SOME TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then we have the tensorial inequality

$$\begin{aligned}\left(f'(A)\otimes 1\right)(A\otimes 1-1\otimes B) &\geq f(A)\otimes 1-1\otimes f(B)\\ &\geq (A\otimes 1-1\otimes B)\left(1\otimes f'(B)\right)\end{aligned}$$

and the inequality for Hadamard product

$$\begin{aligned}\left(f'(A)A\right)\circ 1-f'(A)\circ B &\geq [f(A)-f(B)]\circ 1\\ &\geq A\circ f'(B)-\left(f'(B)B\right)\circ 1.\end{aligned}$$

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [4] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

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and if f is continuous on $[0,\infty)$, then [6, p. 173]

$$f(A\otimes B)\geq(\leq)\, f(A)\otimes f(B) \text{ for all } A,\ B\geq 0. \tag{1.2}$$

This follows by observing that, if

$$A=\int_{[0,\infty)} t dE(t) \text{ and } B=\int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)} f(st)\, dE(t)\otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t\in[0,1]$ and

$$A\# B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\# B=B\# A \text{ and } (A\# B)\otimes(B\# A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\# B)\otimes(A\# B)&\leq\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\leq\frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.4}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A\circ B=\mathcal{U}^*(A\otimes B)\,\mathcal{U} \tag{1.5}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [6, p. 173]

$$f(A\circ B)\geq(\leq)\, f(A)\circ f(B) \text{ for all } A,\ B\geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\geq 0$$

and Fiedler inequality

$$A\circ A^{-1}\geq 1 \text{ for } A>0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [7] that $\left(A^2 \circ 1\right)^{1/2}\left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then we have the tensorial inequality

$$\begin{aligned}\left(f'(A) \otimes 1\right)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)\left(1 \otimes f'(B)\right)\end{aligned}$$

and the inequality for Hadamard product

$$\begin{aligned}\left(f'(A)A\right) \circ 1 - f'(A) \circ B &\geq (f(A) - f(B)) \circ 1 \\ &\geq A \circ f'(B) - \left(f'(B)B\right) \circ 1.\end{aligned}$$

2. Main Results

We start to the following result that is related to super/sub-multiplicative tensorial inequalities in (1.2):

Theorem 1. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 \leq r < R$ and $0 \leq A,\ B \leq 1$, then*

$$h(r)\, h(rA \otimes B) \geq h(rA) \otimes h(rB). \tag{2.1}$$

If $R = \infty$, then the inequality (2.1) also holds for $A,\ B \geq 1$. In this case for R, if either $0 \leq A \leq 1$ and $B \geq 1$ or $A \geq 1$ and $0 \leq B \leq 1$, then the reverse inequality in (2.1) holds as well.

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$ and nonnegative weights $(p_i)_{i\in\mathbb{N}}$:

$$\sum_{i=0}^{n} p_i \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i, \tag{2.2}$$

for any $n \in \mathbb{N}$.

Assume that $0 < r < R$. Let $t,\ s \in (0,1)$ and define the sequences $c_i := t^i$, $b_i := s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$\sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rts)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \tag{2.3}$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i,\ \sum_{i=0}^{\infty} a_i (rts)^i,\ \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \to \infty$ in (2.3) we get

$$h(r)\, h(rts) \geq h(rt)\, h(rs) \tag{2.4}$$

for all $0 < r < R$ and $t, s \in [0,1]$.

Consider the function

$$h_r(t) = \frac{h(rt)}{h(r)},\ t \in [0,1].$$

We observe that, by (2.4), the function h_r is super-multiplicative on $[0,1]$ and by making use of (1.2) we derive the desired result (1.2).

The other parts of the theorem follow in a similar way, we omit the details. □

Corollary 1. *With the assumptions of Theorem 1 and if h is operator concave on $[0,R)$, then*

$$h(r)\,h(rA \circ B) \geq h(rA) \circ h(rB) \tag{2.5}$$

for either $0 \leq A$, $B \leq 1$ or A, $B \geq 1$ in the case when $R = \infty$. In this last case for R, if h is operator convex on $[0,\infty)$ and either $0 \leq A \leq 1$ and $B \geq 1$ or $A \geq 1$ and $0 \leq B \leq 1$ then the reverse inequality in (2.5) holds as well.

