Super-additivity and Discreteness

CONTENT

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Introduction

"Straight line is the shortest road between two points" is one of the most accepted facts in our everyday experience; Archimedes just used it to define *straight line*, 200 years B.Ch. However, we may encounter remarkable cases of opposite situation, which form the subject of the present text. This topic is appropriately discussed in terms of inequalities about metrics, which take place on triplets of elements (in particular representing triangles), like follows:

Let X be an arbitrary nonvoid set and $K \subseteq X \times X$ be a *preorder* (*reflexive* and *transitive* binary relation) on X. We say that $\rho: K \to \mathbb{R}_+$ is a *restrained metric* if:

[M_1] $\rho(x, x) = 0 \ \forall x \in X$.

Usually, especially in the case $K = X \times X$ we avoid mentioning "*restrained*".

If, in addition, K is *symmetric* and

[M_2] $\rho(x, y) = \rho(y, x) \ \forall (x, y) \in K$,

then ρ is said to be a *symmetric* metric.

We say that metric ρ is *sub-additive* (briefly s.a.) if

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z) \tag{s.a.}$$

takes place whenever $(x, y), (y, z) \in K$. In the contrary case, when

$$\rho(x, z) \geq \rho(x, y) + \rho(y, z) \tag{S.a.}$$

occurs for arbitrary $(x, y), (y, z) \in K$, we say ρ is *super-additive* (briefly S.a.).

In other words, (X, K, ρ) is a *s.a. metric space* if (s.a.) holds for each triangle with sides in K (possibly for each of its sides). In the opposite case, the logical negation appears, i.e. there exist triplets such that S.a. holds (necessarily only) for one of the sides; then (X, K, ρ) is called *S.a. metric universe*. The distinction between the terms *space* and *universe* is justified by the nature of the applications where we primarily encounter these two types of metrics. More exactly, practice highlighted (s.a.) in our usual Euclidean *space*, while (S.a.) turned out to be the fundamental property of the proper time in relativity.

As a matter of fact, (S.a.) has rarely appeared in classical mathematics while in our daily practice it goes almost unnoticed, or seems extremely strange. Therefore, the purpose of Chapter I is to present several examples of such metrics and reveal their practical meanings. Due to our long habit of associating the notion of *metrics* with the measurement of *distances*, so tacitly with (s.a.), we thought it appropriate to look for more comprehensive terms, which may include both (s.a.) and (S.a.). This is why we interpret the values of functions that mathematically are *metrics* in more common terms of *difficulty* and *hope*, which characterize the transition from one state to another.

It is true, we feel the difficulties subjectively and most of the time we cannot measure them objectively, e.g. the difficulty of formulating and proving a theorem, writing literature, singing music, painting etc. Therefore, in the present text we are particularly interested in identifying some phenomena in which *difficulty*, *chance* and *hope* can be expressed by mathematical formulas and correspond to physical quantities that can be evaluated experimentally.

Starting with hyperbolic numbers (Section I.1) is a good opportunity to see how insignificant is (S.a.) in classical mathematics. Even if the intrinsic (indefinite) norm is a mirror pair of the Euclidean norm of the field of complex numbers, it has no role in the specific calculus; as a rule, its fundamental property – (S.a.) – is not even mentioned in specialized studies.

In Section I.2 we already avoid the danger of "*Testis unus*, *testis nullus*" (*A witness*, *no witness*) highlighting again (S.a.), this time in physics, relative to the *proper time* in the Einsteinian Special Theory of Relativity. Examples closer to our daily experience appear in section I.3, about anisotropic spaces.

In Section I.4 we remark relativist features in living systems relative to a scalar parameter, p. The difficulty arises here due to p-deviations, but living beings have the power to correct them through homeostasis.

In Section I.5 we stress on the fact that S.a. is present in the understanding of each mathematical statement; it explains why a proof diminishes the difficulty. In Boolean algebra of propositions we may evaluate the difficulty of an implication.

Two-dimensional calculus (discussed in Section I.6) is another example of mathematical theory in which S.a. is obvious, but it is useless and systematically neglected in favor of a theory with a classical feature.

Section I.7 refers to several common spaces (including our Euclidean $\mathbb{R}^3$) in which we can construct both s.a. as well as S.a. norms and metrics depending on a parameter. The issue of finding these inequalities in practice remains open.

Section I.8 is a short story about the connection between s.a. and S.a. norms and convex (respectively concave) sets in linear spaces. In this way, we can extend relativistic space-time to events that occur in arbitrary normed space.

Chrono-Geometry, also known as Geometry of Minkowskian space-time, aims to find visible images of some properties specific to a plane of events. Notions like *distance*, *orthogonality* and *angle* receive new meanings, discussed in Section I.9.

In section I.10, we support the assertion that S.a. is an inequality specific to spaces with undefined inner products. The difference between s.a. and S.a. it comes, via the fundamental inequalities (either Cauchy-Bunjakowski-Schwartz or Aczél-Varga), from the type of the subspace $\mathrm{Lin}\{x, y\}$.

In Chapter II we outline a mathematical theory of structures of discreteness, called *horistologies*, from the Greek χωριστοσ = *separate*. Horistological structures were first considered in a report to the *Alexander von Humboldt Stiftung*, which in 1974, under the presidency of Professor Werner Heisenberg, granted a fellowship to study the mathematical structure of space-time. This work was poorly accepted (by H.H.S.) and qualified as "zu gerring". However, the development of the theory was continued mainly within the Department of Applied Mathematics, at the University of Craiova, Romania.

The term "*discrete*" already has many meanings; there is also a "discrete topology", although *topology* is a structure of continuum. Therefore, in Section II.1 we analyze why we look for *structures of discreteness*.

In Section II.2 we define the *horistological* structures. The terminology is inspired from relativity: instead of *neighborhoods* of a point we speak of *perspectives* of an event. Some specific properties such as *causal order*, *premise operator* and operations with such structures are highlighted.

In Section II.3 we identify the morphisms of the category HOR. Discreteness of a function between horistological worlds reduces to carrying perspectives to perspectives, unlike to continuity, where counter images of neighborhoods are neighborhoods. It is similar to bounded functions (i.e. morphisms of category BOR), which carry bounded sets to bounded images.

The emergence, discussed in Section II.4, corresponds to the convergence in topology. It reduces to the discreteness of the function “net”.

The horistological discrete sets are objects of the Section II.5. They play a role similar to open sets in topology. In particular, specifying a class of discrete sets we may recover the horistological structure.

The rest of sections, II.6 – II.10 highlight several applications of S.a. metrics and of structures of discreteness. Section II.6 notices such elements in the classical theories of the living systems. In Section II.7, instability of a system is defined by the discreteness of the evolution function. Section II.8 sketches a possibility of unifying relativity and quantum physics in terms of discreteness. In Section II.9 we see how to construct $\mathbb{R}$ using the u-horistology of $\mathbb{Q}$. Finally, in Section II.10, a remarkable application refers to the duality theory of the L^p spaces with $p < 1$, where the dual space consists of hor-discrete functions.

We refrain from mentioning a general bibliography because of the abundant information on the internet, e.g.

http://cis01.central.ucv.ro/site/cercetare_tbalan.htm

The readers with moderate mathematical culture will have no difficulty in lecturing this text. However, we mention that a lot of details can be found in [BT][1] and Section II.10 is a summary of [CB][2].

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[1] Balan T., *Complements of Hyperbolic Mathematics, from Super-Additivity to Structural Discreteness*, Editura *Universitaria*, Craiova, 2016

[2] Calvert B., *Strictly plus Functionals on a Supr-additive Normed Linear Space*, Anals Univ. Craiova, Series Mathematics, Vol. XVIII (1990), 27-43

Chapter I. Super-additivity

I.1. Hyperbolic numbers

For the first time, I dealt with the reverse inequality of a triangle during the preparation of my license thesis about Clifford Algebras (1962). So, I found out how to organize $\mathbb{R}^2$ as algebras of numbers, depending on the square of $(0, 1)$:
Complex numbers, $\mathbb{C}$, where $(0, 1)^2 = (-1, 0)$, noted $i^2 = -1$;
Hyperbolic numbers, $\mathbb{H}$,where $(0, 1)^2 = (1, 0)$, briefly $j^2 = 1$;
Parabolic numbers, $\mathbb{P}$, where $(0, 1)^2 = (0, 0)$, i.e. $k^2 = 0$.

The structure of $\mathbb{C}$ is well-known: it is a field and its fundamental quadratic form generates the (s.a.) Euclidean metric, hence a topology; the calculus of the functions of a complex variable represents the nucleus of the classical mathematics. In addition, the Frobenius' Theorem states that only $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{K}$ (quaternions) are Clifford algebras with division, of finite dimension, over $\mathbb{R}$. In particular, only $\mathbb{C}$ is algebraically closed. The only deficiency of $\mathbb{C}$ is the lack of an order relation compatible with its algebraic structure.

Most part of these properties fail in the case of $\mathbb{H}$ and $\mathbb{P}$. They are no longer fields. In particular, $\mathbb{H}$ contains two ideals, namely $I_{\mp} = \{\lambda(1 \mp j): \lambda \in \mathbb{R}\}$, divisors of zero, and four idempotent elements, namely 0, 1 and $\frac{1}{2}(1 \mp j)$. Even worse, the intrinsic quadratic form at $z = a + jb$ is $Q(z) = z \cdot \overline{z} = a^2 - b^2$, which is indefinite. Consequently, $|z| = \sqrt{a^2 - b^2}$ makes sense as a real value norm only if $|b| < |a|$ and the resulting metric $\rho(z_1, z_2) = |z_1 - z_2|$ is *super-additive*. The proof of (S.a.) works on the domain of ρ, but simple examples like $z_1 = 0$, $z_2 = 2 + j$ and $z_3 = 4$ are enough convincing:

$$\rho(z_1, z_3) = 4 > 2\sqrt{3} = \rho(z_1, z_2) + \rho(z_2, z_3).$$

Consequently, norm $|\cdot|$ produces no topology on $\mathbb{H}$, hence no theory of continuity and no (Cauchy) calculus.

However, $\mathbb{H}$ has the advantage of being ordered by

$$K = \{(z_1, z_2) \in \mathbb{H}^2: Q(z_1 - z_2) > 0, a_1 < a_2\} \cup \{(z, z): z \in \mathbb{H}\},$$

where $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$. Thus, we may specify that $|\cdot|$ is defined on the cone of *positive* hyperbolic numbers

$$P = K[0] = \{a + jb: a > |b|\} \cup \{0\},$$

and the S.a. metric ρ is defined on K.

Order K extends the strict order of $\mathbb{R}$. In particular, the square of any invertible hyperbolic number is strictly positive, i.e. member of $\dot{P} = P \setminus \{0\}$.

The set of hyperbolic numbers is a very natural framework of the hyperbolic trigonometry. Each $z = x + y\, j \in \dot{P}$ allows the *polar* representation

$$z = |z| (\cosh \omega + j \sinh \omega),$$

where ω = arctanh $\frac{y}{x}$ is the *argument* of z. In fact, $Q(z) > 0$, hence

$$z = \sqrt{Q(z)}\,\left[\left(x/\sqrt{Q(z)}\right) + j(y/\sqrt{Q(z)}\right],$$

so it remains to note $y / x = \tanh \omega$, for some $\omega \in \mathbb{R}$, and apply usual formulas.

Here, the hyperbolic functions appear in their analytic form

$$\cosh \omega = \frac{e^{\omega}+e^{-\omega}}{2},\ \sinh \omega = \frac{e^{\omega}-e^{-\omega}}{2}\ \text{and}\ \tanh \omega = \frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}},$$

i.e. $\cosh \omega = \sum_{n=0}^{\infty} \frac{\omega^{2n}}{(2n)!}$, $\sinh \omega = \sum_{n=0}^{\infty} \frac{\omega^{2n+1}}{(2n+1)!}$ etc. Thus, $|z|$ and Arg z obtain clear geometrical interpretations if we appeal to parametric equations of a hyperbola and measure hyperbolic angles by "*length of the arc* / *radius*".

