TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR SYNCHRONOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if f, g are synchronous and continuous on I and A, B are selfadjoint with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$(f(A)g(A)) \otimes 1 + 1 \otimes (f(B)g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B)$$

and the inequality for Hadamard product

$$(f(A)g(A) + f(B)g(B)) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A).$$

Let either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If A, $B > 0$, then

$$A^{p+q} \otimes 1 + 1 \otimes B^{p+q} \geq A^p \otimes B^q + A^q \otimes B^p,$$

and

$$\left(A^{p+q} + B^{p+q}\right) \circ 1 \geq A^p \circ B^q + A^q \circ B^p.$$

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [4] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

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It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [6, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t \in [0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A \otimes B + B \otimes A) \end{aligned} \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.5}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [6, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [7] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if f, g are synchronous and continuous on I and A, B are selfadjoint with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then

$$(f(A) g(A)) \otimes 1 + 1 \otimes (f(B) g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B)$$

and the inequality for Hadamard product

$$(f(A) g(A) + f(B) g(B)) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A).$$

Let either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If A, $B > 0$, then

$$A^{p+q} \otimes 1 + 1 \otimes B^{p+q} \geq A^p \otimes B^q + A^q \otimes B^p,$$

and

$$\left(A^{p+q} + B^{p+q}\right) \circ 1 \geq A^p \circ B^q + A^q \circ B^p.$$

2. Main Results

We start to the following main result:

Theorem 1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned} &[h(A) f(A) g(A)] \otimes k(B) + h(A) \otimes [k(B) f(B) g(B)] \\ &\geq [h(A) f(A)] \otimes [k(B) g(B)] + [h(A) g(A)] \otimes [k(B) f(B)] \end{aligned} \tag{2.1}$$

or, equivalently

$$\begin{aligned} &(h(A) \otimes k(B)) [(f(A) g(A)) \otimes 1 + 1 \otimes (f(B) g(B))] \\ &\geq (h(A) \otimes k(B)) [f(A) \otimes g(B) + g(A) \otimes f(B)]. \end{aligned} \tag{2.2}$$

If f, g are asynchronous on I, then the inequality reverses in (2.1) and (2.2).

Proof. Assume that f and g are synchronous on I, then

$$f(t) g(t) + f(s) g(s) \geq f(t) g(s) + f(s) g(t)$$

for all $t, s \in I$.

We multiply this inequality by $h(t) k(s) \geq 0$ to get

$$\begin{aligned} &f(t) g(t) h(t) k(s) + h(t) f(s) g(s) k(s) \\ &\geq f(t) h(t) g(s) k(s) + f(s) k(s) g(t) h(t) \end{aligned}$$

for all $t, s \in I$.

If we take the double integral, then we get

$$\int_I \int_I [f(t) g(t) h(t) k(s) + h(t) f(s) g(s) k(s)] \, dE(t) \otimes dF(s) \geq \int_I \int_I [f(t) h(t) g(s) k(s) + f(s) k(s) g(t) h(t)] \, dE(t) \otimes dF(s). \tag{2.3}$$

Observe that

$$\begin{aligned} &\int_I \int_I [f(t) g(t) h(t) k(s) + h(t) f(s) g(s) k(s)] \, dE(t) \otimes dF(s) \\ &= \int_I \int_I f(t) g(t) h(t) k(s) \, dE(t) \otimes dF(s) \\ &+ \int_I \int_I h(t) f(s) g(s) k(s) \, dE(t) \otimes dF(s) \\ &= [h(A) f(A) g(A)] \otimes k(B) + h(A) \otimes [k(B) f(B) g(B)] \end{aligned}$$

and

$$\begin{aligned} &\int_I \int_I [f(t) h(t) g(s) k(s) + f(s) k(s) g(t) h(t)] \, dE(t) \otimes dF(s) \\ &= \int_I \int_I f(t) h(t) g(s) k(s) \, dE(t) \otimes dF(s) \\ &+ \int_I \int_I g(t) h(t) f(s) k(s) \, dE(t) \otimes dF(s) \\ &= [h(A) f(A)] \otimes [k(B) g(B)] + [h(A) g(A)] \otimes [k(B) f(B)]. \end{aligned}$$

By utilizing (2.3) we derive (2.2).