Proof. As in [6, p. 173], by using Davis-Choi-Jensen's inequality we have

$$\begin{aligned} h(r)\,h(rA \circ B) &= h(r)\,h(r\mathcal{U}^*(A \otimes B)\,\mathcal{U}) \geq h(r)\,\mathcal{U}^* h(rA \otimes B)\,\mathcal{U} \\ &\geq \mathcal{U}^*(h(rA) \otimes h(rB))\,\mathcal{U} = h(rA) \circ h(rB). \end{aligned}$$

and the inequality (2.5) is proved. □

We also have the following double inequality for tensorial product of operators:

Theorem 2. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$\begin{aligned} (f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B)). \end{aligned} \tag{2.6}$$

Proof. Using the gradient inequality for the differentiable convex f on I we have

$$f'(t)(t-s) \geq f(t) - f(s) \geq f'(s)(t-s)$$

for all t, $s \in I$.

Assume that

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B.

These imply that

$$\begin{aligned} \int_I \int_I f'(t)(t-s)\,dE(t) \otimes dF(s) &\geq \int_I \int_I (f(t) - f(s))\,dE(t) \otimes dF(s) \\ &\geq \int_I \int_I f'(s)(t-s)\,dE(t) \otimes dF(s). \end{aligned} \tag{2.7}$$

Observe that

$$\begin{aligned}
(2.8)\qquad & \int_I \int_I f'(t)(t-s)\, dE(t) \otimes dF(s) \\
&= \int_I \int_I (f'(t)\, t - f'(t)\, s)\, dE(t) \otimes dF(s) \\
&= \int_I \int_I f'(t)\, t dE(t) \otimes dF(s) - \int_I \int_I f'(t)\, s dE(t) \otimes dF(s) \\
&= (f'(A)\, A) \otimes 1 - f'(A) \otimes B,
\end{aligned}$$

$$\int_I \int_I (f(t) - f(s))\, dE(t) \otimes dF(s) = f(A) \otimes 1 - 1 \otimes f(B)$$

and

$$\begin{aligned}
& \int_I \int_I f'(s)(t-s)\, dE(t) \otimes dF(s) \\
&= \int_I \int_I (t f'(s) - f'(s)\, s)\, dE(t) \otimes dF(s) \\
&= \int_I \int_I t f'(s)\, dE(t) \otimes dF(s) - \int_I \int_I f'(s)\, s dE(t) \otimes dF(s) \\
&= A \otimes f'(B) - 1 \otimes (f'(B)\, B)
\end{aligned}$$

and by (2.8) we derive the inequality of interest:

$$\begin{aligned}
(2.9)\qquad (f'(A)\, A) \otimes 1 - f'(A) \otimes B &\geq f(A) \otimes 1 - 1 \otimes f(B) \\
&\geq A \otimes f'(B) - 1 \otimes (f'(B)\, B).
\end{aligned}$$

Now, by utilizing the tensorial property

$$(XU) \otimes (YV) = (X \otimes Y)(U \otimes V),$$

for any X, U, Y, $V \in B(H)$, we have

$$(f'(A)\, A) \otimes 1 = (f'(A) \otimes 1)(A \otimes 1),$$

$$f'(A) \otimes B = (f'(A) \otimes 1)(1 \otimes B),$$

$$A \otimes f'(B) = (A \otimes 1)(1 \otimes f'(B))$$

and

$$1 \otimes (f'(B)\, B) = 1 \otimes (B f'(B)) = (1 \otimes B)(1 \otimes f'(B)).$$