Solving algebraic equations in $\mathbb{H}$ presents strange aspects in comparison to the same problem in $\mathbb{C}$. For example, for all $n \in \mathbb{N} \setminus \{0, 1\}$, equation $z^n = z$ has exactly four solutions, namely $z_1 = 0$, $z_2 = 1$, $\mathfrak{z} = \frac{1}{2}(1 + j)$ and $\bar{\mathfrak{z}} = \frac{1}{2}(1 - j)$, which are fixed points relative to rising to a natural power.

As usually, neglecting its intrinsic S.a. metric, which cannot equip $\mathbb{H}$ with a topology, we may develop calculus by using the Euclidean topology produced by the norm $\| x + y j \| = \sqrt{x^2 + y^2}$, or equivalently, $/ x + y j / = | x | + | y |$. Thus, the role of the unit circle from $\mathbb{C}$ is played in $\mathbb{H}$ by the *unit square* centered at 0,

$$\lozenge(0, 1) = \{ z \in \mathbb{H} : / z / < 1\}.$$

In particular, $\lim_{n\to\infty} z^n = 0 \Leftrightarrow z \in \lozenge(0, 1)$.

Another way of developing calculus on $\mathbb{H}$ is to *split* it into two copies of $\mathbb{R}$, which results by a change of base $\{1, j\} \to \{\mathfrak{z}, \bar{\mathfrak{z}}\}$. In fact, $a + jb = \alpha\,\mathfrak{z} + \beta\,\bar{\mathfrak{z}}$ is a change of coordinates $(a, b) \to (a + b, a - b)$, the ideals I_+ and I_- are algebraically closed ($I_\mp + I_\mp \subseteq I_\mp$, $I_\mp \cdot I_\mp \subseteq I_\mp$) and orthogonal ($\mathfrak{z} \cdot \bar{\mathfrak{z}} = 0$), so we obtain $\mathbb{H} \cong \mathbb{R} \oplus \mathbb{R}$. Using this *idempotent* basis, each $z = \alpha\,\mathfrak{z} + \beta\,\bar{\mathfrak{z}}$ has the powers $z^n = \alpha^n\,\mathfrak{z} + \beta^n\,\bar{\mathfrak{z}}$. This property allows extending analytic functions from $\mathbb{R}$ to $\mathbb{H}$ by a simple rule:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ \to\ f(\alpha\,\mathfrak{z} + \beta\,\bar{\mathfrak{z}}) = f(\alpha)\mathfrak{z} + f(\beta)\,\bar{\mathfrak{z}}.$$

On this way we obtain the analytic hyperbolic functions

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^{\alpha}\,\mathfrak{z} + e^{\beta}\,\bar{\mathfrak{z}},\ \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = (\cos \alpha)\,\mathfrak{z} + (\cos \beta)\,\bar{\mathfrak{z}}\ \text{etc.}$$

In conclusion, $\mathbb{H}$ is a clear example of mathematical domain where we may remark the presence of a S.a. metric, but it was completely neglected in favor of a classical theory of continuum. In the subsequent Sections we'll see that (S.a.) is much more than an unfortunate accident between numbers, which, as some say, for our peace of mind, we should overlook.

I.2. Space-time

Learning that *proper time* in the Einsteinian (Restrained) Theory of Relativity is measured by an S.a. metric was very encouraging for me; it was the first evidence that S.a. also takes place in the physical reality. This is known as *twin's paradox*, where term "*paradox*" shows very well how S.a. was viewed at the beginning. Despite the initial difficult acceptance, Relativity was developed as an "art" of distinguishing the invariant aspects (like speed of light (noted c) and proper time), which appear the same for all observers, from the subjective perceptions of our universe (e.g. temporal and spatial coordinates of events).

The simplest relativist universe, $W = \mathbb{R} \times \mathbb{R}$, consists of events that happen on a straight line. It is partially ordered by the *causality* (*accessibility*),

$$K = \{(e_1, e_2) \in W^2 \colon c(t_2 - t_1) > |\, s_2 - s_1|\} \cup \{(e, e) \colon e \in W\},$$

consisting of pairs of events that can be experienced successively by an inertial observer. In this universe it is thoroughly established that whenever the events $e_0 = (t_0, s_0/c)$ and $e = (t, s/c)$ are in relation of causality, the *proper time* between them has the expression

$$\tau(e_0, e) = \sqrt{(t-t_0)^2 - \frac{1}{c^2}(s-s_0)^2}\,. \tag{*}$$

It is easy to prove that $\tau : K \to \mathbb{R}_+$ is a (S.a.) metric.

Function $\Phi : W \to \mathbb{H}$, which carries $e = (t, \frac{s}{c}) \in W$ to $\Phi(e) = t + \frac{s}{c} j = z \in \mathbb{H}$, is an algebraic and (S.a.) metric isomorphism between W and $\mathbb{H}$. Thus, relation K from $\mathbb{H}$ represents causality in W, $I_+ \cup I_-$ is the light cone of vertex $(0, 0)$, $\dot{P}$ is the future cone, the norm of the hyperbolic number $z = t + \frac{s}{c} j$ coincides with $\tau(0, e)$. The change of speed $0 \to v$, at the event $(0, 0)$, i.e. the corresponding acceleration, is represented via notation $\frac{v}{c} = \tanh\omega$ by the argument ω in the trigonometric form of the hyperbolic number $z = t + \frac{s}{c} j$. Correspondingly, its modulus represents proper time.

The Lorentz transformations in W take the form $(t, \frac{s}{c}) \to \left(\theta, \frac{\sigma}{c}\right)$, where

$$\begin{cases} \theta = \left(t - \frac{v}{c}\frac{s}{c}\right)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \\ \frac{\sigma}{c} = \left(-\frac{v}{c} t + \frac{s}{c}\right)\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \end{cases} \tag{Lorentz}$$

The same notation, $v/c = \tanh\omega$, leads to the formulas

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \cosh\omega \text{ and } \frac{v}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \sinh\omega,$$

hence transformations (Lorentz) become *hyperbolic rotation* in $\mathbb{H}$, of angle $-\omega$,

$$\begin{cases} \theta = t\cosh\omega - \frac{s}{c}\sinh\omega \\ \frac{\sigma}{c} = -t\sinh\omega + \frac{s}{c}\cosh\omega\,. \end{cases}$$

On this way we obtain a "visual" interpretation of the Lorentz transformations in W, as well as physical interpretations of the algebraic operations in $\mathbb{H}$.

Besides S.a., which contradicts the usual vision about metrics, the relativist theory of space-time was hardly accepted because of its poor connection with our everyday experience. However, space-time becomes more familiar if instead of cool measurements of distances and moments we interpret *space* as *difficulty* and *time* as *hope*. More exactly, if at $e_0 = (t_0, \frac{s_0}{c})$ we decided to travel to another location marked $\frac{s}{c}$ and be there at the moment t ($> t_0$), this means we wish to live event $e = (t, \frac{s}{c})$ in the future of e_0. Progressing from e_0 to e tacitly assumes a *difficulty* at least because generally we have to change our state at rest (relative to our own laboratory) into a movement with an adequate speed. In practice, because there is no inertial movement, we have to maintain this speed up to the desired location, hence additional difficulty stems from the need to defeat gravity, friction forces etc. Theoretically (in isotropic spaces, during inertial movements, without friction etc.), *difficulty* to progress from e_0 to e depends only on distance, so that an adequate expression seems to be

$$đ(e_0, e) = \frac{|s-s_0|}{c}. \tag{1}$$

Moving with a constant speed v means $s - s_0 = v\,(t - t_0)$, hence

$$đ(e_0, e) = \frac{|v|}{c}(t - t_0).$$

Obviously, the physical dimension of đ is T = *time*, hence the events $e = (t, \frac{s}{c})$ have homogeneous components. In particular, if "*1 second*" is the unit of time, then the unit of space is "*1 light-second*" = 3000.000 km. *Difficulty* $đ(e_0, e)$ is a number between 0, if we remain at rest, and $t - t_0$, which corresponds to a travel at the limit speed c.

On the other hand, in order for us to overcome this difficulty, the relativistic time shows how long we have to live (exist, resist). According to (*), we have to *hope* that we live exactly (or possibly longer) than this proper time between e_0 and e, which transforms $\tau(e_0, e)$ from (*) into

$$ħ(e_0, e) = \sqrt{(t - t_0)^2 - đ^2(e_0, e)}. \tag{2}$$

Similarly, in more details

$$ħ(e_0, e) = \sqrt{1 - \frac{v^2}{c^2}}\,(t - t_0).$$

The physical dimension of ħ is T = *time* too. Contrarily to difficulty, the *hope* to reach e takes the greatest value, namely $t - t_0$, if the entire time we remain at rest ($v = 0$), and vanishes if the needed speed is $v = c$.

In spite of a simple (Pythagorean) connection between *hope* and *difficulty*,

$$ħ^2(e_0, e) + đ^2(e_0, e) = (t - t_0)^2,$$

which "*splits*" time into difficulty and hope, *difficulty* alone isn't enough to deduce *hope*. For example, even constant speeds, but different (say $v_1 > v_2$), may correspond to the same difficulty whenever $|v_1|\ (t_1 - t_0) = s = |v_2|\ (t_2 - t_0)$, while the hope is greater for the longer necessary time (in this case $t_2 - t_0$).

Because the values of difficulty and hope are bounded by 0 and $t - t_0$, we may refer to their *complementary* metrics, *co-difficulty* and *co-hope*, defined by:

$$co\text{đ}(e_0, e) = (t - t_0) - \text{đ}(e_0, e) \text{ and } co\hbar(e_0, e) = (t - t_0) - \hbar(e_0, e).$$

Appropriate practical meanings for cođ are *chance*, *luck*, while that of $co\hbar$ are *shortage* or *give up*.

This change of terminology does not affect the well-known theory; this is a more *subjective* representation of the universe of events, which allows the observation of relativistic phenomena in other fields of study. From a pure physical point of view, the *subjectivity* of the *difficulty* and *chance* results from their dependence on the system of reference; in particular, for an inertial system that is already moving relative to our laboratory along a straight line of equation $s = vt$, there is no difficulty to reach the event $e = (t, s/c)$. But more generally, using these subjective terms is a good interpretation of the philosophy that stresses on the parallel between what *happens* according to the laws of physics and what we (and all living beings) *intend* to do. For living systems, we may remark relativist aspects in the behavior of each parameter (what we try to sustain in Section I.4. about homeostasis).

Terminology "*difficulty – hope*" becomes even closer to our usual experience if instead of c – the speed of light – we refer to some "more terrestrial" maximal speed ¢, e.g. speed of the sound in the air (340m/s), if we are able to develop only subsonic movements. Formally, this is a simple replacement of c by ¢ in (1) and (2), which preserves the isomorphism with $\mathbb{H}$. In this case, *difficulty* must contain additional terms, depending on actual conditions of travel, so that the fundamental inequalities may change. Anyway, we may easily accept inequality (S.a.) for *hope*: the greatest chance of reaching an event is to progress towards it rectilinearly.

The fundamental inequalities relative to *difficulty* and *hope* are immediate: being proportional to the usual metric that measures *distances*, đ satisfies (s.a.). On the other hand, just like τ (based on s.a. of đ), ħ satisfies (S.a.). Obviously, the complementary metrics verify opposite inequalities, namely: cođ is S.a. if and only if đ is s.a., respectively coħ is s.a. if and only if ħ is S.a.

In the proof of the fundamental inequalities there are some hidden hypotheses:

(*i*) The space – relativist or not – is *isotropic*;

(*ii*) The movement is *inertial*.

The case when hypothesis (*i*) fails is the subject of the next section.

If hypothesis (*ii*) fails, then besides the inertial movement between two locations, there are a lot of other paths, which correspond to different roads: inertial in pieces, with detours etc. Consequently, we have to extend formulas (1) and (2), which remain valid only on small (infinitesimal) pieces of road. Because the general idea of difficulty and hope assumes additivity relative to the concatenation of arcs, we may follow the classical technique of constructing integrals. In particular, the difficulty of traveling a particular road naturally depends on the length of that road.

Let $v\colon [t_0, t_1] \to \mathbb{R}^3$ be a continuous function, where $v(\theta) = (v_x(\theta), v_y(\theta), v_z(\theta))$ represents the speed of our system of reference in the (isotropic) three-dimensional space at moment $\theta \in [t_0, t_1]$. Usually, we obtain the components of v by deriving in the parametric equations of the road. The formula for *difficulty* becomes

$$đ(e_0, v, e) = \frac{1}{c}\int_{t_0}^{t} \sqrt{< v(\theta), v(\theta) >}\ d\theta, \tag{3}$$

where $< . , . >$ is the scalar product of $\mathbb{R}^3$ (the integral is the length of the road).

