dE(t) \otimes dF(s) \\ &\le \frac{1}{8} M (\ln M - \ln m)^2 \int_m^M \int_m^M dE(t) \otimes dF(s) . \end{aligned} \tag{2.4}$$

Now, observe that, by (1.11)

$$\begin{aligned}&\int_m^M\int_m^M(\ln t-\ln s)^2\,dE(t)\otimes dF(s)\\&=\int_m^M\int_m^M\left(\ln^2 t-2\ln t\ln s+\ln^2 s\right)dE(t)\otimes dF(s)\\&=\int_m^M\int_m^M\ln^2 t dE(t)\otimes dF(s)+\int_m^M\int_m^M\ln^2 s dE(t)\otimes dF(s)\\&-2\int_m^M\int_m^M\ln t\ln s dE(t)\otimes dF(s)\\&=\left(\ln^2 A\right)\otimes 1+1\otimes\left(\ln^2 B\right)-2\ln A\otimes\ln B,\end{aligned}$$

$$\begin{aligned}&\int_m^M\int_m^M\left[(1-\nu)t+\nu s-t^{1-\nu}s^\nu\right]dE(t)\otimes dF(s)\\&=(1-\nu)\int_m^M\int_m^M tdE(t)\otimes dF(s)+\nu\int_m^M\int_m^M sdE(t)\otimes dF(s)\\&-\int_m^M\int_m^M t^{1-\nu}s^\nu dE(t)\otimes dF(s)\\&=(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^\nu\end{aligned}$$

and

$$\int_m^M\int_m^M dE(t)\otimes dF(s)=1\otimes 1=1.$$

By employing (2.4) we then get the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 2,*

$$\begin{aligned}0&\le\frac{1}{2}m\nu(1-\nu)\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\&\le[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^\nu\\&\le\frac{1}{2}M\nu(1-\nu)\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\&\le\frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2\end{aligned}\tag{2.5}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0&\le\frac{1}{8}m\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\&\le\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\\&\le\frac{1}{8}M\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\&\le\frac{1}{8}M(\ln M-\ln m)^2.\end{aligned}\tag{2.6}$$

Remark 1. *If we take $B = A$ in Corollary 1, then we get*

$$
\begin{aligned}
0 &\leq m\nu(1-\nu)\left[\left(\ln^2 A\right)\circ 1 - \ln A \circ \ln A\right] \leq A\circ 1 - A^{1-\nu}\circ A^{\nu} \\
&\leq M\nu(1-\nu)\left[\left(\ln^2 A\right)\circ 1 - \ln A \circ \ln A\right] \\
&\leq \frac{1}{2}\nu(1-\nu) M(\ln M - \ln m)^2
\end{aligned}
\tag{2.7}
$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\leq \frac{1}{4}m\left[\left(\ln^2 A\right)\circ 1 - \ln A\circ\ln A\right] \leq A\circ 1 - A^{1/2}\circ A^{1/2} \\
&\leq \frac{1}{4}M\left[\left(\ln^2 A\right)\circ 1 - \ln A\circ\ln A\right] \leq \frac{1}{8}M(\ln M - \ln m)^2 .
\end{aligned}
\tag{2.8}
$$

Theorem 3. *With the assumptions of Theorem 2 we have*

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left(A^2\otimes 1 + 1\otimes B^2 - 2A\otimes B\right) \\
&\leq (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left(A^2\otimes 1 + 1\otimes B^2 - 2A\otimes B\right) \leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2
\end{aligned}
\tag{2.9}
$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\leq \frac{m}{8M^2}\left(A^2\otimes 1 + 1\otimes B^2 - 2A\otimes B\right) \\
&\leq \frac{A\otimes 1 + 1\otimes B}{2} - A^{1/2}\otimes B^{1/2} \\
&\leq \frac{M}{8m^2}\left(A^2\otimes 1 + 1\otimes B^2 - 2A\otimes B\right) \leq \frac{M}{8m^2}(M-m)^2 .
\end{aligned}
\tag{2.10}
$$

Proof. We observe that

$$
0 < \frac{1}{\max\{a,b\}} \leq \frac{\ln a - \ln b}{a-b} \leq \frac{1}{\min\{a,b\}},
$$

which implies that

$$
0 < \frac{1}{\max^2\{a,b\}} \leq \left(\frac{\ln a - \ln b}{a-b}\right)^2 \leq \frac{1}{\min^2\{a,b\}}
$$

for all $a, b > 0$.