Therefore

$$\begin{aligned}
(f'(A)\, A) \otimes 1 - f'(A) \otimes B &= (f'(A) \otimes 1)(A \otimes 1) - (f'(A) \otimes 1)(1 \otimes B) \\
&= (f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B)
\end{aligned}$$

and

$$\begin{aligned}
A \otimes f'(B) - 1 \otimes (f'(B)\, B) &= (A \otimes 1)(1 \otimes f'(B)) - (1 \otimes B)(1 \otimes f'(B)) \\
&= (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B))
\end{aligned}$$

and by (2.9) we derive (2.6). $\square$

Corollary 2. *With the assumptions of Theorem 2 and if $A_j \in B(H)$ with spectra $\operatorname{Sp}(A_j) \subset I$, $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$\left(\sum_{j=1}^{n} p_j f'(A_j) A_j\right) \otimes 1 - \left(\sum_{j=1}^{n} p_j f'(A_j)\right) \otimes B \tag{2.10}$$
$$\geq \left(\sum_{j=1}^{n} p_j f(A_j)\right) \otimes 1 - 1 \otimes f(B)$$
$$\geq \left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes B\right)(1 \otimes f'(B)).$$

In particular, we have

$$\left(\sum_{j=1}^{n} p_j f'(A_j) A_j\right) \otimes 1 - \left(\sum_{j=1}^{n} p_j f'(A_j)\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) \tag{2.11}$$
$$\geq \left(\sum_{j=1}^{n} p_j f(A_j)\right) \otimes 1 - 1 \otimes f\left(\sum_{j=1}^{n} p_j A_j\right)$$
$$\geq \left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right)\left(1 \otimes f'\left(\sum_{j=1}^{n} p_j A_j\right)\right).$$

Proof. From Theorem 2 we have

$$(f'(A_j) A_j) \otimes 1 - f'(A_j) \otimes B \geq f(A_j) \otimes 1 - 1 \otimes f(B)$$
$$\geq (A_j \otimes 1 - 1 \otimes B)(1 \otimes f'(B))$$

for $j \in \{1, ..., n\}$.

If we multiply by $p_j \geq 0$, $j \in \{1, ..., n\}$ and then sum from 1 to n, then we get

$$\sum_{j=1}^{n} p_j (f'(A_j) A_j) \otimes 1 - \sum_{j=1}^{n} p_j f'(A_j) \otimes B$$
$$\geq \sum_{j=1}^{n} p_j f(A_j) \otimes 1 - \sum_{j=1}^{n} p_j (1 \otimes f(B))$$
$$\geq \sum_{j=1}^{n} p_j (A_j \otimes 1 - 1 \otimes B)(1 \otimes f'(B))$$
$$= \left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes B\right)(1 \otimes f'(B)),$$

for a selfadjoint operator B with $\operatorname{Sp}(B) \subset I$, which gives (2.10).

Since $\operatorname{Sp}(A_j) \subset I$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, hence $\operatorname{Sp}\left(\sum_{j=1}^{n} p_j A_j\right) \subset I$ and by taking $B = \sum_{j=1}^{n} p_j A_j$ in (2.10), we get (2.11). □

Remark 1. *With the assumptions of Corollary 2 and if*

$$\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 = 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right), \tag{2.12}$$

then

$$\begin{aligned}\left(\sum_{j=1}^{n} p_j f'(A_j) A_j\right) \otimes 1 - \left(\sum_{j=1}^{n} p_j f'(A_j)\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) \\ \geq \left(\sum_{j=1}^{n} p_j f(A_j)\right) \otimes 1 - 1 \otimes f\left(\sum_{j=1}^{n} p_j A_j\right) \geq 0.\end{aligned} \tag{2.13}$$

Theorem 3. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}(f'(A) A) \circ 1 - f'(A) \circ B &\geq (f(A) - f(B)) \circ 1 \\ &\geq A \circ f'(B) - (f'(B) B) \circ 1.\end{aligned} \tag{2.14}$$