Now, by making use of the tensorial property

$$(XU) \otimes (YV) = (X \otimes Y)(U \otimes V),$$

for any X, U, Y, $V \in B(H)$, we obtain

$$\begin{aligned} &[h(A) f(A) g(A)] \otimes k(B) + h(A) \otimes [k(B) f(B) g(B)] \\ &= (h(A) \otimes k(B)) [(f(A) g(A)) \otimes 1] + (h(A) \otimes k(B)) [1 \otimes (f(B) g(B))] \\ &= (h(A) \otimes k(B)) [(f(A) g(A)) \otimes 1 + 1 \otimes (f(B) g(B))] \end{aligned}$$

and

$$\begin{aligned} &[h(A) f(A)] \otimes [k(B) g(B)] + [h(A) g(A)] \otimes [k(B) f(B)] \\ &= (h(A) \otimes k(B)) (f(A) \otimes g(B)) + (h(A) \otimes k(B)) (g(A) \otimes f(B)) \\ &= (h(A) \otimes k(B)) [f(A) \otimes g(B) + g(A) \otimes f(B)], \end{aligned}$$

which proves (2.2). □

Remark 1. *With the assumptions of Theorem 1 and if we take $k = h$, then we get*

$$[h(A) f(A) g(A)] \otimes h(B) + h(A) \otimes [h(B) f(B) g(B)] \geq [h(A) f(A)] \otimes [h(B) g(B)] + [h(A) g(A)] \otimes [h(B) f(B)], \tag{2.4}$$

where f, g are synchronous and continuous on I and h is nonnegative and continuous on the same interval.

Moreover, if we take $h \equiv 1$ in (2.4), then we get

$$(f(A) g(A)) \otimes 1 + 1 \otimes (f(B) g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B), \tag{2.5}$$

where f, g are synchronous and continuous on I

Corollary 1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$
\begin{aligned}
&k(B)\circ[h(A)f(A)g(A)]+h(A)\circ[k(B)f(B)g(B)] \\
&\geq[h(A)f(A)]\circ[k(B)g(B)]+[k(B)f(B)]\circ[h(A)g(A)].
\end{aligned} \tag{2.6}
$$

If f, g are asynchronous on I, then the inequality reverses in (2.6).
In particular, we have

$$
\begin{aligned}
&h(B)\circ[h(A)f(A)g(A)]+h(A)\circ[h(B)f(B)g(B)] \\
&\geq[h(A)f(A)]\circ[h(B)g(B)]+[h(B)f(B)]\circ[h(A)g(A)]
\end{aligned} \tag{2.7}
$$

and

$$
(f(A)g(A)+(f(B)g(B)))\circ 1\geq f(A)\circ g(B)+f(B)\circ g(A). \tag{2.8}
$$

Proof. If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.1), we get

$$
\begin{aligned}
&\mathcal{U}^*([h(A)f(A)g(A)]\otimes k(B))\mathcal{U} \\
&+\mathcal{U}^*(h(A)\otimes[k(B)f(B)g(B)])\mathcal{U} \\
&\geq\mathcal{U}^*([h(A)f(A)]\otimes[k(B)g(B)])\mathcal{U} \\
&+\mathcal{U}^*([h(A)g(A)]\otimes[k(B)f(B)])\mathcal{U}
\end{aligned}
$$

which is equivalent to (2.6). □

Corollary 2. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A_j, B_j are selfadjoint with spectra* $\operatorname{Sp}(A_j), \operatorname{Sp}(B_j) \subset I$ *and* $p_j, q_j \geq 0$, $j\in\{1,...,n\}$, *then*

$$
\begin{aligned}
&\left(\sum_{j=1}^n p_j h(A_j)f(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_i k(B_i)\right) \\
&+\left(\sum_{j=1}^n p_j h(A_j)\right)\otimes\left(\sum_{i=1}^n q_i k(B_i)f(B_i)g(B_i)\right) \\
&\geq\left(\sum_{j=1}^n p_j h(A_j)f(A_j)\right)\otimes\left(\sum_{i=1}^n q_i k(B_i)g(B_i)\right) \\
&+\left(\sum_{j=1}^n p_j h(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_i k(B_i)f(B_i)\right).
\end{aligned} \tag{2.9}
$$