By making use of (1.9) we derive

$$
\begin{aligned}
&\frac{1}{2}\nu(1-\nu)(b-a)^2\frac{\min\{a,b\}}{\max^2\{a,b\}} \\
&\leq \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2\min\{a,b\} \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\
&\leq \frac{1}{2}\nu(1-\nu)(b-a)^2\frac{\max\{a,b\}}{\min^2\{a,b\}}.
\end{aligned}
\tag{2.11}
$$

If $t,\ s \in [m,M] \subset (0,\infty)$, then by (2.11) we get

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)(t-s)^2 \leq (1-\nu)t + \nu s - t^{1-\nu}s^{\nu} \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(t-s)^2 .
\end{aligned}
\tag{2.12}
$$

If

$$A=\int_{m}^{M}tdE\left(t\right)\text{ and }B=\int_{m}^{M}sdF\left(s\right)$$

are the spectral resolutions of A and B, then by taking in (2.12) the double integral $\int_{m}^{M}\int_{m}^{M}$ over $dE\left(t\right)\otimes dF\left(s\right)$, we get

$$\begin{aligned}0&\leq\frac{m}{2M^{2}}\nu\left(1-\nu\right)\int_{m}^{M}\int_{m}^{M}\left(t-s\right)^{2}E\left(t\right)\otimes dF\left(s\right)\\&\leq\int_{m}^{M}\int_{m}^{M}\left[\left(1-\nu\right)t+\nu s-t^{1-\nu}s^{\nu}\right]E\left(t\right)\otimes dF\left(s\right)\\&\leq\frac{M}{2m^{2}}\nu\left(1-\nu\right)\int_{m}^{M}\int_{m}^{M}\left(t-s\right)^{2}E\left(t\right)\otimes dF\left(s\right).\end{aligned}\tag{2.13}$$

Since, by (1.11)

$$\begin{aligned}&\int_{m}^{M}\int_{m}^{M}\left(t-s\right)^{2}E\left(t\right)\otimes dF\left(s\right)\\&=\int_{m}^{M}\int_{m}^{M}\left(t^{2}-2ts+s^{2}\right)E\left(t\right)\otimes dF\left(s\right)\\&=\int_{m}^{M}\int_{m}^{M}t^{2}E\left(t\right)\otimes dF\left(s\right)+\int_{m}^{M}\int_{m}^{M}s^{2}E\left(t\right)\otimes dF\left(s\right)\\&-\int_{m}^{M}\int_{m}^{M}2tsE\left(t\right)\otimes dF\left(s\right)\\&=A^{2}\otimes1+1\otimes B^{2}-2A\otimes B,\end{aligned}$$

then by (2.13) we derive the first part of (2.9).

The last part follows by the fact that

$$\left(t-s\right)^{2}\leq\left(M-m\right)^{2}$$

for all $t,\,s\in\left[m,M\right]$. □

Corollary 2. *With the assumptions of Theorem 2, we have the following inequalities for the Hadamard product*

$$\begin{aligned}0&\leq\frac{m}{M^{2}}\nu\left(1-\nu\right)\left(\frac{A^{2}+B^{2}}{2}\circ1-A\circ B\right)\\&\leq\left[\left(1-\nu\right)A+\nu B\right]\circ1-A^{1-\nu}\circ B^{\nu}\\&\leq\frac{M}{m^{2}}\nu\left(1-\nu\right)\left(\frac{A^{2}+B^{2}}{2}\circ1-A\circ B\right)\leq\frac{M}{2m^{2}}\nu\left(1-\nu\right)\left(M-m\right)^{2}\end{aligned}\tag{2.14}$$

for all $\nu\in\left[0,1\right]$.