The *hope* to reach e depends on the difficulty that we encounter along the chosen road. For fixed e_0, e_1 and v, we naturally extend (2) to

$$ħ(e_0, v, e_1) = \int_{t_0}^{t_1} \sqrt{1 - \frac{v^2(t)}{c^2}}\ dt. \tag{4}$$

Being defined by integrals, both difficulty and hope is additive relative to the concatenation of routs. Simple examples show that in the general case difficulty and hope correlate differently, i.e. the Pythagorean relation fails. However, the smallest difficulty (respectively the greatest hope) corresponds to the inertial progress between events. In addition, the Pythagorean relation between ħ and đ holds for some mean values. In fact, the mean value theorem in (4) leads to

$$ħ(e_0, v, e_1) = \sqrt{(t_1 - t_0)^2 - đ^2(e_0, v^*, e_1)}\,,$$

where $v^* = v(t^*)$ is a mean value of the function v. Under conditions about the mean values of đ, this is useful in proving inequality (S.a.) for ħ.

The well-known *Space-time*, noted $W = \mathbb{R} \times \mathbb{R}^n$, where $n = 1, 2, 3$, is the standard universe of events in the classical relativist theories. Besides events of the form e = (*time*, *space*), the above study of fundamental inequalities about space and time has highlighted other types of events. So, if $n = 1$, they take the form e = (*time*, *difficulty*), which form the universe $Đ = \mathbb{R} \times \mathbb{R}_+$. Because *speed* is essential in characterizing movement by *difficulty* and *hope*, it is also useful to consider events of the form ε = (*time*, *speed*); their universe will be noted $\Xi = \mathbb{R} \times \mathbb{R}^n$, where $n = 1, 2, 3$. Other types of events will appear in the subsequent Section I.4., where, in particular, we try to describe living beings as relativist dynamical systems in terms of difficulty and hope.

The operations of derivation and integration realize a connection between the universes W and Ξ. The definition of the *difficulty* – đ in formula (1) – realizes a function from W to Đ, which allows a description of the movement in terms of difficulty. In the case $n = 1$, the converse relation is working, i.e. if we know the course of the difficulties, then we may deduce what happens in W and Ξ.

I.3. Anisotropic spaces

So far, inequality (S.a.) was visible only for ħ and cođ, where it derived from the (s.a.) of đ. From now on, we'll see that đ may satisfy (S.a.) too.

As we saw in the previous section, it is often meaningful for each distance ℓ between two points A and B to know the difficulty of covering this distance. This remains valid in our everyday practice too, in Newtonian space and time, difficulty being evaluated by no matter what (energy, fuel, money, etc.). In the simplest case the difficulty is proportional to the distance, but in anisotropic spaces, the difficulty also depends on the direction, so that although the distance verifies inequality (s.a.), the difficulty may check (S.a.).

Example 1 (*Mountain roads*). Let $đ_0$ (>0) be the difficulty of covering the unit distance along a horizontal line. A natural formula for the *difficulty* of traveling a road of length ℓ, from A to B, under the inclination φ, is

$$đ(A, B) = \frac{đ_0 \ell}{1 - k \sin^2 \varphi},$$

where constant k characterizes the vertical direction; we take $k \in (0, 1)$ to avoid infinity. For example, if $k = 0.9$, then the vertical displacement, when φ is either $+ 90^0$ or $- 90^0$, is ten times more difficult than the horizontal one; if $\varphi = 60^0$ it is ~ 3 times greater etc. For a fixed ℓ, the minimal value of $đ(A, B)$, namely $đ_0\, \ell$, corresponds to the horizontal displacement, which is most comfortable.

In particular, let ABC be an equilateral triangle of side ℓ, on a plane of slope 60^0, such that the inclination of AC be $\varphi_1 = 60^0$ too. In an orthogonal system of reference in this plane whose Ox axis is horizontal, the coordinates of the vertices are $A = (0, 0)$, $B = (\frac{\sqrt{3}}{2}, \frac{1}{2})\ell$ and $C = (0, \ell)$. Using the inclination of the other two sides, which is $\varphi_2 = \varphi_3 = \arcsin \sqrt{3}/4$, we obtain

$$đ(A, C) = \frac{4đ_0 \ell}{4 - 3k} \text{ and } đ(A, B) = đ(B, C) = \frac{16đ_0 \ell}{16 - 3k}.$$

It is easy to see that đ satisfies inequality (S.a.) whenever $k > 16/21$. In this case, (S.a.) justifies the use of serpentine roads in mountains.

Obviously, time may influence the choice of the most convenient road, hence we have to consider the events $e_A = (t_A, A)$, $e_B = (t_B, B)$ and $e_C = (t_C, C)$, which correspond to the presence of an observer at these points. The formula of the difficulty to progress from e_A to e_B rewrites $đ(A, B)$ as

$$đ(e_A, e_B) = \frac{đ_0 v}{1 - k \sin^2 \varphi} (t_B - t_A),$$

where v denotes the speed of the movement. Considering that a direct climbing on AC takes as long as the detour through B, it follows that the speed along AC is half of v. Consequently, the difficulties are

$$đ(e_A, e_C) = \frac{2đ_0 v}{4 - 3k} (t_C - t_A) \text{ and } đ(e_A, e_B) = đ(e_B, e_C) = \frac{16đ_0 v}{16 - 3k} (t_B - t_A),$$

hence the previous inequality (S.a.) appears again.

The events e_A, e_B and e_C are essential to evaluate the *hope* of passing from one event to another. Using formula (2) from the previous section, i.e.

$$ħ(e_A, e_B) = \sqrt{(t_B - t_A)^2 - đ^2(e_A, e_B)} \text{ etc.,}$$

we obtain

$$ħ(e_A, e_B) = ħ(e_B, e_C) = \sqrt{t^2 - đ^2(A, B)} \text{ and } ħ(e_A, e_C) = \sqrt{4t^2 - đ^2(A, C)},$$

where $t = t_B - t_A = t_C - t_B$ (> 0). Because đ satisfies (S.a.), it follows that

$$ħ(e_A, e_C) < ħ(e_A, e_B) + ħ(e_B, e_C)$$

whenever $k > 16/21$. Consequently, the hope to progress from e_A to e_C is greater if we follow the serpentine $\widehat{ABC}$.

Example 2 (*Lifeguard*). As we know, swimming speed, u, is much lower than running speed (even on sand), v, say $v = ku$, $k > 1$. From this point of view, the sea shore is an anisotropic space. It is also normal to accept that time is of the essence in saving a life after an accident; a shorter required time means a higher chance of success, so less difficulty. Therefore, it is possible to achieve more success by covering a greater distance.

For example, let s be the distance from point A of the lifeguard to the place B, where a tourist enters the see. After swimming distance ℓ, at point C, the tourist needs help. The question is which route is better: swimming directly from A to C, or running on AB and then swimming on BC? It is easy to see that

$$\frac{1}{u}\sqrt{s^2 + \ell^2} > \frac{s}{v} + \frac{\ell}{u},$$

i.e. the detour $\widehat{ABC}$ is advisable whenever $s > \frac{2k}{k^2-1}\ell$.

The same conclusion results if we reason in terms of difficulties of the form (1) in the previous section. If ¢ is the limit of the running speed (e.g. 10m/s) and ϵ is an upper bound of the swimming speed (e.g. 2m/s), then the difficulties to cover distances AB, BC and AC are:

$$đ(A, B) = \frac{s}{¢},\ đ(B, C) = \frac{\ell}{\epsilon} \text{ and } đ(A, C) = \frac{1}{\epsilon}\sqrt{s^2 + \ell^2}.$$

Similarly, if $s > \frac{2\lambda}{1-\lambda^2}\ell$, where $\lambda = \frac{\epsilon}{¢}$, then $đ(A, C) > đ(A, B) + đ(B, C)$, hence difficulty satisfies (S.a.) in the triangle ABC. For ħ we obtain (s.a.).

Generally, if đ satisfies (S.a.), then ħ must verify (s.a.). To verify this property it is enough to consider $đ(e_A, e_B) = đ(e_B, e_C) = d$, $đ(e_A, e_C) = 2d + \varepsilon$ with $\varepsilon > 0$ (to mark the hypothesis that đ satisfies (S.a.)) and $t_B - t_A = t_C - t_B = t$ (> 0), such that $t_C - t_A = 2t$. If we suppose (S.a.) for ħ, i.e.

$$\sqrt{4t^2 - (2d + \varepsilon)^2} > 2\sqrt{t^2 - d^2},$$

it follows that $\varepsilon(4d + \varepsilon) < 0$, which is impossible.

I.4. Homeostasis

To observe relativistic aspects outside of Einsteinian relativity, it is useful to extend the classical notion of *event*, represented by pairs (*moment*, *place*), to triplets of the form {*moment*, *place*, *what happens*}. This corresponds to the idea that "*whenever* (*at any moment*) and *everywhere* (*at any place*) *something is happening*". The resulting universe of events will be

$$W = T \times S \times \mathcal{A},$$

where T (generally $\subseteq \mathbb{R}$) is *time*, S (frequently $\mathbb{R}^3$) is *space* and $\mathcal{A}$ is a set of *assertions* (e.g. a Boolean algebra of propositions that describe facts).

One of the advantages of considering this extended type of events is the possibility to describe the evolution of a phenomenon in a fixed place. For example, the *speed events* in the previous section correspond to the case where *place* is the laboratory of a moving observer and the *parameter* of interest is its speed in relation to our own laboratory. Likewise, *place* can be our laboratory, $\mathcal{O}$, a dynamic system in it, a living being, etc.; the parameters can be physical, chemical, etc. In the sequel, we will focus on a single scalar parameter p, in a living system, $\mathcal{L}$. *Place* being fixed, we will describe the evolution of p in $\mathcal{L}$ by pairs of the form $\varepsilon = (\theta$ (= *time*), $p)$, called *p-events*, or events of "*color*" p. We note their universe by Ξ.

It is well known that a *standard* scalar parameter, noted p, of a living system, has an optimal, most comfortable value, say p_0. For each class, $\mathfrak{C}$, of living beings (e.g. species, clan etc.), there are some extreme values of the parameter p, noted p^- and p^+, such that life is possible only if $p^- < p < p^+$. For example: the sanguine *pH* has the optimal value $p_0 = 7.4$, while $p^- = 6.8$, and $p^+ = 7.8$; the systolic blood pressure has $p_0 = 120$, $p^- = 80$ and $p^+ = 220$ (in mmHg); the optimal human glucose level is $p_0 = 80$, while $p^- = 0$, and $p^+ = 2000$ (measured in mg/100cc) etc. Obviously, all these values are idealizations: living systems may live pretty well in a *band of normality* around p_0; disease corresponds to some pathological subintervals of (p^-, p^+), and may be of type *hypo* or *hyper*, etc.; in the present context we will neglect such details.

The parallel with the relativist speed-events becomes obvious if we transform the interval (p^-, p^+) into $(-¢, ¢)$ and consider that the optimal reference value is that of our system, chosen $p_0 = 0$. Thus, function $f_¢ : \mathbb{R} \to \mathbb{R}$, of values

$$f_¢(p) = \frac{¢}{p_+ - p_-}\,(2p - p_+ - p_-),$$

carries (p^-, p^+) into $(-¢, ¢)$. If p_0 lies at the middle of (p^-, p^+), then $f_¢(p_0) = 0$; otherwise we may use function $f_{p_0}: \mathbb{R} \to \mathbb{R}$, of the form

$$f_{p_0}(p) = \frac{p - p_0}{¢^2 - p_0^2}\,(p\,p_0 + ¢^2),$$

which transforms $(-¢, ¢)$ into itself and $f_{p_0}(p_0) = 0$.

If $p \in (-¢, ¢)$, we say that $\varepsilon = (\theta, p)$ is a *proper event* of the class $\mathfrak{C}$. Between the proper events there exists a relation of *accessibility*, defined by

$$\mathscr{A} = \{(\varepsilon_1, \varepsilon_2) \in \Xi^2 : |p_1| < ¢,\ |p_2| < ¢,\ \theta_1 < \theta_2\},$$

consisting of those pairs of events $\varepsilon_1 = (\theta_1, p_1)$ and $\varepsilon_2 = (\theta_2, p_2)$, which can be lived successively by a living system.