In particular,

$$\left(\sum_{j=1}^{n}p_jh\left(A_j\right)f\left(A_j\right)g\left(A_j\right)\right)\otimes\left(\sum_{i=1}^{n}q_ih\left(B_i\right)\right)+\left(\sum_{j=1}^{n}p_jh\left(A_j\right)\right)\otimes\left(\sum_{i=1}^{n}q_ih\left(B_i\right)f\left(B_i\right)g\left(B_i\right)\right)\geq\left(\sum_{j=1}^{n}p_jh\left(A_j\right)f\left(A_j\right)\right)\otimes\left(\sum_{i=1}^{n}q_ih\left(B_i\right)g\left(B_i\right)\right)+\left(\sum_{j=1}^{n}p_jh\left(A_j\right)g\left(A_j\right)\right)\otimes\left(\sum_{i=1}^{n}q_ih\left(B_i\right)f\left(B_i\right)\right)\tag{2.10}$$

and, if $\sum_{j=1}^{n}p_j=\sum_{j=1}^{n}q_j=1$, *then*

$$\left(\sum_{j=1}^{n}p_jf\left(A_j\right)g\left(A_j\right)\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}q_if\left(B_i\right)g\left(B_i\right)\right)\geq\left(\sum_{j=1}^{n}p_jf\left(A_j\right)\right)\otimes\left(\sum_{i=1}^{n}q_ig\left(B_i\right)\right)+\left(\sum_{j=1}^{n}p_jg\left(A_j\right)\right)\otimes\left(\sum_{i=1}^{n}q_if\left(B_i\right)\right).\tag{2.11}$$

Proof. We have from (2.1) that

$$\left[h\left(A_j\right)f\left(A_j\right)g\left(A_j\right)\right]\otimes k\left(B_i\right)+h\left(A_j\right)\otimes\left[k\left(B_i\right)f\left(B_i\right)g\left(B_i\right)\right]\geq\left[h\left(A_j\right)f\left(A_j\right)\right]\otimes\left[k\left(B_i\right)g\left(B_i\right)\right]+\left[h\left(A_j\right)g\left(A_j\right)\right]\otimes\left[k\left(B_i\right)f\left(B_i\right)\right]$$

for all $i,j\in\{1,...,n\}$.

If we multiply by $p_jq_i\geq 0$ and sum over j, $i\in\{1,...,n\}$, then we get

$$\sum_{j,i=1}^{n}p_jq_i\left[h\left(A_j\right)f\left(A_j\right)g\left(A_j\right)\right]\otimes k\left(B_i\right)+\sum_{j,i=1}^{n}p_jq_ip_jq_ih\left(A_j\right)\otimes\left[k\left(B_i\right)f\left(B_i\right)g\left(B_i\right)\right]\geq\sum_{j,i=1}^{n}p_jq_i\left[h\left(A_j\right)f\left(A_j\right)\right]\otimes\left[k\left(B_i\right)g\left(B_i\right)\right]+\sum_{j,i=1}^{n}p_jq_i\left[h\left(A_j\right)g\left(A_j\right)\right]\otimes\left[k\left(B_i\right)f\left(B_i\right)\right]$$

and by using the properties of tensorial product we derive (2.9). □

Remark 2. *If we take* $B_i = A_i$ *and* $p_i = q_i$, $i \in \{1, ..., n\}$, *then we get*

$$\left(\sum_{i=1}^{n} p_i f(A_i) g(A_i)\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i f(A_i) g(A_i)\right) \geq \left(\sum_{i=1}^{n} p_i f(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i g(A_i)\right) + \left(\sum_{i=1}^{n} p_i g(A_i)\right) \otimes \left(\sum_{i=1}^{n} p_i f(A_i)\right), \tag{2.12}$$

where f, g *are synchronous and continuous on* I *and* A_i *are selfadjoint with spectra* $\operatorname{Sp}(A_i) \subset I$, $p_i \geq 0$ *for* $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^{n} p_i = 1$.