In particular,

$$\begin{aligned}0&\leq\frac{m}{4M^{2}}\left(\frac{A^{2}+B^{2}}{2}\circ1-A\circ B\right)\leq\frac{A+B}{2}\circ1-A^{1/2}\circ B^{1/2}\\&\leq\frac{M}{4m^{2}}\left(\frac{A^{2}+B^{2}}{2}\circ1-A\circ B\right)\leq\frac{M}{8m^{2}}\left(M-m\right)^{2}.\end{aligned}\tag{2.15}$$

Remark 2. *If we take $B = A$ in Corollary 2, then we get*

$$0 \le \frac{m}{M^2}\nu(1-\nu)\left(A^2\circ 1 - A\circ A\right) \le A - A^{1-\nu}\circ A^{\nu} \le \frac{M}{m^2}\nu(1-\nu)\left(A^2\circ 1 - A\circ A\right) \le \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 \tag{2.16}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \le \frac{m}{4M^2}\left(A^2\circ 1 - A\circ A\right) \le A\circ 1 - A^{1/2}\circ A^{1/2} \le \frac{M}{4m^2}\left(A^2\circ 1 - A\circ A\right) \le \frac{M}{8m^2}(M-m)^2. \tag{2.17}$$

Further, we also have:

Theorem 4. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \le A,\ B \le M$, then*

$$0 \le (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \le M\nu(1-\nu)\left(\frac{A^{-1}\otimes B + A\otimes B^{-1}}{2} - 1\right) \tag{2.18}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \le \frac{A\otimes 1 + 1\otimes B}{2} - A^{1/2}\otimes B^{1/2} \le \frac{1}{4}M\left(\frac{A^{-1}\otimes B + A\otimes B^{-1}}{2} - 1\right). \tag{2.19}$$

Proof. Recall that if $a,\ b > 0$ and

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } a \ne b, \\ \\ b \text{ if } a = b \end{cases}$$

is the *logarithmic mean* and $G(a,b) := \sqrt{ab}$ is the *geometric mean*, then $L(a,b) \ge G(a,b)$ for all $a,\ b > 0$.

Then from (1.9) we have for $a \ne b$ that

$$\begin{aligned}(1-\nu)a + \nu b - a^{1-\nu}b^{\nu} &\le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \\ &= \frac{1}{2}\nu(1-\nu)(b-a)^2\left(\frac{\ln a - \ln b}{b-a}\right)^2 \max\{a,b\} \\ &\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{ab}\max\{a,b\} \\ &= \frac{1}{2}\nu(1-\nu)\left(\frac{b}{a} + \frac{a}{b} - 2\right)\max\{a,b\},\end{aligned}$$

which implies that

$$(1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\left(\frac{b}{a} + \frac{a}{b} - 2\right)\max\{a,b\} \tag{2.20}$$

for all $a,\ b > 0$.

If $t,\ s\in[m,M]\subset(0,\infty)$, then by (2.20) we get

$$(2.21)\qquad (1-\nu)\,t+\nu s-t^{1-\nu}s^{\nu}\le\frac{1}{2}\nu\,(1-\nu)\left(\frac{s}{t}+\frac{t}{s}-2\right)\max\{t,s\}$$
$$\le\frac{1}{2}M\nu\,(1-\nu)\left(\frac{s}{t}+\frac{t}{s}-2\right).$$

By taking in (2.21) the double integral $\int_m^M\int_m^M$ over $dE\,(t)\otimes dF\,(s)$, we get

$$(2.22)\qquad \int_m^M\int_m^M\left[(1-\nu)\,t+\nu s-t^{1-\nu}s^{\nu}\right]dE\,(t)\otimes dF\,(s)$$
$$\le\frac{1}{2}M\nu\,(1-\nu)\int_m^M\int_m^M\left(\frac{s}{t}+\frac{t}{s}-2\right)dE\,(t)\otimes dF\,(s).$$

Since

$$\int_m^M\int_m^M\left(\frac{s}{t}+\frac{t}{s}-2\right)dE\,(t)\otimes dF\,(s)$$
$$=\int_m^M\int_m^M t^{-1}sE\,(t)\otimes dF\,(s)+\int_m^M\int_m^M ts^{-1}dE\,(t)\otimes dF\,(s)$$
$$-\int_m^M\int_m^M dE\,(t)\otimes dF\,(s)$$
$$=A^{-1}\otimes B+A\otimes B^{-1}-2,$$

hence by (2.22) we derive (2.18). $\square$

Corollary 3. *With the assumptions of Theorem 4, we have the inequalities for the Hadamard product*

$$(2.23)\qquad 0\le[(1-\nu)\,A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}$$
$$\le M\nu\,(1-\nu)\left(\frac{A^{-1}\circ B+A\circ B^{-1}}{2}-1\right)$$

for all $\nu\in[0,1]$.