Proof. If we multiply the inequality (2.9) to the left with $\mathcal{U}^*$ and at the right with $\mathcal{U}$, we get

$$\begin{aligned}&\mathcal{U}^* [(f'(A) A) \otimes 1 - f'(A) \otimes B] \mathcal{U} \\ &\geq \mathcal{U}^* [f(A) \otimes 1 - 1 \otimes f(B)] \mathcal{U} \\ &\geq \mathcal{U}^* [A \otimes f'(B) - 1 \otimes (f'(B) B)] \mathcal{U},\end{aligned}$$

namely

$$\begin{aligned}&\mathcal{U}^* ((f'(A) A) \otimes 1) \mathcal{U} - \mathcal{U}^* (f'(A) \otimes B) \mathcal{U} \\ &\geq \mathcal{U}^* (f(A) \otimes 1) \mathcal{U} - \mathcal{U}^* (1 \otimes f(B)) \mathcal{U} \\ &\geq \mathcal{U}^* (A \otimes f'(B)) \mathcal{U} - \mathcal{U}^* (1 \otimes (f'(B) B)) \mathcal{U}.\end{aligned}$$

Using representation (1.5) we get

$$\begin{aligned}(f'(A) A) \circ 1 - f'(A) \circ B &\geq f(A) \circ 1 - 1 \circ f(B) \\ &\geq A \circ f'(B) - 1 \circ (f'(B) B),\end{aligned} \tag{2.15}$$

which gives (2.14). □

Remark 2. *If $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for the separable Hilbert space H, then, under the assumptions of Theorem 3, we have*

$$\begin{aligned}&\langle f'(A) A e_j, e_j\rangle - \langle f'(A) e_j, e_j\rangle \langle B e_j, e_j\rangle \\ &\geq \langle f(A) e_j, e_j\rangle - \langle f(B) e_j, e_j\rangle \\ &\geq \langle A e_j, e_j\rangle \langle f'(B) e_j, e_j\rangle - \langle f'(B) B e_j, e_j\rangle,\end{aligned} \tag{2.16}$$

for all $j \in \mathbb{N}$.

Corollary 3. *With the assumptions of Theorem 3 and if* $A_j \in B(H)$ *with spectra* $\operatorname{Sp}(A_j) \subset I$, $p_j \geq 0$ *for* $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$\left(\sum_{j=1}^{n} p_j f'(A_j) A_j\right) \circ 1 - \left(\sum_{j=1}^{n} p_j f'(A_j)\right) \circ B \tag{2.17}$$
$$\geq \left(\sum_{j=1}^{n} p_j f(A_j) - f(B)\right) \circ 1$$
$$\geq \left(\sum_{j=1}^{n} p_j A_j\right) \circ f'(B) - (f'(B) B) \circ 1.$$

In particular,

$$\left(\sum_{j=1}^{n} p_j f'(A_j) A_j\right) \circ 1 - \left(\sum_{j=1}^{n} p_j f'(A_j)\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right) \tag{2.18}$$
$$\geq \left(\sum_{j=1}^{n} p_j f(A_j)\right) \circ 1 - f\left(\sum_{j=1}^{n} p_j A_j\right) \circ 1$$
$$\geq \left(\sum_{j=1}^{n} p_j A_j\right) \circ f'\left(\sum_{j=1}^{n} p_j A_j\right) - \left(f'\left(\sum_{j=1}^{n} p_j A_j\right) \sum_{j=1}^{n} p_j A_j\right) \circ 1.$$

Proof. If we replace in (2.14) $B = A_j$, multiply by p_j and sum over j from 1 to n, then we get (2.17).

The inequality (2.18) follows by taking $B = \sum_{j=1}^{n} p_j A_j$ in (2.17). □

3. Some Examples

Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. We have the following examples

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \; z \in D(0,1); \tag{3.1}$$
$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \; z \in \mathbb{C};$$
$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \; z \in \mathbb{C};$$
$$h(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \; z \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^{n}=\exp(z)\qquad z\in\mathbb{C},\tag{3.2}$$

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1);$$

$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$

and

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1)\tag{3.3}$$

$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}z^{n},\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