By (2.12) we also have the inequality for the Hadamard product

$$\left(\sum_{i=1}^{n} p_i f(A_i) g(A_i)\right) \circ 1 \geq \left(\sum_{i=1}^{n} p_i f(A_i)\right) \circ \left(\sum_{i=1}^{n} p_i g(A_i)\right), \tag{2.13}$$

where f, g *are synchronous and continuous on* I *and* A_i *are selfadjoint with spectra* $\operatorname{Sp}(A_i) \subset I$, $p_i \geq 0$ *for* $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^{n} p_i = 1$.

We also have:

Theorem 2. *Let* $f, g : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *be continuous on* $[m, M]$ *and differentiable on* (m, M) *with* $g'(t) \neq 0$ *for* $t \in (m, M)$. *Assume that*

$$-\infty < \gamma = \inf_{t \in (m,M)} \frac{f'(t)}{g'(t)}, \quad \sup_{t \in (m,M)} \frac{f'(t)}{g'(t)} = \Gamma < \infty,$$

and A, B *selfadjoint operators with spectra* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$, *then for any continuous and nonnegative function* h *defined on* $[m, M]$,

$$\begin{aligned}
&\gamma\left[\left(h(A) g^2(A)\right) \otimes h(B) + h(A) \otimes \left(h(B) g^2(B)\right) - 2\left(g(A) h(A)\right) \otimes \left(h(B) g(B)\right)\right] \\
&\leq [h(A) f(A) g(A)] \otimes h(B) + h(A) \otimes [h(B) f(B) g(B)] \\
&\quad - [h(A) f(A)] \otimes [h(B) g(B)] - [h(A) g(A)] \otimes [h(B) f(B)] \\
&\leq \Gamma\left[\left(h(A) g^2(A)\right) \otimes h(B) + h(A) \otimes \left(h(B) g^2(B)\right) - 2\left(g(A) h(A)\right) \otimes \left(h(B) g(B)\right)\right].
\end{aligned} \tag{2.14}$$

In particular,

$$\begin{aligned}
&\gamma\left[g^2(A) \otimes 1 + 1 \otimes g^2(B) - 2g(A) \otimes g(B)\right] \\
&\leq [f(A) g(A)] \otimes 1 + 1 \otimes [f(B) g(B)] - f(A) \otimes g(B) - g(A) \otimes f(B) \\
&\leq \Gamma\left[g^2(A) \otimes 1 + 1 \otimes g^2(B) - 2g(A) \otimes g(B)\right].
\end{aligned} \tag{2.15}$$

Proof. Using the Cauchy mean value theorem, for all $t, s \in [m, M]$ with $t \neq s$ there exists ξ between t and s such that

$$\frac{f(t) - f(s)}{g(t) - g(s)} = \frac{f'(\xi)}{g'(\xi)} \in [\gamma, \Gamma].$$

Therefore

$$\gamma [g(t) - g(s)]^2 \leq [f(t) - f(s)][g(t) - g(s)] \leq \Gamma [g(t) - g(s)]^2$$

for all $t,\, s\in[m,M]$, which is equivalent to

$$\begin{aligned}&\gamma\left[g^{2}(t)-2g(t)\,g(s)+g^{2}(s)\right]\\&\leq f(t)\,g(t)+f(s)\,g(s)-f(t)\,g(s)-f(s)\,g(t)\\&\leq\Gamma\left[g^{2}(t)-2g(t)\,g(s)+g^{2}(s)\right]\end{aligned}$$

for all $t,\, s\in[m,M]$.

If we multiply by $h(t)\,h(s)\geq 0$, then we get

$$\begin{aligned}&\gamma\left[h(t)\,g^{2}(t)\,h(s)-2g(t)\,h(t)\,h(s)\,g(s)+h(t)\,h(s)\,g^{2}(s)\right]\\&\leq h(t)\,f(t)\,g(t)\,h(s)+h(t)\,h(s)\,f(s)\,g(s)\\&-h(t)\,f(t)\,h(s)\,g(s)-h(t)\,g(t)\,h(s)\,f(s)\\&\leq\Gamma\left[h(t)\,g^{2}(t)\,h(s)-2g(t)\,h(t)\,h(s)\,g(s)+h(t)\,h(s)\,g^{2}(s)\right]\end{aligned}$$

for all $t,\, s\in[m,M]$.