In particular,

$$(2.24)\qquad 0\le\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\le\frac{1}{4}M\left(\frac{A^{-1}\circ B+A\circ B^{-1}}{2}-1\right).$$

We observe that, if we take $B=A$ in Corollary 3, then we get

$$(2.25)\qquad 0\le A\circ 1-A^{1-\nu}\circ A^{\nu}\le M\nu\,(1-\nu)\left(A^{-1}\circ A-1\right)$$

for all $\nu\in[0,1]$.

In particular,

$$(2.26)\qquad 0\le A\circ 1-A^{1/2}\circ A^{1/2}\le\frac{1}{8}M\left(A^{-1}\circ A-1\right).$$

We also have the following multiplicative results:

Theorem 5. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \leq A,\ B \leq M$, then*

$$
\begin{aligned}
A^{1-\nu} \otimes B^{\nu} &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^{2}\right] A^{1-\nu} \otimes B^{\nu} \\
&\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^{2}\right] A^{1-\nu} \otimes B^{\nu}
\end{aligned} \tag{2.27}
$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
A^{1-\nu} \otimes B^{\nu} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^{2}\right] A^{1/2} \otimes B^{1/2} \\
&\leq \frac{A \otimes 1 + 1 \otimes B}{2} \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^{2}\right] A^{1/2} \otimes B^{1/2}.
\end{aligned} \tag{2.28}
$$

Proof. Since

$$
\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\max\{a,b\} - \min\{a,b\}}{\max\{a,b\}}\right)^2 = \left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2
$$

and

$$
\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\} - \min\{a,b\}}{\min\{a,b\}}\right)^2 = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2,
$$

hence by (1.10) we derive

$$
\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\
&\leq \frac{(1-\nu)a + \nu b}{a^{1-\nu} b^{\nu}} \\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right].
\end{aligned} \tag{2.29}
$$

If $t,\ s \in [m, M] \subset (0, \infty)$, then by (2.29) we get

$$
\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^{2}\right] t^{1-\nu} s^{\nu} \\
&\leq (1-\nu) t + \nu s \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^{2}\right] t^{1-\nu} s^{\nu}.
\end{aligned} \tag{2.30}
$$

Now, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.30), we derive the desired result (2.27). □

Corollary 4. *With the assumptions of Theorem 5, we have the inequalities for Hadamard product*

$$A^{1-\nu} \circ B^{\nu} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] A^{1-\nu} \circ B^{\nu} \tag{2.31}$$
$$\leq (1-\nu) A + \nu B$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] A^{1-\nu} \circ B^{\nu}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \circ B^{1/2} \leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \circ B^{1/2} \tag{2.32}$$
$$\leq \frac{A+B}{2} \circ 1$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \circ B^{1/2}.$$

If we take $B = A$ in Corollary 4, then we get the following inequalities for one operator A satisfying the condition $0 < m \leq A \leq M$,

$$A^{1-\nu} \circ A^{\nu} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] A^{1-\nu} \circ A^{\nu} \tag{2.33}$$
$$\leq A \circ 1$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] A^{1-\nu} \circ A^{\nu}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \circ A^{1/2} \leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \circ A^{1/2} \tag{2.34}$$
$$\leq A \circ 1$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \circ A^{1/2}.$$

3. Inequalities for Sums

We also have the following inequalities for sums of operators:

Proposition 1. *Assume that* $0<m\leq A_i,\, B_j\leq M$ *and* $p_i,\, q_j\geq 0$ *for* $i\in\{1,...,n\}$, $j\in\{1,...,k\}$, *and put* $P_n:=\sum_{i=1}^n p_i$, $Q_k:=\sum_{j=1}^k q_j$, *then*