Assume that $0<r<1$ and $0\le A,\ B\le 1$, then by (2.1) for $h(z)=(1-z)^{-1}$ we get

$$(1-r)^{-1}(1-rA\otimes B)^{-1}\ge(1-rA)^{-1}\otimes(1-rB)^{-1},\tag{3.4}$$

for $h(z)=\ln(1-z)^{-1}$ we obtain

$$\ln(1-r)^{-1}\ln(1-rA\otimes B)^{-1}\ge\ln(1-rA)^{-1}\otimes\ln(1-rB)^{-1},\tag{3.5}$$

while for $h(z)=\sin^{-1}(z)$ we derive

$$\sin^{-1}(r)\sin^{-1}(rA\otimes B)\ge\sin^{-1}(rA)\otimes\sin^{-1}(rB).\tag{3.6}$$

If $r>0$ and either $0\le A,\ B\le 1$ or $A,\ B\ge 1$, then by (2.1) for $h(z)=\exp z$ we get

$$\exp(r(1+A\otimes B))\ge\exp(rA)\otimes\exp(rB).\tag{3.7}$$

If either $0\le A\le 1$ and $B\ge 1$ or $A\ge 1$ and $0\le B\le 1$ then the reverse inequality in (3.7) holds as well.

By (2.1) for $h(z)=\cosh z$ or $\sinh z$ we get

$$\cosh(r)\cosh(rA\otimes B)\ge\cosh(rA)\otimes\cosh(rB)\tag{3.8}$$

or

$$\sinh(r)\sinh(rA\otimes B)\ge\sinh(rA)\otimes\sinh(rB)\tag{3.9}$$

for either $0\le A,\ B\le 1$ or $A,\ B\ge 1$.

If either $0\le A\le 1$ and $B\ge 1$ or $A\ge 1$ and $0\le B\le 1$, then the reverse inequality in (3.8) or (3.8) holds as well.

If we take the convex function $f(t)=-\ln t$, $t>0$, then from (2.9) for $A,\ B>0$ we get

$$1-A^{-1}\otimes B\le(\ln A)\otimes 1-1\otimes(\ln B)\le A\otimes B^{-1}-1.\tag{3.10}$$

From (2.11) we get

$$\left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - 1 \tag{3.11}$$
$$\geq 1 \otimes \ln\left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j \ln A_j\right) \otimes 1$$
$$\geq \left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1\right)\left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^{-1}\right),$$

where $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

Moreover, if the condition (2.12) is satisfied, then

$$\left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - 1 \tag{3.12}$$
$$\geq 1 \otimes \ln\left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j \ln A_j\right) \otimes 1 \geq 0$$

From (2.14) we get

$$A^{-1} \circ B - 1 \geq (\ln B - \ln A) \circ 1 \geq 1 - A \circ B^{-1} \tag{3.13}$$

for A, $B > 0$.

If $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ then by (2.18) we derive

$$\left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right) - 1 \tag{3.14}$$
$$\geq \ln\left(\sum_{j=1}^{n} p_j A_j - \sum_{j=1}^{n} p_j \ln A_j\right) \circ 1$$
$$\geq \left(\sum_{j=1}^{n} p_j A_j\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right)^{-1} - 1 \geq 0$$

The last inequality follows by Fiedler inequality $B \circ B^{-1} \geq 1$, see for instance [6, p. 176].

If we take the convex function $f(t) = t \ln t$, $t > 0$, then from (2.9) for A, $B > 0$ we get

$$((\ln A) \otimes 1 + 1)(A \otimes 1 - 1 \otimes B) \geq (A \ln A) \otimes 1 - 1 \otimes (B \ln B) \tag{3.15}$$
$$\geq (A \otimes 1 - 1 \otimes B)(1 \otimes \ln B + 1).$$