This implies that

$$\begin{aligned}&\gamma\int_{m}^{M}\int_{m}^{M}\left[h(t)\,g^{2}(t)\,h(s)-2g(t)\,h(t)\,h(s)\,g(s)+h(t)\,h(s)\,g^{2}(s)\right]\\&\times dE(t)\otimes dF(s)\\&\leq\int_{m}^{M}\int_{m}^{M}[h(t)\,f(t)\,g(t)\,h(s)+h(t)\,h(s)\,f(s)\,g(s)\\&-h(t)\,f(t)\,h(s)\,g(s)-h(t)\,g(t)\,h(s)\,f(s)]\,dE(t)\otimes dF(s)\\&\leq\Gamma\int_{m}^{M}\int_{m}^{M}\left[h(t)\,g^{2}(t)\,h(s)-2g(t)\,h(t)\,h(s)\,g(s)+h(t)\,h(s)\,g^{2}(s)\right]\\&\times dE(t)\otimes dF(s)\end{aligned}$$

and by performing the calculations as in the proof of Theorem 1, we derive (2.14). □

Corollary 3. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}&\gamma\left[h(B)\circ\left(h(A)\,g^{2}(A)\right)+h(A)\circ\left(h(B)\,g^{2}(B)\right)\right.\\&\left.-2\,(g(A)\,h(A))\circ(h(B)\,g(B))\right]\\&\leq h(B)\circ[h(A)\,f(A)\,g(A)]+h(A)\circ[h(B)\,f(B)\,g(B)]\\&-[h(A)\,f(A)]\circ[h(B)\,g(B)]-[h(A)\,g(A)]\circ[h(B)\,f(B)]\\&\leq\Gamma\left[h(B)\circ\left(h(A)\,g^{2}(A)\right)+h(A)\circ\left(h(B)\,g^{2}(B)\right)\right.\\&\left.-2\,(g(A)\,h(A))\circ(h(B)\,g(B))\right].\end{aligned}\tag{2.16}$$

In particular,

$$\begin{aligned}&\gamma\left[\left[g^{2}(A)+g^{2}(B)\right]\circ 1-2g(A)\circ g(B)\right]\\&\leq[f(A)\,g(A)+[f(B)\,g(B)]]\circ 1-f(A)\circ g(B)-g(A)\circ f(B)\\&\leq\Gamma\left[\left[g^{2}(A)+g^{2}(B)\right]\circ 1-2g(A)\circ g(B)\right].\end{aligned}\tag{2.17}$$

We also have:

Corollary 4. *With the assumptions of Theorem 2 and if A_j are selfadjoint with spectra* $\operatorname{Sp}(A_j) \subset I$ *and* $p_j \geq 0,\ j \in \{1, ..., n\}$, *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$\begin{aligned}
(2.18)\qquad & \gamma \left\{ \left(\sum_{i=1}^{n} p_i g^2(A_i) \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i g^2(A_i) \right) \right. \\
& \left. -2 \left(\sum_{i=1}^{n} p_i g(A_i) \right) \otimes \left(\sum_{i=1}^{n} p_i g(A_i) \right) \right\} \\
& \leq \left(\sum_{i=1}^{n} p_i f(A_i) g(A_i) \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i f(A_i) g(A_i) \right) \\
& - \left(\sum_{i=1}^{n} p_i f(A_i) \right) \otimes \left(\sum_{i=1}^{n} p_i g(A_i) \right) \\
& - \left(\sum_{i=1}^{n} p_i g(A_i) \right) \otimes \left(\sum_{i=1}^{n} p_i f(A_i) \right) \\
& \leq \Gamma \left\{ \left(\sum_{i=1}^{n} p_i g^2(A_i) \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i g^2(A_i) \right) \right. \\
& \left. -2 \left(\sum_{i=1}^{n} p_i g(A_i) \right) \otimes \left(\sum_{i=1}^{n} p_i g(A_i) \right) \right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
(2.19)\qquad & \gamma \left[\left(\sum_{i=1}^{n} p_i g^2(A_i) \right) \circ 1 - \left(\sum_{i=1}^{n} p_i g(A_i) \right) \circ \left(\sum_{i=1}^{n} p_i g(A_i) \right) \right] \\
& \leq \left(\sum_{i=1}^{n} p_i f(A_i) g(A_i) \right) \circ 1 - \left(\sum_{i=1}^{n} p_i f(A_i) \right) \circ \left(\sum_{i=1}^{n} p_i g(A_i) \right) \\
& \leq \Gamma \left[\left(\sum_{i=1}^{n} p_i g^2(A_i) \right) \circ 1 - \left(\sum_{i=1}^{n} p_i g(A_i) \right) \circ \left(\sum_{i=1}^{n} p_i g(A_i) \right) \right].
\end{aligned}$$