$$\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left[Q_k\left(\sum_{i=1}^n p_i A_i^2\right)\otimes 1+P_n 1\otimes\left(\sum_{j=1}^k q_j B_j^2\right)\right. \\
&\left.-2\left(\sum_{i=1}^n p_i A_i\right)\otimes\left(\sum_{j=1}^k q_j B_j\right)\right] \\
&\leq (1-\nu) Q_k\left(\sum_{i=1}^n p_i A_i\right)\otimes 1+\nu P_n 1\otimes\left(\sum_{j=1}^k q_j B_j\right) \\
&-\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_j B_j^{\nu}\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left[Q_k\left(\sum_{i=1}^n p_i A_i^2\right)\otimes 1+P_n 1\otimes\left(\sum_{j=1}^k q_j B_j^2\right)\right. \\
&\left.-2\left(\sum_{i=1}^n p_i A_i\right)\otimes\left(\sum_{j=1}^k q_j B_j\right)\right] \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n Q_k
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
0 &\leq (1-\nu) Q_k\left(\sum_{i=1}^n p_i A_i\right)\otimes 1+\nu P_n 1\otimes\left(\sum_{j=1}^k q_j B_j\right) \\
&-\left(\sum_{i=1}^n p_i A_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_j B_j^{\nu}\right) \\
&\leq M\nu(1-\nu) \\
&\times\left[\frac{\left(\sum_{i=1}^n p_i A^{-1}\right)\otimes\left(\sum_{j=1}^k q_j B\right)+\left(\sum_{i=1}^n p_i A\right)\otimes\left(\sum_{j=1}^k q_j B^{-1}\right)}{2}\right. \\
&\left.-P_n Q_k\right].
\end{aligned} \tag{3.2}$$

Proof. From (2.9) we get

$$\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left(A_i^2\otimes 1+1\otimes B_j^2-2A_i\otimes B_j\right) \\
&\leq (1-\nu) A_i\otimes 1+\nu 1\otimes B_j-A_i^{1-\nu}\otimes B_j^{\nu} \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left(A_i^2\otimes 1+1\otimes B_j^2-2A_i\otimes B_j\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2
\end{aligned}$$

for all for $i\in\{1,...,n\}$, $j\in\{1,...,k\}$ and $\nu\in[0,1]$.

If we multiply by $p_i q_j \geq 0$ and sum, then we get

$$0 \leq \frac{m}{2M^2}\nu(1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \tag{3.3}$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right]$$
$$\leq \frac{M}{2m^2}\nu(1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right)$$
$$\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i.$$

Observe that

$$\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i^2 \otimes 1 + \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i 1 \otimes B_j^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i \otimes B_j$$
$$= Q_k \left(\sum_{i=1}^{n} p_i A_i^2\right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j^2\right)$$
$$- 2\left(\sum_{i=1}^{n} p_i A_i\right) \otimes \left(\sum_{j=1}^{k} q_j B_j\right)$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left[(1-\nu) A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right]$$
$$= (1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i \otimes 1 + \nu \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i 1 \otimes B_j$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i^{1-\nu} \otimes B_j^{\nu}$$
$$= (1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j\right)$$
$$- \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right).$$

By (3.3) we then get the desired result (3.1).

The inequality (3.2) follows in a similar way from (2.18). □

Corollary 5. *With the assumptions of Proposition 1, we have the Hadamard product inequalities*

$$
\begin{aligned}
(3.4)\qquad 0 &\leq \frac{m}{2M^2}\nu\left(1-\nu\right)\left[\left(Q_k\left(\sum_{i=1}^{n}p_iA_i^2\right)+P_n\left(\sum_{j=1}^{k}q_jB_j^2\right)\right)\circ 1\right.\\
&\left.-2\left(\sum_{i=1}^{n}p_iA_i\right)\circ\left(\sum_{j=1}^{k}q_jB_j\right)\right]\\
&\leq \left[\left(1-\nu\right)Q_k\left(\sum_{i=1}^{n}p_iA_i\right)+\nu P_n\left(\sum_{j=1}^{k}q_jB_j\right)\right]\circ 1\\
&-\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^{k}q_jB_j^{\nu}\right)\\
&\leq \frac{M}{2m^2}\nu\left(1-\nu\right)\left[\left(Q_k\left(\sum_{i=1}^{n}p_iA_i^2\right)+P_n\left(\sum_{j=1}^{k}q_jB_j^2\right)\right)\circ 1\right.\\
&\left.-2\left(\sum_{i=1}^{n}p_iA_i\right)\circ\left(\sum_{j=1}^{k}q_jB_j\right)\right]\\
&\leq \frac{M}{2m^2}\nu\left(1-\nu\right)\left(M-m\right)^2P_nQ_k
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\qquad 0 &\leq \left[\left(1-\nu\right)Q_k\left(\sum_{i=1}^{n}p_iA_i\right)+\nu P_n\left(\sum_{j=1}^{k}q_jB_j\right)\right]\circ 1\\
&-\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^{k}q_jB_j^{\nu}\right)\\
&\leq M\nu\left(1-\nu\right)\\
&\times\left[\frac{\left(\sum_{i=1}^{n}p_iA^{-1}\right)\circ\left(\sum_{j=1}^{k}q_jB\right)+\left(\sum_{i=1}^{n}p_iA\right)\circ\left(\sum_{j=1}^{k}q_jB^{-1}\right)}{2}\right.\\
&\left.-P_nQ_k\right].
\end{aligned}
$$