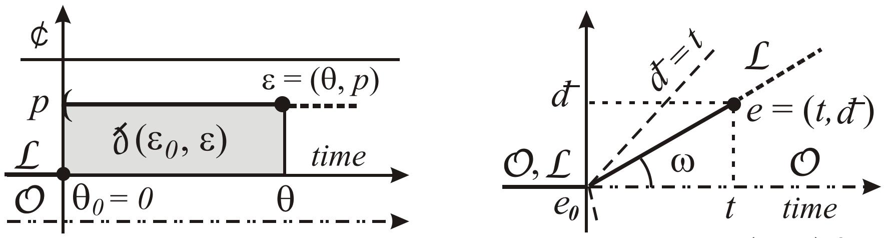
If a living system, $\mathcal{L}$, lives with a constant value of p, we say that it is *inertial* (is implied, relative to p). Let $\mathfrak{I}$ be the set of all inertial living systems.

Let p_0 be the optimal value of p in the inertial system $\mathcal{L}$. Living between the moments θ_0 and θ by a constant p-deviation ($p \neq p_0$) produces a difficulty to $\mathcal{L}$; like in STR, a natural expression of this *difficulty* is

$$ð([\varepsilon_0, p, \varepsilon]) \stackrel{\text{def}}{=} \frac{|p - p_0|}{¢}(\theta - \theta_0), \tag{1}$$

where $\varepsilon_0 = (\theta_0, p_0)$ and $\varepsilon = (\theta, p)$.

Like in space-time, according to (1), the physical dimension of ð is *time*. If we note $\theta - \theta_0 = t$ and $ð([\varepsilon_0, p, \varepsilon]) = đ$, then we may describe the evolution of $\mathcal{L}$ in terms of events of the form $e = (t, đ) \in Đ$ (respectively $W = \mathbb{R}^2$). In spite of the formal similarity of the formulas of difficulty in space-time and living systems, there are essentially different meanings. While in space-time difficulty refers to a preexistent physical quantity – space – wherefrom we derive the usual speed, there is no physical quantity whose derivative is p. Consequently, the difficulty ð, caused by a p-deviation, signifies a new physical quantity.



a) Ξ; $ð = (p - p_0)\theta/¢$ *b*) W; $\tanh \omega = (p - p_0)/¢$

Fig. 1.

Let the event $\varepsilon_0 = (0, p_0)$ represent the jump to a p-deviation in system $\mathcal{L}$. The resulting difficulty suffered by $\mathcal{L}$ up to ε is marked by Fig.1 in both universes of events, Ξ and W. The analogy with special relativity is obvious.

To survive, the attitude of $\mathcal{L}$ towards a difficulty arises from the hope of overcoming it, which is done by homeostasis. Like in relativity, the *hope* of $\mathcal{L}$ to survive from $\varepsilon_0 = (\theta_0, p_0)$ to $\varepsilon = (\theta, p)$, facing a constant p-deviation, is

$$\hbar([\varepsilon_0, p, \varepsilon]) \stackrel{\text{def}}{=} \sqrt{(\theta - \theta_0)^2 - ð^2([\varepsilon_0, p, \varepsilon])} =$$

$$= \frac{1}{¢}\sqrt{¢^2 - (p - p_0)^2} \cdot (\theta - \theta_0). \tag{2}$$

In the case of a constant p-deviation, the same Pythagorean relation shows a “decomposition” of the time into *difficulty* and *hope*:

$$ð^2([\varepsilon_0, p, \varepsilon]) + \hbar^2([\varepsilon_0, p, \varepsilon]) = (\theta - \theta_0)^2.$$

Because difficulty is a positive number in both hypo and hyper p-deviations, it follows that function ð is additive, hence "=" holds in (s.a.). This is enough for $\hbar$ to satisfy (S.a.).

Similarly, we may introduce the *co-difficulty* and the *co-hope*, for which we find properties similar to those in relativist framework. However, the profond relativist character of the living systems derives from the relativist principles. Specifically, we speak of *relativist living systems* (RLS for short) when they are subjected to axioms similar to those of special relativity:

[PR] Principle of Relativity. All inertial living systems similarly perceive the events in one another; each of them may be system of reference.

[PH] Principle of Homeostasis. The fatal limit, ¢, is the same for the entire class $\mathfrak{I}$; by homeostasis, each living system tries to avoid the values $\pm$¢.

Consequently, we may develop a relativist theory of RLS like follows:

The change of the referential system, respectively the adaptation of a living system to another optimal value, generates different perception of the events. So, we are led to the problem of establishing how do change the components of an event (t, đ) if instead of $\mathcal{O}$, the measurements are made by another RLS, $\mathcal{L}$, with a "new" optimal value. The resulting formulas are analogous to the *Lorentz transformations*, but their meaning is different:

Lemma. The change of coordinates of an event, (t, đ) → ($t^{\#}$, đ$^{\#}$), caused by the transition from $\mathcal{O}$ to $\mathcal{L}$, has the form

$$\begin{cases} t^{\#} = \alpha t + \beta đ \\ đ^{\#} = \beta t + \alpha đ. \end{cases} \tag{L}$$

Proof. Since $\mathcal{O}$ and $\mathcal{L}$ are inertial living systems, the change of coordinates shall be linear, i.e.

$$\begin{cases} t^{\#} = \alpha t + \beta đ \\ đ^{\#} = \gamma t + \delta đ. \end{cases}$$

According to the Principles of the RLS, $\mathcal{O}$ and $\mathcal{L}$ do simultaneously perceive the fatal values $\pm$¢. In terms of difficulties, this means

$$[t = đ \Leftrightarrow t^{\#} = đ^{\#}] \text{ and } [t = -đ \Leftrightarrow t^{\#} = -đ^{\#}].$$

Algebraically, $t^{\#} - đ^{\#} = \lambda(t - đ)$ and $t^{\#} + đ^{\#} = \mu(t + đ)$, wherefrom

$$t^{\#} = \frac{\lambda+\mu}{2} t + \frac{\mu-\lambda}{2} đ \text{ and } đ^{\#} = \frac{\mu-\lambda}{2} t + \frac{\lambda+\mu}{2} đ.$$

Consequently, the formulas (L) hold with $\alpha = \frac{\lambda+\mu}{2}$ and $\beta = \frac{\mu-\lambda}{2}$. ■

Theorem (Lorentz). If $\mathcal{O}$ perceives $\mathcal{L}$ as a constant p-deviation, then the other events observed by these systems are connected by the formulas

$$\begin{cases} t^{\#} = \left(t - \frac{p}{¢} đ\right)\left(1 - \frac{p^2}{¢^2}\right)^{-1/2} \\ đ^{\#} = \left(đ - \frac{p}{¢} t\right)\left(1 - \frac{p^2}{¢^2}\right)^{-1/2}. \end{cases} \tag{Lorentz}$$

Proof. System $\mathcal{O}$ evaluates the p-deviation of $\mathcal{L}$ as a difficulty đ = $\frac{p}{¢}t$, while for $\mathcal{L}$ there is no difficulty, đ$^{\#}$ = 0. The second relation (L) gives

$$\beta = -\frac{p}{¢}\,\alpha. \qquad \text{(I)}$$

According to the same relation, at the moment $t = 0$, $\mathcal{O}$ evaluates a difficulty đ$^{\#}$ = $\Delta \neq 0$, of $\mathcal{L}$, as being

$$đ = \Delta / \alpha. \qquad \text{(II)}$$

A difficulty đ = Δ, of $\mathcal{O}$, measured by $\mathcal{L}$ at the moment $t^{\#} = 0$, according to the first relation (L), is perceived by $\mathcal{O}$ at the moment

$$t = \frac{p}{¢}\,\Delta. \qquad \text{(III)}$$

In accordance to the second relation (L), system $\mathcal{L}$ estimates difficulty đ = Δ, felt by $\mathcal{O}$ at the moment $t = \frac{p}{¢}\,\Delta$, given by (III), at the value

$$đ^{\#} = \beta\,\frac{p}{¢}\,\Delta + \alpha\,\Delta. \qquad \text{(IV)}$$

The Principle of relativity says that evaluations (II) and (IV) shall coincide, so we obtain

$$\alpha = \frac{1}{\sqrt{1-\left(\frac{p}{¢}\right)^2}}\,. \qquad \text{(V)}$$

From (I) we deduce β, and (L) become (Lorentz). ■

It is well known that many results of Einstein's Special Theory of Relativity are consequences of Lorentz's theorem. Formulating similar properties for RLS is a simple exercise that we leave to the reader.

If $\mathcal{L}$ has a non-inertial evolution during the period $[\theta_0, \theta_1]$ of life, then we need to know function $\varphi : [\theta_0, \theta_1] \to (-¢, ¢)$, for which $p = \varphi(\theta)$ at each $\theta \in [\theta_0, \theta_1]$. Supposing that φ is piecewise continuous, we may define the *difficulty* of $\mathcal{L}$ to live a φ-interval $[\theta_0, \varphi, \theta] = \{(\tau, \varphi(\tau)) \in \Xi : \tau \in [\theta_0, \theta]\}$ by an integral:

$$ð([\theta_0, \varphi, \theta]) \overset{\text{def}}{=} \frac{1}{¢}\int_{\theta_0}^{\theta} |\varphi(\tau) - p_0|\,d\tau. \qquad (3)$$

Like in Relativity, an appropriate expression of *hope* is

$$\hbar([\theta_0, \varphi, \theta_1]) \overset{\text{def}}{=} \int_{\theta_0}^{\theta_1} \sqrt{1 - \frac{1}{¢^2}(\varphi(\theta) - p_0)^2}\; d\theta. \qquad (4)$$

Obviously, the Pythagorean relation between ð and ħ is not valid any more. However, it takes place for some mean values of ð and ħ. Thus, the mean value theorem of the integral (4) shows that

$$\hbar([\theta_0, \varphi, \theta_1]) = \sqrt{(\theta_1 - \theta_0)^2 - ð^2([\theta_0, p^*, \theta_1])}\ ,$$

where $p^* = \varphi(\theta^*)$ is a mean value of φ. If the additivity of ð remains valid for mean values too, it follows that (S.a.) holds for ħ in the general case of non-inertial living systems.

I.5. Mathematical Logic

It is interesting that mathematicians have nonchalantly ignored inequality of super-additivity even though it stood in front of them at every moment of mathematical activity. Undoubtedly, they noticed that understanding each mathematical statement presents a degree of *difficulty*, which can be diminished by demonstrations. It is true that measuring such a *difficulty* is itself very difficult. One way to assess the difficulty of proving a statement in mathematics (but also in other fields) can be to count the intermediate facts between hypothesis (noted H) and conclusion (say C). Regardless of how we measure the difficulty, this function reverses the well-known "triangle rule".

More exactly, let đ(H, C) be the difficulty of proving a theorem that states that the hypothesis H implies the conclusion C. If F is an intermediate property (fact, logical formula, etc.) contained in the proof, i.e. $H \Rightarrow F \Rightarrow C$, then

$$đ(H, C) \geq đ(H, F) + đ(F, C).$$

This inequality even justifies performing the demonstration (sketched in Fig.1.).

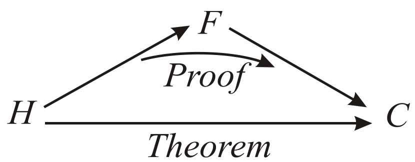


Fig. 1. Why proof?

Even if the *HFC* triplet is far from a geometric (visible) triangle, the super-additivity of the functional đ is obvious.

The subjectivism of appreciating a difficulty in carrying out a spiritual activity – understanding, study, conviction – derives from the strong dependence on previously formed knowledge (experience, conceptions, religion), or even on genetically acquired ones. Specialists also appreciate with great subjectivity the difficulty of proving a theorem (solving a problem, etc.), also depending on the knowledge they have previously acquired. Among the evaluation options we notice checking the time required for the demonstration and the length of the text, especially in formal logic and computer programming.