From (2.11) we get

$$\begin{aligned}(3.16)\quad &\left(\sum_{j=1}^{n} p_j A_j \ln A_j + \sum_{j=1}^{n} p_j A_j\right)\otimes 1 \\ &- \left(\sum_{j=1}^{n} p_j \ln\left(A_j\right) + 1\right)\otimes\left(\sum_{j=1}^{n} p_j A_j\right) \\ &\geq \left(\sum_{j=1}^{n} p_j A_j \ln A_j\right)\otimes 1 - 1\otimes\left[\left(\sum_{j=1}^{n} p_j A_j\right)\ln\left(\sum_{j=1}^{n} p_j A_j\right)\right] \\ &\geq \left(\left(\sum_{j=1}^{n} p_j A_j\right)\otimes 1 - 1\otimes\left(\sum_{j=1}^{n} p_j A_j\right)\right) \\ &\times\left(1\otimes\ln\left(\sum_{j=1}^{n} p_j A_j\right) + 1\right),\end{aligned}$$

where $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

From (2.14) we get

$$\begin{aligned}(3.17)\quad (A\ln A + A)\circ 1 - (\ln A + 1)\circ B &\geq (A\ln A - B\ln B)\circ 1 \\ &\geq A\circ(\ln B + 1) - (B\ln B + B)\circ 1\end{aligned}$$

for A, $B > 0$.

From (2.18) we get

$$\begin{aligned}(3.18)\quad &\left(\sum_{j=1}^{n} p_j A_j \ln\left(A_j\right)\right)\circ 1 - \left(\sum_{j=1}^{n} p_j \ln\left(A_j\right)\right)\circ\left(\sum_{j=1}^{n} p_j A_j\right) \\ &\geq \left(\sum_{j=1}^{n} p_j A_j \ln A_j - \left(\sum_{j=1}^{n} p_j A_j\right)\ln\left(\sum_{j=1}^{n} p_j A_j\right)\right)\circ 1 \\ &\geq \left(\sum_{j=1}^{n} p_j A_j\right)\circ\ln\left(\sum_{j=1}^{n} p_j A_j\right) - \left[\ln\left(\sum_{j=1}^{n} p_j A_j\right)\sum_{j=1}^{n} p_j A_j\right]\circ 1,\end{aligned}$$

where $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

If we write the inequality (2.6) for the convex function $f(t) = t^r$, $r \in (-\infty, 0) \cup [1, \infty)$, then we get

$$\begin{aligned}(3.19)\quad r\left(A^{r-1}\otimes 1\right)(A\otimes 1 - 1\otimes B) &\geq A^r\otimes 1 - 1\otimes B^r \\ &\geq r(A\otimes 1 - 1\otimes B)\left(1\otimes B^{r-1}\right),\end{aligned}$$

for A, $B > 0$.

For $r = 2$, we get

$$\begin{aligned}(3.20)\quad 2(A\otimes 1)(A\otimes 1 - 1\otimes B) &\geq A^2\otimes 1 - 1\otimes B^2 \\ &\geq 2(A\otimes 1 - 1\otimes B)(1\otimes B),\end{aligned}$$

while for $r=-1$ we get

$$\left(A^{-2}\otimes 1\right)(1\otimes B-A\otimes 1)\geq A^{-1}\otimes 1-1\otimes B^{-1} \geq (1\otimes B-A\otimes 1)\left(1\otimes B^{-2}\right), \tag{3.21}$$

for $A,\ B>0$.

From (2.11) we derive

$$\begin{aligned} &r\left[\left(\sum_{j=1}^{n}p_jA_j^r\right)\otimes 1-\left(\sum_{j=1}^{n}p_jA_j^{r-1}\right)\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\right] \\ &\geq\left(\sum_{j=1}^{n}p_jA_j^r\right)\otimes 1-1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)^r \\ &\geq r\left(\left(\sum_{j=1}^{n}p_jA_j\right)\otimes 1-1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\right)\left(1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)^{r-1}\right), \end{aligned} \tag{3.22}$$

where $A_j>0$ and $p_j\geq 0$ for $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$.