If we take $k=n$, $p_i=q_i$ and $B_i=A_i$, then we get the simpler inequalities

$$
\begin{aligned}
(3.6)\qquad 0 &\leq \frac{m}{M^2}\nu\left(1-\nu\right)\\
&\times\left[P_n\left(\sum_{i=1}^{n}p_iA_i^2\right)\circ 1-\left(\sum_{i=1}^{n}p_iA_i\right)\circ\left(\sum_{i=1}^{n}p_iA_i\right)\right]\\
&\leq P_n\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1-\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right)
\end{aligned}
$$

$$\leq \frac{M}{2m^2}\nu(1-\nu)$$
$$\times\left[P_n\left(\sum_{i=1}^n p_iA_i^2\right)\circ 1-\left(\sum_{i=1}^n p_iA_i\right)\circ\left(\sum_{i=1}^n p_iA_i\right)\right]$$
$$\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n^2$$

and

$$\begin{aligned} 0 &\leq P_n\left(\sum_{i=1}^n p_iA_i\right)\circ 1-\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^n p_iA_i^{\nu}\right) \\ &\leq M\nu(1-\nu)\left[\left(\sum_{i=1}^n p_iA^{-1}\right)\circ\left(\sum_{i=1}^n p_iA\right)-P_n^2\right], \end{aligned} \tag{3.7}$$

for all $\nu\in[0,1]$, provided that $0<m\leq A_i\leq M$ and $p_i\geq 0$ for $i\in\{1,...,n\}$.

We also have the multiplicative inequalities:

Proposition 2. *With the assumptions of Proposition 2,*

$$\begin{aligned} &\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_jB_j^{\nu}\right) \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_jB_j^{\nu}\right) \\ &\leq (1-\nu)Q_k\left(\sum_{i=1}^n p_iA_i\right)\otimes 1+\nu P_n 1\otimes\left(\sum_{j=1}^k q_jB_j\right) \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_jB_j^{\nu}\right) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^k q_jB_j^{\nu}\right) \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^k q_jB_j^{\nu}\right) \\ &\leq (1-\nu)Q_k\left(\sum_{i=1}^n p_iA_i\right)\circ 1+\nu P_n 1\circ\left(\sum_{j=1}^k q_jB_j\right) \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\left(\sum_{i=1}^n p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^k q_jB_j^{\nu}\right), \end{aligned} \tag{3.9}$$

for all $\nu\in[0,1]$.

If we take $k=n$, $p_i=q_i$ and $B_i=A_i$ in (3.9), then we get the simpler inequalities

$$\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right) \tag{3.10}$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{j=1}^{k}q_jB_j^{\nu}\right)$$
$$\leq P_n\left(\sum_{i=1}^{n}p_iA_i\right)\circ 1$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\left(\sum_{i=1}^{n}p_iA_i^{1-\nu}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{\nu}\right),$$

for all $\nu\in[0,1]$, provided that $0<m\leq A_i\leq M$ and $p_i\geq 0$ for $i\in\{1,...,n\}$.

References

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, **63** (2015), Issue 3, 622-635.
[2] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.
[3] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.
[4] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.
[5] S. S. Dragomir, A note on Young's inequality, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* **111** (2017), no. 2, 349-354. Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].
[6] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, *Transyl. J. Math. Mec.* **8** (2016), No.1, 45-49. Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. . [https://rgmia.org/papers/v18/v18a131.pdf].
[7] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.*, **5** (2011), 21-31.
[8] S. Furuichi, Refined Young inequalities with Specht's ratio, *Journal of the Egyptian Mathematical Society* **20**(2012) , 46–49.
[9] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535
[10] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[11] S. Furuichi and N. Minculete, Alternative reverse inequalities for Young's inequality, *J. Math Inequal.* **5** (2011), Number 4, 595–600.
[12] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241
[13] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269
[14] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.
[15] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.
[16] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[17] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.
[18] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[19] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[20] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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