However, there is a simple case in which we can indicate an objective way of assessing the difficulty of understanding an implication, namely that of logical implication in Boolean algebra of sentences. Here, the relation of implication, denoted $A \Rightarrow B$, is defined by the fact that "*B is true whenever A is true*", A and B being obtained by various logical operations from several constituent sentences P_1, P_2, ..., P_n. In order to find that the implication is taking place, it is sufficient to compare the truth tables of sentences A and B, from the point of view of the cases when they are true.

The same tables allow us to find the number of „intermediate” sentences, X, defined by $A \Rightarrow X \Rightarrow B$. If we admit that an implication is all the more difficult to establish as there are more intermediate sentences between A and B, then the number of these sentences can be considered a measure of the *difficulty* of understanding that implication. More exactly,

$$đ(A, B) = card\ \{X: A \Rightarrow X \Rightarrow B\}.$$

For example, if for the sentence A the value "*true*" appears a times and for B it appears b times, then between A and B can be interspersed

$$đ(A, B) = 2^{b-a} - 1$$

sentences, where $b \geq a$ takes place because of $A \Rightarrow B$.

To align with the general theory, we mention that

$$\mathcal{A} = \{(A, B): A \Rightarrow B\}$$

is a strict order relation, and the function $đ: \mathcal{A} \to \mathbb{R}_+$ is a (S.a.) metric. In fact,

$$đ(A, C) \geq đ(A, B) + đ(B, C)$$

takes place whenever $(A, B), (B, C) \in \mathcal{A}$ because

$$2^{c-a} - 1 \geq 2^{b-a} - 1 + 2^{c-b} - 1,$$

which finally reduces to $(2^{b-a} - 1)(2^{c-b} - 1) \geq 0$.

Although the theory of relativity is far from that of Boolean algebras, there is a resemblance between S.a. from these fields, namely relativistic proper time, respectively the difficulty of logical implication (sketched in Fig. 2.). Primarily, relation $\mathcal{A}$ of implication is similar to causality.

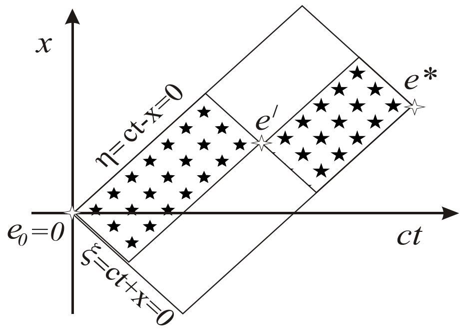


Fig. 2. Counting events

If we note $ct + x = \xi$ and $ct - x = \eta$, then the proper time between the space-time events $e_0 = (0, 0)$ and $e = (ct, x)$ is $\tau(e_0, e) = \frac{1}{c}\sqrt{\xi\eta}$ (compare to (*) in Section I.2). The area $\xi\eta$ (for which the inequality S.a. is evident in Fig.2) is a measure of the content of events between e_0 and e. If space-time is quantified (granular), these events can be counted, as well as the sentences in a Boolean interval.

I.6. Two-dimensional Calculus

The study of the elastic plate's deformations has inspired mathematicians to develop a theory known as *Two-dimensional Calculus* or *Hyperbolic Analysis*. For me, it was the second sign of destiny to dedicate myself to the study of S.a., especially because I could also benefit from the support of exceptional mentors.

The main idea was to replace the usual increment $\Delta x = x - x_0$ of the classical calculus on $\mathbb{R}$ by a two-dimensional increment $\Delta_2((x_0, y_0), (x, y)) = \Delta x\, \Delta y$ in $\mathbb{R}^2$ and the resulted increment of a function, $\Delta f(x_0, x) = f(x) - f(x_0)$, by

$$\Delta_2 f((x_0, y_0), (x, y)) = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0).$$

In this way, mathematicians obtained a lot of specific results similar to those in the classical calculus on continuity, convergence, derivation and integration of the functions of two real variables. A typical example is the replacement of the second order mixed derivative $\frac{\partial^2 f}{\partial x \partial y}$ of a function defined on a domain from $\mathbb{R}^2$ by a „*direct*" *two-dimensional derivative*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \lim_{\Delta x \Delta y \to 0} \frac{f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0)}{(x-x_0)(y-y_0)}.$$

In fact, hyperbolic calculus has faced serious criticism because in this theory the notion of limit has no topological character. While in classical analysis the distance $|x - x_0| = \rho(x_0, x)$ is the value of a common (s.a.) metric, the area

$$|\Delta x\, \Delta y| = \rho_2((x_0, y_0), (x, y))$$

is no longer given by a metric, or, more precisely, is the value of a S.a. metric (as seen in Fig. 1.).

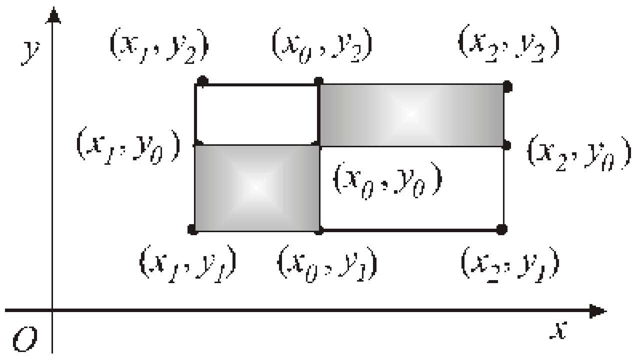


Fig. 1.

Property S.a. of the metric ρ_2 was of no use in the two-dimensional calculus. However, ρ_2 can be considered as a quaternary restrained s.a. metric, defined on quartets of points in the plane, which are vertices of rectangles with sides parallel to the coordinate axes. Thus it is revealed that the hyperbolic analysis has its own topological structure, called *second order topology*. To illustrate the effort required to avoid S.a. and to construct a classical calculus, based on topological structures, we will sketch how quaternary metrics generate second-order topologies. The advantage of these general structures is the structural unification of most of the results of the hyperbolic analysis.

By *quaternary relation*, also called *2-relation*, in the arbitrary non-void set X, we understand any part of X^4, generically noted ${}^2\mathrm{R}$. A 2-relation ${}^2\mathrm{R}$ in X is called *2-preorder* if it is reflexive (i.e. ${}^2\mathrm{R} \supseteq {}^2\text{д} = \{(x, x, x, x) : x \in X\}$) and transitive (i.e. ✠ $({}^2\mathrm{R}) \subseteq {}^2\mathrm{R}$, where ✠ denotes the quaternary composition of 2-relations). If ${}^2\mathrm{R}$ is a 2-preorder, then function ${}^2\rho : {}^2\mathrm{R} \to \mathbb{R}_+$ is a *restrained pseudo quasi metric of the second order* on X (briefly 2-r-p-q-metric) if:

[${}^2\mathrm{M}_{1\mathrm{R}}$] ${}^2\rho({}^2\text{д}) = \{0\}$;

[${}^2\mathrm{M}_{2\mathrm{R}}$] If $(x_{11}, x_{10}, x_{01}, x_{00}) \in {}^2\mathrm{R}$, $(x_{10}, x_{12}, x_{00}, x_{02}) \in {}^2\mathrm{R}$, $(x_{01}, x_{00}, x_{21}, x_{20}) \in {}^2\mathrm{R}$ and $(x_{00}, x_{02}, x_{20}, x_{22}) \in {}^2\mathrm{R}$, then ${}^2\rho(x_{11}, x_{12}, x_{21}, x_{22}) \le {}^2\rho(x_{11}, x_{10}, x_{01}, x_{00}) + {}^2\rho(x_{10}, x_{12}, x_{00}, x_{02}) + {}^2\rho(x_{01}, x_{00}, x_{21}, x_{20}) + {}^2\rho(x_{00}, x_{02}, x_{20}, x_{22})$.

The triplet $(\mathrm{X}, {}^2\mathrm{R}, {}^2\rho)$ is a r-p-q-metric space of the second order.

We say that filter ${}^2\mathscr{U} \subseteq \mathscr{P}(X^4)$ is a *second order uniformity* (briefly *2-u-topology*) on X if the following conditions hold:

[${}^2\mathrm{U}_1$] ${}^2\text{д} \subseteq {}^2\mathrm{U}$ for all ${}^2\mathrm{U} \in {}^2\mathscr{U}$ (reflexivity);

[${}^2\mathrm{U}_2$] ${}^2\mathrm{U} \in {}^2\mathscr{U} \Rightarrow {}^2\mathrm{U}^{/} \cap {}^2\mathrm{U}^{//} \in {}^2\mathscr{U}$ (symmetry);

[${}^2\mathrm{U}_3$] $\forall\, {}^2\mathrm{U} \in {}^2\mathscr{U}\ \exists\, {}^2\mathrm{V} \in {}^2\mathscr{U}$ such that ✠ $({}^2\mathrm{V}) \subseteq {}^2\mathrm{U}$.

The pair $(X, {}^2\mathscr{U})$ is a *2-uniform* (topological) space. It may be generated by a 2-r-p-q-metric and generates a *second order topology*. Family ${}^2\mathscr{V}(x) \subseteq \mathscr{P}(X^3)$ is a system of *second order neighborhoods* (briefly *2-neighborhoods*) of x if:

[${}^2\mathrm{N}_1$] $(x, x, x) \in {}^2V$ for each ${}^2V \in {}^2\mathscr{V}(x)$;

[${}^2\mathrm{N}_2$] If ${}^2V \in {}^2\mathscr{V}(x)$ and ${}^2U \supseteq {}^2V$, then ${}^2U \in {}^2\mathscr{V}(x)$;

[${}^2\mathrm{N}_3$] If ${}^2U, {}^2V \in {}^2\mathscr{V}(x)$, then ${}^2U \cap {}^2V \in {}^2\mathscr{V}(x)$;

[${}^2\mathrm{N}_4$] For every ${}^2V \in {}^2\mathscr{V}(x_{11})$ there exists ${}^2W_1 \in {}^2\mathscr{V}(x_{11})$ such that for every $(x_{10}, x_{01}, x_{00}) \in {}^2W_1$, there exist ${}^2W_2 \in {}^2\mathscr{V}(x_{10})$, ${}^2W_3 \in {}^2\mathscr{V}(x_{01})$ and ${}^2W_4 \in {}^2\mathscr{V}(x_{00})$ such that $(x_{12}, x_{21}, x_{22}) \in {}^2V$ holds whenever
$(x_{12}, x_{00}, x_{02}) \in {}^2W_2$, $(x_{00}, x_{21}, x_{20}) \in {}^2W_3$ and $(x_{02}, x_{20}, x_{22}) \in {}^2W_4$.

The function ${}^2\tau : X \to \mathscr{P}(\mathscr{P}(X^3))$, which attaches to each $x \in X$ a *system of 2-neighborhoods* of x, i.e. ${}^2\tau(x) = {}^2\mathscr{V}(x)$, is a *second order topology*.

Let $(X, {}^2\tau_X)$ and $(Y, {}^2\tau_Y)$ be 2-topological spaces, $A \subseteq X$, a be a 2-accumulation point of A, and $f : A \to Y$. We say that the element ${}^2\ell \in Y$ is the *2-limit* of f at a, and we note ${}^2\ell = 2 - \lim_{x \to a} f(x)$, if

$$\forall\, V^2\tau_Y({}^2\ell)\ \exists\, U \in {}^2\tau_X(a) \text{ such that } f_{\mathrm{III}}((U \setminus \{(a, a, a)\}) \cap A^3 \subseteq V .$$

If $a \in A$ and $2 - \lim_{x \to a} f(x) = f(a)$, then we say that f is *2-continuous* at a.

The other notions (2-convergence, 2-derivative, etc.) are defined by analogy to the usual calculus.

I.7. $\boldsymbol{\ell_p}$ and $\boldsymbol{\mathcal{L}_p}$ with $\boldsymbol{0 < p < 1}$

From a strictly mathematical point of view, it is easy to arrange some formulas so as to obtain S.a. norms and metrics, even in spaces that are usually endowed with classical topological structures. Now, a few demonstrations are outlined to combat some extremist views that S.a. it is not correct (exaggerating the fact that it is so rarely used).