Proof. From (2.15) we get

$$\begin{aligned}
& \gamma \left[g^2(A_i) \otimes 1 + 1 \otimes g^2(A_j) - 2g(A_i) \otimes g(A_j) \right] \\
& \leq [f(A_i) g(A_i)] \otimes 1 + 1 \otimes [f(A_j) g(A_j)] \\
& - f(A_i) \otimes g(A_j) - g(A_i) \otimes f(A_j) \\
& \leq \Gamma \left[g^2(A_i) \otimes 1 + 1 \otimes g^2(A_j) - 2g(A_i) \otimes g(A_j) \right]
\end{aligned}$$

for all $i, j \in \{1, ..., n\}$.

If we multiply by $p_i p_j \geq 0$ and sum, then we get

$$\begin{aligned}
&\gamma \sum_{i,j=1}^{n} p_i p_j \left[g^2(A_i) \otimes 1 + 1 \otimes g^2(A_j) - 2g(A_i) \otimes g(A_j)\right] \\
&\leq \sum_{i,j=1}^{n} p_i p_j \{[f(A_i) g(A_i)] \otimes 1 + 1 \otimes [f(A_j) g(A_j)] \\
&- f(A_i) \otimes g(A_j) - g(A_i) \otimes f(A_j)\} \\
&\leq \Gamma \sum_{i,j=1}^{n} p_i p_j \left[g^2(A_i) \otimes 1 + 1 \otimes g^2(A_j) - 2g(A_i) \otimes g(A_j)\right],
\end{aligned}$$

which gives (2.18). □

3. Some Examples

Let either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$ and $r \in \mathbb{R}$. If $A,\ B > 0$, then from (2.4) we get

$$A^{r+p+q} \otimes B^r + A^r \otimes B^{r+p+q} \geq A^{r+p} \otimes B^{r+q} + A^{r+q} \otimes B^{r+p}, \tag{3.1}$$

while from (2.6) we obtain

$$A^{r+p+q} \circ B^r + A^r \circ B^{r+p+q} \geq A^{r+p} \circ B^{r+q} + A^{r+q} \circ B^{r+p}. \tag{3.2}$$

If one of the parameters p, q is in $(-\infty, 0)$ while the other in $(0, \infty)$, then the inequality reverses in (3.1) and (3.2).

If we take $q = p$, then we get

$$A^{r+2p} \otimes B^r + A^r \otimes B^{r+2p} \geq 2A^{r+p} \otimes B^{r+p}, \tag{3.3}$$

and

$$A^{r+2p} \circ B^r + A^r \circ B^{r+2p} \geq 2A^{r+p} \circ B^{r+p} \tag{3.4}$$

for $p, r \in \mathbb{R}$ and $A,\ B > 0$.

If we take $q = -p$, then we get

$$2A^r \otimes B^r \geq A^{r+p} \otimes B^{r-p} + A^{r-p} \otimes B^{r+p}, \tag{3.5}$$

while from (2.6) we obtain

$$2A^r \circ B^r \geq A^{r+p} \circ B^{r-p} + A^{r-p} \circ B^{r+p}, \tag{3.6}$$

for $p, r \in \mathbb{R}$ and $A,\ B > 0$.