For $r=2$ we get

$$\begin{aligned} &2\left[\left(\sum_{j=1}^{n}p_jA_j^2\right)\otimes 1-\left(\sum_{j=1}^{n}p_jA_j\right)\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\right] \\ &\geq\left(\sum_{j=1}^{n}p_jA_j^2\right)\otimes 1-1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)^2 \\ &\geq 2\left(\left(\sum_{j=1}^{n}p_jA_j\right)\otimes 1-1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\right)\left(1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\right), \end{aligned} \tag{3.23}$$

while for $r=-1$, we get

$$\begin{aligned} &\left(\sum_{j=1}^{n}p_jA_j^{-2}\right)\otimes\left(\sum_{j=1}^{n}p_jA_j\right)-\left(\sum_{j=1}^{n}p_jA_j^{-1}\right)\otimes 1 \\ &\geq\left(\sum_{j=1}^{n}p_jA_j^{-1}\right)\otimes 1-1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)^{-1} \\ &\geq\left(1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)-\left(\sum_{j=1}^{n}p_jA_j\right)\otimes 1\right)\left(1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)^{-2}\right). \end{aligned} \tag{3.24}$$

From (2.14) written for the convex function $f(t)=t^r$, $r\in(-\infty,0)\cup[1,\infty)$, we get

$$r\left(A^r\circ 1-A^{r-1}\circ B\right)\geq (A^r-B^r)\circ 1\geq r\left(A\circ B^{r-1}-B^r\circ 1\right), \tag{3.25}$$

for $A,\ B>0$.

For $r=2$ we get

$$2\left(A^2\circ 1-A\circ B\right)\geq\left(A^2-B^2\right)\circ 1\geq 2\left(A\circ B-B^2\circ 1\right), \tag{3.26}$$

while for $r=-1$, we get

$$A^{-2}\circ B-A^{-1}\circ 1\geq \left(A^{-1}-B^{-1}\right)\circ 1\geq B^{-1}\circ 1-A\circ B^{-2}, \tag{3.27}$$

for A, $B>0$.

If $A_j>0$ and $p_j\geq 0$ for $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$, then by (2.18)

$$\begin{aligned} &r\left[\left(\sum_{j=1}^{n}p_jA_j^r\right)\circ 1-\left(\sum_{j=1}^{n}p_jA_j^{r-1}\right)\circ\left(\sum_{j=1}^{n}p_jA_j\right)\right] \\ &\geq\left(\sum_{j=1}^{n}p_jA_j^r-\left(\sum_{j=1}^{n}p_jA_j\right)^r\right)\circ 1 \\ &\geq r\left[\left(\sum_{j=1}^{n}p_jA_j\right)\circ\left(\sum_{j=1}^{n}p_jA_j\right)^{r-1}-\left(\sum_{j=1}^{n}p_jA_j\right)^r\circ 1\right]. \end{aligned} \tag{3.28}$$

For $r=2$, then we get

$$\begin{aligned} &2\left[\left(\sum_{j=1}^{n}p_jA_j^2\right)\circ 1-\left(\sum_{j=1}^{n}p_jA_j\right)\circ\left(\sum_{j=1}^{n}p_jA_j\right)\right] \\ &\geq\left(\sum_{j=1}^{n}p_jA_j^2-\left(\sum_{j=1}^{n}p_jA_j\right)^2\right)\circ 1 \\ &\geq 2\left[\left(\sum_{j=1}^{n}p_jA_j\right)\circ\left(\sum_{j=1}^{n}p_jA_j\right)-\left(\sum_{j=1}^{n}p_jA_j\right)^2\circ 1\right], \end{aligned} \tag{3.29}$$

while for $r=-1$ we get

$$\begin{aligned} &\left(\sum_{j=1}^{n}p_jA_j^{-2}\right)\circ\left(\sum_{j=1}^{n}p_jA_j\right)-\left(\sum_{j=1}^{n}p_jA_j^{-1}\right)\circ 1 \\ &\geq\left(\sum_{j=1}^{n}p_jA_j^{-1}-\left(\sum_{j=1}^{n}p_jA_j\right)^{-1}\right)\circ 1 \\ &\geq\left(\sum_{j=1}^{n}p_jA_j\right)^{-1}\circ 1-\left(\sum_{j=1}^{n}p_jA_j\right)\circ\left(\sum_{j=1}^{n}p_jA_j\right)^{-2}. \end{aligned} \tag{3.30}$$

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