For each strictly positive p, ℓ_p is an extension of $\mathbb{R}^n$, which denotes the linear space of sequences $x{:}\mathbb{N} \to \mathbb{R}_+$ such that $\sum_{k\in\mathbb{N}}(x_k)^p$ is convergent relative to the usual topology of $\mathbb{R}$. The norm of ℓ_p is defined by $\|x\|_p = [\sum_{k\in\mathbb{N}}(x_k)^p]^{\frac{1}{p}}$. If $p \geq 1$, then $\|\cdot\|_p$ is an usual, s.a. norm, but in the case $0 < p < 1$ it satisfies S.a. In this case, a simple proof of the inequality

$$\|x + y\|_p \geq \|x\|_p + \|y\|_p$$

uses the concavity of the function $f(x) = x^p$. In fact, for each k component of x and y, and for each $t \in (0,1)$, the hypothesis f-concave gives

$$(x_k + y_k)^p = \left(t\,\frac{x_k}{t} + (1-t)\,\frac{y_k}{1-t}\right)^p \geq t\,\frac{x_k^p}{t^p} + (1-t)\,\frac{y_k^p}{(1-t)^p}\,.$$

What remains is taking the sum (finite in $\mathbb{R}^n$ and implicitly convergent in ℓ_p) and replace $t = \frac{\|x\|_p}{\|x\|_p + \|y\|_p}$.

A similar reason, based on the convexity of $g(x) = x^p$ in the case $p \geq 1$, proves inequality s.a. of the norm $\|\cdot\|_p$. The Euclidean norm corresponds to $p = 2$, but unfortunately, at least as far as I know, there is no field of application for its pair S.a., $\|\cdot\|_{\frac{1}{2}}: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of values $\|x\|_{\frac{1}{2}} = (\sqrt{x_1} + \sqrt{x_2})^2$.

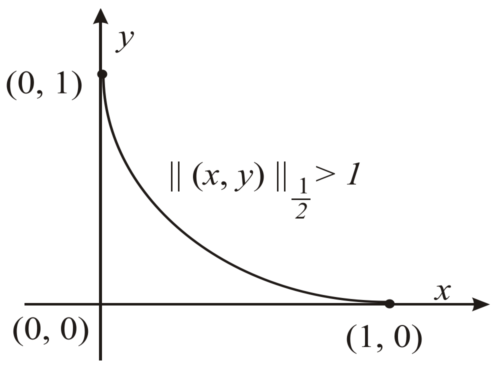


Fig. 1. $\|\cdot\|_{\frac{1}{2}}$ S.a.

Figure 1 presents a geometrical interpretation of the property S.a. of $\|\cdot\|_{\frac{1}{2}}$, namely the convexity of the set $P_r = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : \|(x, y)\|_{\frac{1}{2}} > r\}$, $r > 0$. In this case, their complements $N_r = \mathbb{R}_+ \times \mathbb{R}_+ \setminus P_r$, even *concave*, form a basis of neighborhoods for the Euclidean topology retrained to the product order.

We meet a similar situation in abstract measure spaces (X, $\mathscr{A}$, μ). As usually, $\mathcal{L}_1(X, \mathscr{A}, \mu)$, briefly $\mathcal{L}_1$, consists of real (or complex) functions defined μ-a.e. and integrable on X. For each real $p > 0$, $\mathcal{L}_p$ denotes the set of functions for which $|f|^p \in \mathcal{L}_1$. This linear space is naturally endowed with the norm

$$\|f\|_p = \left[\int_X |f|^p d\mu\right]^{\frac{1}{p}},$$

which behaves differently depending on p: If $p \geq 1$, $\|\cdot\|_p$ is an usual s.a. norm, but if $0 < p < 1$, then it satisfies S.a. The difference comes from the Hölder inequality, which depends on p.

An important topic in the measure theory concerns the *conjugate* (*adjoint*, *dual*) spaces; if E is a normed space, then its conjugate, noted E^*, is a Banach space consisting of all linear and continuous functionals on E. If $\frac{1}{p} + \frac{1}{p'} = 1$, wherefrom $p' = \frac{p}{p-1}$, then $\mathcal{L}_{p'}$ is the conjugate of $\mathcal{L}_p$. This is a consequence of the fact that if $g \in \mathcal{L}_{p'}$ and L_g is defined by

$$L_g(f) = \int_X f\bar{g}\, d\mu,$$

then L_g is a linear and continuous functional on $\mathcal{L}_p$, and $\|L_g\| = \|g\|_{p'}$. The main ingredient of the proof is the Hölder's inequality in the case $p \geq 1$,

$$\left| \int_X f\bar{g}\, d\mu \right| \leq \|f\|_p \, \|g\|_{p'} \qquad \text{(Hölder s.a.)}$$

In the case of $0 < p < 1$ we obtain the reversed Hölder inequality,

$$\left| \int_X f\bar{g}\, d\mu \right| \geq \|f\|_p \, \|g\|_{p'} \, . \qquad \text{(Hölder S.a.)}$$

This inequality derives from (Hölder s.a.) if we note $q = \frac{1}{p}$, consider $\varphi = g^{-\frac{1}{q}}$ and $\psi = g^{\frac{1}{q}} f^{\frac{1}{q}}$, such that $\varphi \in \mathcal{L}_{q'}$ and $\psi \in \mathcal{L}_q$ with $q, q' > 1$.

Finally, (Hölder S.a.) shows that $\|\cdot\|_p$ is a S.a. norm, which blocks the theory in the case $p \geq 1$. In fact, there is no more topology, no continuous functionals and no classical conjugate. That is why the efforts to study the conjugate of an $\mathcal{L}_p$ space with $0 < p < 1$ in the classical sense had poor results. In the next Chapter we'll see a coherent solution of this problem in the sense of a conjugate space consisting of discrete functions in horistological framework.

I.8. Concave sets

The topological structures on an arbitrary linear space X operate with special types of sets. Thus, a part $A \subseteq X$ is *convex* if for all $x, y \in A$ we have

$$[x, y] = \{(1 - \lambda)\, x + \lambda\, y : \lambda \in [0, 1]\} \subseteq A\,.$$

A set A is *balanced* if

$$[x \in A \text{ and } |\, \alpha \,| \leq 1] \Rightarrow [\alpha\, x \in A]\,.$$

Finally, A is *absorbent* if for all $x \in X$ there exists $\mu > 0$ such that

$$[\nu \in \Gamma, |\, \nu \,| \leq \mu\,] \Rightarrow [\nu\, x \in A]\,.$$

Consequently, if $\|\cdot\| : X \to \mathbb{R}_+$ is a semi-norm, then

$$A = \{x \in X : \|\, x \,\| < 1\}$$

is convex, balanced and absorbent.

Conversely, if A is a convex, balanced and absorbent part of X, then (the so-called *Minkowski functional*) $p : X \to \mathbb{R}_+$, of values

$$p(x) = \inf \{\lambda \in \mathbb{R}_+ : x \in \lambda\, A\}$$

is a (s.a.) semi-norm.

Similarly, we may use convexity and additional types of sets to generate S.a. Minkowski functionals on ordered real linear spaces.

Let R be a preorder on the real linear space X, and let $\mathrm{P} = \mathsf{R}[0]$ be the cone attached to R. We say that a part A of X is *eventual in* P if $0 \notin A \subseteq P$ and

$$\forall x \in \mathrm{P} \setminus \{0\} \;\; \exists \alpha \in \mathbb{R}_+ \text{ such that } [\beta \geq \alpha] \Rightarrow [\beta\, x \in A]\,.$$

A set $A \subseteq$ P is *withdrawable* in P if

$$\forall x \in \mathrm{P} \;\exists \mu \in \mathbb{R}_+ \text{ such that } [\nu \leq \mu] \Rightarrow [\nu\, x \notin A]\,.$$

It is easy to see that if $\nmid \cdot \nmid\, : \mathrm{P} \to \mathbb{R}_+$ is a S.a. norm on X, then set

$$A = \{x \in \mathrm{P} : \nmid x \nmid\, > 1\}$$

is convex, eventual and withdrawable in P. Conversely, if a set $A \subset \mathrm{P}$ is convex, eventual and withdrawable, then the functional $q : \mathrm{P} \to \mathbb{R}_+$, of values

$$q(x) = \sup \{\lambda \in \mathbb{R}_+ : x \in \lambda\, A\}$$

is a S.a. norm.

Using restrained norms we may extend the relativist space-time to a *normed universe of events* happening in a real normed space $(X, \|\cdot\|)$, noted $W = \mathbb{R} \times X$. In this universe of events we distinguish the *causality*

$$\mathrm{K} = \{((t, x), (s, y)) \in W^2 : s - t > \|\, x - y \,\|\},$$

on which $\|\cdot\|_t : \mathrm{K}[0] \to \mathbb{R}_+$, of values $\|\,(t, x)\,\|_t = \sqrt{t^2 - \|x\|^2}$, is a S.a. norm.

Restricting S.a. norms and metrics to some preorders is a necessary process, but it may be applied to s.a. norms and metrics too. Because there are restrained s.a. norms that cannot be extended to the entire space, the result is a possibility to describe more topological properties in terms of metrics.

I.9. Chrono-Geometry

Many geometrical properties of the Euclidean plane correspond to similar properties in the plane $W = \mathbb{R} \times \mathbb{R}$ of events happening on a straight line. Term *chrono-geometry* intends warning that these properties involve both space and time, even if we can't draw time. Concretely, the problem is to replace the Euclidean scalar product of the points $x = (x_1, x_2)$, $y = (y_1, y_2)$, i.e.

$$< x \mid y > = x_1 y_1 + x_2 y_2,$$

by the indefinite inner product of the events $e = (t, x)$, $\varepsilon = (s, y)$, namely

$$(e \mid \varepsilon) = ts - xy .$$

This replacement naturally requires reconsideration of all derived notions, like *distance*, *orthogonality*, *angle*, *bisector* of a segment or of an angle, etc. Even if we may formulate twin problems, the anisotropy of $(W, (\, . \mid . \,))$ will permanently make the difference, so that Chrono-Geometry cannot be obtained from the Euclidean geometry by simple change of notation. The novelty is that we obtain properties with physical interpretations in terms of inertial observers, spatial positions and temporal sequences.

First of all, relative to the origin, we distinguish three types of events: spatial, temporal and light. Consequently, there are three types of vectors. In particular, two orthogonal vectors must be of different types or both light; one of them is an inertial observer and the other consists of simultaneous spatial positions from its point of view. Orthogonality, meaning $(e \mid \varepsilon) = 0$, has nothing to do with a particular angle, since the notion of *angle* makes sense only for pairs of half straight lines that meet the hyperbola branch from the future cone. A serious consequence is the reconsideration of the notion of a *triangle*; it cannot consist of three angles, so we may consider only *three-events* or *three-laterals*.

In metric framework, “distance” takes a temporal feature; thus, the radius of a hyperbola shows how long we have to live from the center to an arbitrary event on the hyperbola. Power of a point, rules of congruence, tangency etc. need proofs in new reformulations.

Chrono-geometry is a kind of hyperbolic geometry, where the role of circles is played by hyperbolas. For example, the problem of four equal circles becomes a property of hyperbolas: “If three hyperbolas of equal radiuses have a common event, then the other three pairwise intersections belong to a hyperbola of the same radius.” A simple proof follows by using hyperbolic numbers.

Similarly, the Fermat-Torricelli problem of finding points that minimize the sum of distances to several fixed points in a plane has nice counterparts in the universe of events; the interpretation says that if we intend to send information from e_1 to both e_2 and e_3, by particles / observers or signals, then we shall find where and when to split the message. Torricelli event gives the solution.

The list of such problems seems endless.

I.10. Inner Products

Inner products represent extensions of the well-known scalar products, which concentrate the usual geometric properties. Generally, a function

$$(\,.\, | \,.\,) : X \times X \to \Gamma$$

is an *inner product* on a linear space X over Γ if the following conditions hold:

[IP$_1$] $(\alpha x + \beta y \,|\, z) = \alpha\,(x \,|\, z) + \beta\,(y \,|\, z)$, $\forall\, x, y, z \in X$, $\forall\, \alpha, \beta \in \Gamma$ (linearity);

[IP$_2$] $(x \,|\, y) = \overline{(y \,|\, x)}$, $\forall\, x, y \in X$ (conjugate symmetry).