Assume that $A_j > 0$, $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then by (2.12) we get

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \\
&\geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q}\right) + \left(\sum_{i=1}^{n} p_i A_i^{q}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p}\right),
\end{aligned} \tag{3.7}$$

if either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If one of the parameters p, q is in $(-\infty, 0)$ while the other in $(0, \infty)$, then the inequality reverses in (3.7).

In particular, we derive

$$\left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \tag{3.8}$$

and

$$2 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{-p}\right) + \left(\sum_{i=1}^{n} p_i A_i^{-p}\right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p}\right). \tag{3.9}$$

From (2.13) we obtain

$$\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) \circ 1 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q}\right), \tag{3.10}$$

if either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If one of the parameters p, q is in $(-\infty, 0)$ while the other in $(0, \infty)$, then the inequality reverses in (3.10).

In particular, we have

$$\left(\sum_{i=1}^{n} p_i A_i^{2p}\right) \circ 1 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \tag{3.11}$$

and

$$1 \geq \left(\sum_{i=1}^{n} p_i A_i^{p}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{-p}\right), \tag{3.12}$$

for $p \in \mathbb{R}$, $A_j > 0$, $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

Consider the functions $f(t) = t^p$, $g(t) = t^q$ defined on $(0, \infty)$. Then $f'(t) = pt^{p-1}$, $g'(t) = qt^{q-1}$ for $t > 0$ and

$$\frac{f'(t)}{g'(t)} = \frac{p}{q} t^{p-q}, \ t > 0.$$

Assume that either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. Then $\frac{p}{q} > 0$ and $\frac{f'(t)}{g'(t)}$ is increasing for $p > q$ and decreasing for $p < q$ and constant 1 for $p = q$.

Assume that $0 < m \leq A, B \leq M$, then

$$\inf_{t\in[m,M]} \frac{f'(t)}{g'(t)} = \frac{p}{q} m^{p-q} \text{ and } \sup_{t\in[m,M]} \frac{f'(t)}{g'(t)} = \frac{p}{q} M^{p-q} \text{ for } p > q$$

and

$$\inf_{t\in[m,M]} \frac{f'(t)}{g'(t)} = \frac{p}{q} M^{p-q} \text{ and } \sup_{t\in[m,M]} \frac{f'(t)}{g'(t)} = \frac{p}{q} m^{p-q} \text{ for } p < q.$$

Assume that either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$ and $0 < m \leq A, B \leq M$. From (2.15) we get for $p > q$ that

$$\begin{aligned} 0 &\leq \frac{p}{q} m^{p-q} \left(A^{2q} \otimes 1 + 1 \otimes B^{2q} - 2A^q \otimes B^q\right) \\ &\leq A^{p+q} \otimes 1 + 1 \otimes B^{p+q} - A^p \otimes B^q - A^q \otimes B^p \\ &\leq \frac{p}{q} M^{p-q} \left(A^{2q} \otimes 1 + 1 \otimes B^{2q} - 2A^q \otimes B^q\right) \end{aligned} \tag{3.13}$$

and for $p < q$

$$0 \leq \frac{p}{q} M^{p-q} \left(A^{2q} \otimes 1 + 1 \otimes B^{2q} - 2A^q \otimes B^q \right) \tag{3.14}$$
$$\leq A^{p+q} \otimes 1 + 1 \otimes B^{p+q} - A^p \otimes B^q - A^q \otimes B^p$$
$$\leq \frac{p}{q} m^{p-q} \left(A^{2q} \otimes 1 + 1 \otimes B^{2q} - 2A^q \otimes B^q \right).$$

From (2.17) we also have the inequalities for the Hadamard product for $p > q$ that

$$0 \leq \frac{p}{q} m^{p-q} \left(\left(A^{2q} + B^{2q} \right) \circ 1 - 2A^q \circ B^q \right) \tag{3.15}$$
$$\leq \left(A^{p+q} + B^{p+q} \right) \circ 1 - A^p \circ B^q - A^q \circ B^p$$
$$\leq \frac{p}{q} M^{p-q} \left(\left(A^{2q} + B^{2q} \right) \circ 1 - 2A^q \circ B^q \right)$$

and for $p < q$

$$0 \leq \frac{p}{q} M^{p-q} \left(\left(A^{2q} + B^{2q} \right) \circ 1 - 2A^q \circ B^q \right) \tag{3.16}$$
$$\leq \left(A^{p+q} + B^{p+q} \right) \circ 1 - A^p \circ B^q - A^q \circ B^p$$
$$\leq \frac{p}{q} m^{p-q} \left(\left(A^{2q} + B^{2q} \right) \circ 1 - 2A^q \circ B^q \right).$$