The pair $(X, (\,.\, | \,.\,))$ denotes an *inner product space*. If, in addition,

[IP$_3$] $[(x \,|\, y) = 0$ for all $y \in X] \Rightarrow [x = 0]$ (non-degeneracy),

then $(\,.\, | \,.\,)$ is a *non-degenerate* inner product, respectively $(X, (\,.\, | \,.\,))$ is a *non-degenerate* inner product space. In the contrary case, x is called *isotropic* vector, and $(X, (\,.\, | \,.\,))$ is a *degenerate* inner product space. X^0 denotes the set of all isotropic vectors, and forms the *isotropic part* (*subspace*) of X.

In connection with (S.a.), we notice the universe of the events that may occur in a real (pre)Hilbert space $(\mathsf{S}, < . \,|\, . >)$. Let $W = \mathbb{R} \times \mathsf{S}$ be the universe of these events, organized as a linear space. Such an event is a pair $e = (t, x)$, where $t \in \mathbb{R}$ means *time*, and $x \in \mathsf{S}$ is *place*. The functional $(\,.\, | \,.\,) : W \times W \to \mathbb{R}$, of values

$$(e_1 \,|\, e_2) = c^2\, t_1\, t_2 - < x_1 \,|\, x_2 > ,$$

where $c \in \mathbb{R}_+^*$, $e_k = (t_k, x_k)$, $k = 1, 2$, is an inner product on W.

If X is a complex linear space of finite dimension, $n \in \mathbb{N}^*$, then each inner product $(\,.\, | \,.\,)$ may be represented by a Hermitian matrix $A = (a_{i\,j})$, which means $a_{ij} = \overline{a_{ji}}$, such that $(X, (\,.\, | \,.\,))$ is isomorphic to $(\mathbb{C}^n, (\,.\, | \,.\,)_A)$, namely

$$(x \,|\, y) = ((x_1, x_2, \ldots, x_n) \,|\, (y_1, y_2, \ldots, y_n))_A = \sum_{i,j=1}^{n} a_{ij} x_i \overline{y_j} \,.$$

If X is a real linear space, then A is simply symmetric.

Property $(x \,|\, x) \in \mathbb{R}$ offers the possibility of classifying the vectors of the inner product space $(X, (\,.\, | \,.\,))$ according to the sign of $(x \,|\, x)$. Thus, $x \in X$ is:

- *positive* if $(x \,|\, x) > 0$;
- *negative* if $(x \,|\, x) < 0$;
- *neutral* if $(x \,|\, x) = 0$;

From here results a classification of spaces. We say that $(X, (\,.\, | \,.\,))$ is:

- *indefinite* if there exist $x, y \in X$ such that $(x \,|\, x) > 0$ and $(y \,|\, y) < 0$;
- *positive semi-definite* if $(x \,|\, x) \geq 0$ for all $x \in X$;
- *negative semi-definite* if $(x \,|\, x) \leq 0$ for all $x \in X$;
- *positive definite* if $(x \,|\, x) > 0$ for all $x \in X$, $x \neq 0$;
- *negative definite* if $(x \,|\, x) < 0$ for all $x \in X$, $x \neq 0$;
- *neutral* if $(x \,|\, x) = 0$ for all $x \in X$.

This classification extends to linear subspaces and other parts of X:

$$\mathfrak{P}^{+} = \{x \in X : (x \mid x) \geq 0\};\ \mathfrak{P}^{++} = \{x \in X : (x \mid x) > 0\} \cup \{0\};$$
$$\mathfrak{P}^{-} = \{x \in X : (x \mid x) \leq 0\};\ \mathfrak{P}^{--} = \{x \in X : (x \mid x) < 0\} \cup \{0\};$$
$$\mathfrak{P}^{0} = \{x \in X : (x \mid x) = 0\};\ \mathfrak{P}^{00} = \{x \in \mathfrak{P}^{0} : x \neq 0\}.$$

The inner product space (X, (. | .)) is indefinite if and only if $\mathfrak{P}^0 + \mathfrak{P}^0 = X$, but $\mathfrak{P}^0 \neq X$. In the contrary case, (X, (. | .)) is semi-definite if each neutral vector is isotropic, i.e. $\mathfrak{P}^0 = X^0$.

The universe $W = \mathbb{R} \times \mathsf{S}$ is an example of *indefinite* inner product space, where we may recognize a lot of properties of the relativist space-time. In particular,

$$\mathrm{K} = \{(e_1, e_2) \in W^2 : e_2 - e_1 \in \mathfrak{P}^{++},\ t_2 > t_1\}$$

represents causality (accessibility). The signal relation involves neutral events

$$\Sigma = \{(e_1, e_2) \in W^2 : e_2 - e_1 \in \mathfrak{P}^{0},\ t_2 \geq t_1\},$$

and the relation of incidence refers to negative events

$$\Xi = \{(e_1, e_2) \in W^2 : e_2 - e_1 \in \mathfrak{P}^{--} \setminus \{0\}\}.$$

In all studies on scalar products (positively defined inner products) the notions of quadratic form, $Q(x) = \langle x \mid x \rangle$, and the norm $\|x\| = \sqrt{Q(x)}$, are mentioned. Then, the Cauchy-Bunjakowski-Schwartz inequality

$$|(x \mid y)|^2 \leq (x \mid x)(y \mid y) \tag{*}$$

is proved, with the help of which it is shown that $\rho(x, y) = \| x - y \|$ is a s.a. metric. So, (X, < . | . >) is a linear topological space. If it is also complete relative to the uniform topology generated by ρ, it is called a Hilbert space.

Unexpectedly, no monograph on indefinite products mentions the presence in these spaces of the inverse inequality, (S.a.), perhaps because it generates no topology, hence no functional analysis. However, a classical theory is available in the decomposable (Krein) spaces, where $X = L \overset{\perp}{\oplus} M \overset{\perp}{\oplus} N$, with $L \subset \mathfrak{P}^{--}$, $M \subset \mathfrak{P}^{0}$, and $N \subset \mathfrak{P}^{++}$. The completeness of these subspaces relative to the intrinsic uniform topologies (and $M = \{0\}$) transforms X into a Hilbert space.

The study of the fundamental inequality in indefinite inner product spaces requires a couple of auxiliary results:

Lemma 1. Let (X, (. | .)) be an inner product space, and $x, y \in X$. If the inner product subspace $\mathrm{Lin}\{x, y\}$ is indefinite, then:

a) x and y are linearly independent, and

b) there exist $\lambda_1, \lambda_2 \in \Gamma$ such that $x + \lambda_1 y \in \mathfrak{P}^{++}$ and $x + \lambda_2 y \in \mathfrak{P}^{--}$.

Lemma 2. Let (X, (. | .)) be a finite dimensional complex inner product space, and let $\underline{B} = \{e_1, e_2, \ldots, e_n\}$ be a base of X, such that

$$(e_k \mid e_l) \in \mathbb{R},\ \forall\, k, l \in \{1, 2, \ldots, n\}.$$

If we endow the real linear space $Y = \mathrm{Lin}_{\mathbb{R}}\{\underline{B}\}$ with the restriction of the inner product (. | .) to $Y \times Y$, noted $(. \mid .)_{Y \times Y}$, then (Y, $(. \mid .)_{Y \times Y}$) has the same nature as (X, (. | .)).

Now, we can show that the inequality opposite to (*), known as Aczél - Varga inequality, is specific to indefinite inner product spaces.

Theorem. Let $(X, (\,.\,|\,.\,))$ be an arbitrary inner product space, and $x, y\in X$. If the inner product subspace $\mathrm{Lin}\{x, y\}$ is indefinite, then

$$|(x\,|\,y)|^2 > (x\,|\,x)\,(y\,|\,y)\,. \qquad (**)$$

Proof. According to Lemma 1.*a*), vectors x and y are linearly independent, hence $\underline{B} = \{x, y\}$ is a base of $\mathrm{Lin}\{x, y\}$. Reformulating part *b*), of the same lemma, the function $f : \Gamma \to \mathbb{R}$, of values $f(\lambda) = (x + \lambda\, y \,|\, x + \lambda\, y)$ vanishes twice. If $\Gamma = \mathbb{R}$, then $f(\lambda)$ takes the form

$$f(\lambda) = (x\,|\,x) + 2\lambda\,(x\,|\,y) + \lambda^2\,(y\,|\,y),$$

and (**) follows from the behavior of the $\mathrm{sign}\, f(\lambda)$.

If $\Gamma = \mathbb{C}$, then

$$f(\lambda) = (x\,|\,x) + \lambda\,(y\,|\,x) + \overline{\lambda}\,(x\,|\,y) + |\lambda|^2\,(y\,|\,y),$$

and we shall reduce the problem to the case $\Gamma = \mathbb{R}$. With this purpose, let us distinguish two situations:

Case 1. $(x\,|\,y) = 0$. In this case, we have

$$f(\lambda) = (x\,|\,x) + |\lambda|^2\,(y\,|\,y),$$

hence the change of sign is possible exactly if

$$(x\,|\,x)\,(y\,|\,y) < 0.$$

Obviously, this is a particular form of (**).

Case 2. $(x\,|\,y) \neq 0$. We have $(x\,|\,y) = \rho\,(\cos\theta + i\sin\theta)$, where $\rho > 0$. If we consider the auxiliary element $y^* = (\cos\theta + i\sin\theta)\,y$, then we obtain

$$(y^*\,|\,y^*) = (y\,|\,y), \text{ and } (x\,|\,y^*) = (y^*\,|\,x) = \rho \in\mathbb{R}.$$

Because y^* and y are collinear, $\underline{B}^* = \{x, y^*\}$ is a base of $\mathrm{Lin}\{x, y\}$ too, and the hypotheses of the Lemma 2 are fulfilled. According to this lemma, the inner product spaces $(\mathrm{Lin}\{x, y\}, (\,.\,|\,.\,))$ and $(Y, (\,.\,|\,.\,)_{Y\times Y})$, where $\mathrm{Lin}_{\mathbb{R}}\{\underline{B}^*\} = Y$, have the same nature. In particular, Y is a linear space over $\mathbb{R}$, and $(\,.\,|\,.\,)_{Y\times Y}$ is an indefinite inner product. Following Lemma 1, trinomial $f^* : \mathbb{R} \to\mathbb{R}$, of values

$$f^*(\lambda) = (x + \lambda\, y^* \,|\, x + \lambda\, y^*) = (x\,|\,x) + 2\lambda\,\rho + \lambda^2\,(y\,|\,y),$$

has two roots. This property of f^* holds if and only if $\rho^2 - (x\,|\,x)\,(y\,|\,y) > 0$, which becomes (**) since $\rho = |(x\,|\,y)|$. Q.E.D.

The above proof is useful to see how important the fact that the inner product is undefined is to obtain the inequality (**).

Orthogonality is another important topic about pairs of vectors. Even if we start from the same definition, surprises appear in the spaces with the undefined inner product. We have already seen that there must be self-orthogonal vectors and the connection with the notion of an angle becomes different.

In arbitrary inner product spaces, we say that the elements x, y are *orthogonal* if $(x\,|\,y) = 0$, and we note $x \perp y$. Two non-void sets A, B are orthogonal if $x \perp y$ holds for all $x\in A$ and $y\in B$. In this case we note $A \perp B$. In particular, if $A = \{x_0\}$, then we note $x_0 \perp B$ instead of $A \perp B$, and we say that x_0 is orthogonal to B. The *orthogonal companion* of A is defined by $A^{\perp} = \{x\in X : x \perp A\}$.

Pythagoras theorem holds in arbitrary inner product spaces $(X, (. | .))$, namely if $x_1, x_2, \ldots, x_n \in X$, $x = x_1 + x_2 + \ldots + x_n$ and $x_k \perp x_l$ for all $k \neq l$, then

$$(x \mid x) = \sum_{k=1}^{n} (x_k \mid x_k).$$

We may notice interesting features of orthogonality in universes of events that happen in a scalar product space $(\mathsf{S}, < . | . >)$, noted earlier $W = \mathbb{R} \times \mathsf{S}$:

- If $e \perp \varepsilon$ and $e \in \mathfrak{P}^{++} \setminus \{0\}$, then $\varepsilon \in \mathfrak{P}^{--} \setminus \{0\}$;
- If $\dim \mathsf{S} > 1$ and e is a neutral vector, then $e^{\perp}$ is a negative semi-definite degenerate subspace;
- If $\dim \mathsf{S} = 1$, then $e^{\perp} = \mathrm{Lin}\{e\}$ is a neutral subspace.

The proof is directly based on (**).