Assume that either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$ and $0 < m \leq A_j \leq M$, $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$. By (2.18) we get for $p > q$

$$0 \leq \frac{p}{q} m^{p-q} \left\{ \left(\sum_{i=1}^n p_i A_i^{2q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^n p_i A_i^{2q} \right) \right. \tag{3.17}$$
$$\left. -2 \left(\sum_{i=1}^n p_i A_i^q \right) \otimes \left(\sum_{i=1}^n p_i A_i^q \right) \right\}$$
$$\leq \left(\sum_{i=1}^n p_i A_i^{p+q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^n p_i A_i^{p+q} \right)$$
$$- \left(\sum_{i=1}^n p_i A_i^p \right) \otimes \left(\sum_{i=1}^n p_i A_i^q \right) - \left(\sum_{i=1}^n p_i A_i^q \right) \otimes \left(\sum_{i=1}^n p_i A_i^p \right)$$
$$\leq \frac{p}{q} M^{p-q} \left\{ \left(\sum_{i=1}^n p_i A_i^{2q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^n p_i A_i^{2q} \right) \right.$$
$$\left. -2 \left(\sum_{i=1}^n p_i A_i^q \right) \otimes \left(\sum_{i=1}^n p_i A_i^q \right) \right\}$$

and for $p < q$

$$0 \le \frac{p}{q} M^{p-q} \left\{ \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) -2 \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right\}$$
$$\le \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) - \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q} \right) - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{p} \right)$$
$$\le \frac{p}{q} m^{p-q} \left\{ \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{n} p_i A_i^{2q} \right) -2 \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \otimes \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right\}. \tag{3.18}$$

Also, by (2.19) we get for $p > q$

$$0 \le \frac{p}{q} m^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right]$$
$$\le \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right)$$
$$\le \frac{p}{q} M^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right], \tag{3.19}$$

while for $p < q$

$$0 \le \frac{p}{q} M^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right]$$
$$\le \left(\sum_{i=1}^{n} p_i A_i^{p+q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{p} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right)$$
$$\le \frac{p}{q} m^{p-q} \left[\left(\sum_{i=1}^{n} p_i A_i^{2q} \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{q} \right) \right]. \tag{3.20}$$

Consider the exponential functions $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha, \beta \in \mathbb{R}$. If $\alpha\beta > 0$ then the functions have the same monotonicity. If $\alpha\beta < 0$ they have different monotonicity.

If $\alpha\beta > 0$ and A, B are selfadjoint operators, then by (2.5) we get

$$\exp[(\alpha+\beta) A] \otimes 1 + 1 \otimes \exp[(\alpha+\beta) B]$$
$$\ge \exp(\alpha A) \otimes \exp(\beta B) + \exp(\beta A) \otimes \exp(\alpha B), \tag{3.21}$$

and

$$\exp[(\alpha+\beta)A]\circ 1+1\circ\exp[(\alpha+\beta)B] \geq \exp(\alpha A)\circ\exp(\beta B)+\exp(\beta A)\circ\exp(\alpha B). \tag{3.22}$$

If $\alpha\beta<0$, then the reverse inequality holds in (3.21) and (3.22).

If we take $f(t)=t^p$ and $g(t)=\ln t$, we also have the logarithmic inequalities

$$(A^p\ln A)\otimes 1+1\otimes(B^p\ln B)\geq A^p\otimes\ln B+\ln A\otimes B^p, \tag{3.23}$$

and

$$(A^p\ln A+B^p\ln B)\circ 1\geq A^p\circ\ln B+\ln A\circ B^p, \tag{3.24}$$

for A, $B>0$ and $p>0$. If $p<0$, then the inequality reverses in (3.23) and (3.24).

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