Defining the *angle* between two vectors, say x and y, depends on the nature of the subspace $\mathrm{Lin}\{x, y\}$, which shows what inequality holds, either (*) or (**). If $\mathrm{Lin}\{x, y\}$ is semi-definite, then (*) allows defining a *circular* angle by

$$cos\alpha = \frac{<x|y>}{\|x\|\|y\|}.$$

In this case, $x \perp y$ reduces to $\omega = \pi/2$, hence orthogonality really means ορθή γωνία. In more details, α is the ratio between an arc of a circle and the radius.

If $\mathrm{Lin}\{x, y\}$ is indefinite, then (**) leads to an *hyperbolic* angle, defined by

$$cosh\omega = \frac{(x|y)}{\|x\|\|y\|}.$$

Obviously, this formula works only for vectors of the same type, e.g. $(x \mid x) > 0$ and $(y \mid y) > 0$ and has no connection with $x \perp y$, which is impossible. Similarly, ω is the ratio between an arc of a hyperbola and the radius.

The most consistent part of the functional analysis on indefinite inner product spaces deals with linear operators. In particular, a surprising result is Zeeman's Theorem, which establishes that the Lorentz group of the Special Einsteinian Relativity derives from causality. Unexpectedly, a qualitative structure, namely the causal order between events, derived from light, determines a quantitative structure, described by the intrinsic indefinite metric of the space-time, which measures proper time. More philosophically, *light* determines *time*.

Theorem (*Zeeman*). If $W = \mathbb{R} \times \mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}^3$, then $\mathscr{C} = \mathscr{L}$, where $\mathscr{C}$ represents the *causality group* (causal automorphisms of W) and $\mathscr{L}$ is the *extended Lorentz group* (generated by Lorentz transformations, translations and dilations).

This theorem has a lot of extensions, e.g. to Krein spaces with $\dim W \geq 3$. So, if f is a neutral automorphism, then it is a J – continuous affine transformation of X, for which there exists $c \in \mathbb{R} \setminus \{0\}$, such that the equality

$$(f(x) - f(y) \mid f(x) - f(y)) = c\ (x - y \mid x - y)$$

holds for all $x, y \in X$. In addition, if $\varkappa_+ \neq \varkappa_-$, then $c > 0$.

Chapter II. Structural Discreteness

II.1. Why Structures?

There are several reasons to look for structures of the *discrete*. Among the most general we may mention:

- *Each paradox in a theory generates another theory*. In particular, (S.a.) goes against classical topology and mathematical analysis, so naturally, it requires a theory of its own.
- Even practitioners accept that „*Nothing is more practical than a good theory*" (attributed to Ludwig Bolzmann). That is why so many theories have been constructed and studied, especially in mathematics.
- By identifying some structures, *a series of disparate results from a field of knowledge is unified*. Once accepted, the axioms of any structure lead to a theory that highlights new aspects.
- Mathematics is a conglomerate of theories in which different structures are studied. In particular, continuum theories are based on topological structures; *why wouldn't the discreteness also have its own structures*?
- Last but not least, Einstein himself remarked the discrete character of the universe of events and the lack of adequate mathematical structures. In a letter to Walter Dallenbach (1916), he wrote: „*The problem seems to me how one can formulate statements about a discontinuum without calling upon continuum space-time as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately*."

Going into details, we can notice a number of weaknesses in the way classical theories treat the S.a. inequality and discreteness. Thus, we remark:

Discrete space-time theories.

Initially (~1916), Einstein said: "*There is no more banal statement than that our usual world is a four-dimensional space-time continuum*". An explanation is the Minkowski's complexification, which transforms the universes of events into Euclidean space. Later on, about 1936, especially influenced by quantum physics results, Einstein has changed his mind: "… *perhaps the success of the Heisenberg method points to a purely algebraic description of nature, that is, to the elimination of the continuous functions from physics. Then, however, we must give up, by principle, the space-time continuum.*"

Following Einstein, many scientists have discussed the subject of space-time discreteness and the syntagma "*discrete space-time*" took many meanings. So, Heisenberg proposed the discreteness of the space-time as a way to avoid infinities from quantum field theory. Schild's opinion was "*It seems likely that a physical theory based on a discrete space-time background will be free of the infinities which trouble the contemporary quantum mechanics.*" Snyder remarked that "… *the usual assumptions concerning the continuous nature of space-time are not necessary for Lorentz invariance*".

Quantum representations of the world.

The *discrete – continuous* dispute started around 500BC, when Leucippus and Democritus sustained that matter is made by indivisible units, called *atoms*. We recognize it in the actual *wave – particle* duality. Quantum Physics, starting with Planck's quanta of energy, extended discreteness to all physical quantities. In particular, Heisenberg's uncertainty principle leads to a shortest measurable length, called the Planck length. Thus, quantum physics rejects continuity.

The above quantization process is inappropriate for sub-additive metrics, i.e. if (X, ρ) is a s.a. metric space, $\varepsilon > 0$ and

$$\rho_\varepsilon(x, y) = \begin{cases} 0 & if \rho(x,y) < \varepsilon \\ \rho(x,y) & if\ \rho(x,y) \geq \varepsilon \end{cases},$$

then ρ_ε brakes s.a. This "discretization" is natural for S.a. metrics, which are restrained anyway. Topologies do not support such discreteness because there may be no minimal neighborhoods.

Topologies do not match events.

The Euclidean topology of $\mathbb{R}^4$ is locally homogeneous, whereas $W = \mathbb{R} \times \mathbb{R}^3$ is not; The group of all homeomorphisms of the 4-dimensional Euclidean space is vast, and of no physical significance for the events in W. However, Zeeman has indicated a way of avoiding these inadequacies. He considered that a topology φ is suitable to the universe $W = \mathbb{R} \times \mathbb{R}^3$ of events if it satisfies the conditions:

[Z_1] φ is not locally homogeneous, and the light cone through any event can be deduced from φ ;

[Z_2] The group of automorphisms of φ is generated by the inhomogeneous Lorentz group and dilatations;

[Z_3] φ induces the 3-dimensional Euclidean topology on each spatial hyper-plane and the 1-dimensional Euclidean topology on each time axis.

Zeeman showed that such a topology exists, he called it the *fine topology*, it being the finest topology of $\mathbb{R}^4$ that satisfies [Z_3]. Unfortunately, the filter of fine-neighborhoods of has no countable basis, hence the fine topology is not metrizable. This is a considerable disadvantage in practice, where the metric measurements are essential. In the ultimate analysis, a topology on W is not necessary to attain properties [Z_1], [Z_2] and [Z_3]; they immediately derive from the intrinsic indefinite inner product of W, from the attached S.a. metric and from structures similar to topologies, but appropriate to the universe of events, such as those in the next section.

II.2. Horistology

To make the comparison with structures of continuum handy, we recall that a topology on a set X is a function $\tau : X \to \mathscr{P}(\mathscr{P}(X))$, which attaches to each $x \in X$ a system of neighborhoods of x, such that:

[N_1] $x \in V$ for each $V \in \tau(x)$;

[N_2] If $V \in \tau(x)$ and $U \supseteq V$, then $U \in \tau(x)$;

[N_3] If $U, V \in \tau(x)$, then $U \cap V \in \tau(x)$;

[N_4] $\forall\, V \in \tau(x)\ \exists\, W \in \tau(x)$ such that $V \in \tau(y)$ holds at each $y \in W$.

Let X be an arbitrary nonvoid set, most often thought of as a universe of relativist events, from where we also will extract the terminology. A *horistology* on X is a function $\chi : X \to \mathscr{P}(\mathscr{P}(X))$ such that:

[h_1] $x \notin P$ for all $P \in \chi(x)$;

[h_2] $[P \in \chi(x)$ and $Q \subseteq P] \Rightarrow [Q \in \chi(x)]$;

[h_3] $[P, Q \in \chi(x)] \Rightarrow [P \cup Q \in \chi(x)]$;

[h_4] $\underset{P \in \chi(x)}{\forall}\ \underset{T \in \chi(x)}{\exists}$ such that $[y \in P$ and $Q \in \chi(y)] \Rightarrow [Q \subseteq T]$.

The pair (X, χ) forms a *horistological universe* (*world*, *space*, etc.), and each set $P \in \chi(x)$ is called χ – *perspective* of x.

There is an obvious parallel between topology and horistology: [N_1] and [h_1] are opposite; unlike [N_2] and [N_3], which define a filter, [h_2] and [h_3] organize $\chi(x)$ as an ideal. Finally, if compared to [N_4], which we generally accept as a refinement of the sub-additivity, the role of [h_4] is to refine super-additivity.

If we have to operate with perspectives of the same "size" at different events, we need uniformization. We say that $\mathscr{H} \subset \mathscr{P}(X^2)$ is a *uniform* (briefly *u*-) *horistology* on X if the following conditions hold:

[UH_1] $\mathsf{P} \cap \text{д} = \varnothing$ for all $\mathsf{P} \in \mathscr{H}$;

[UH_2] $[\mathsf{P} \in \mathscr{H}$ and $\mathsf{Q} \subseteq \mathsf{P}] \Rightarrow [\mathsf{Q} \in \mathscr{H}]$;

[UH_3] If $\mathsf{P}, \mathsf{Q} \in \mathscr{H}$ then $\mathsf{P} \cup \mathsf{Q} \in \mathscr{H}$;

[UH_4] $\underset{\mathsf{P} \in \mathscr{H}}{\forall}\ \underset{\mathsf{T} \in \mathscr{H}}{\exists}$ such that for $\underset{\mathsf{Q} \in \mathscr{H}}{\forall}$ we have $\mathsf{P} \circ \mathsf{Q} \subseteq \mathsf{T}$ and $\mathsf{Q} \circ \mathsf{P} \subseteq \mathsf{T}$.

The elements of $\mathscr{H}$ are called *prospects* and the pair $(X, \mathscr{H})$ forms a *uniform horistological world* (briefly *u.h.* world, universe, space).

Among other structures defined by ideals of sets we mention the *bornologies* – structures of *boundedness*. Thus, family $\text{Б} \subseteq \mathscr{P}(X)$ is a *bornology* on X if:

[B1] $\cup\{B \subseteq X : B \in \text{Б}\} = X$;

[B2] $[B \in \text{Б}$ and $A \subseteq B] \Rightarrow [A \in \text{Б}]$;

[B3] $[B_1, B_2, \dots, B_n \in \text{Б}] \Rightarrow [B_1 \cup B_2 \cup \dots \cup B_n \in \text{Б}]$ for any $n \in \mathbb{N}^*$.

Each horistology has an ideal base $\beta : X \to \mathscr{P}(\mathscr{P}(X))$, which satisfies

$[h_b]$ $\underset{P,Q\in\beta(x)}{\forall}\ \underset{W\in\beta(x)}{\exists}$ such that $P\cup Q\subseteq W$.

If β also satisfies $[h_1]$ and $[h_4]$, then $\chi : X\to\mathscr{P}(\mathscr{P}(X))$, of values

$$\chi(x) = \{\, P\subseteq X : \underset{W\in\beta(x)}{\exists} \text{ such that } P\subseteq W\,\}$$

is a horistology of X. Conversely, each ideal base of a horistology satisfies $[h_1]$, $[h_4]$ and $[h_b]$. The u-horistologies allow a similar definition.

If $\mathscr{H}$ is a u-horistology on X, then $\chi_{\mathscr{H}}: X\to\mathscr{P}(\mathscr{P}(X))$, where

$$\chi_{\mathscr{H}}(x) = \{P\in\mathscr{P}(X) : \underset{\mathsf{H}\in\mathscr{H}}{\exists} \text{ such that } P\subseteq \mathsf{H}[x]\}$$

is a horistology on X (said to be *generated by* $\mathscr{H}$).

Let χ_1 and χ_2 be two horistologies on X. If each χ_1 – perspective of x is a χ_2 – perspective of x, at each $x\in X$, then we note $\chi_1\subseteq\chi_2$, and we say that horistology χ_1 is *coarser* than χ_2, or equivalently, χ_2 is *finer* than χ_1. Similarly, we may compare uniform horistologies.

The *coarsest* horistology on X is defined by $\chi_0(x) = \{\varnothing\}$; if R is an order on X, then $\chi(x) = \mathscr{P}((\mathsf{R}\setminus д)[x])$ is the finest horistology relative to R. Remarkable examples of (u-) horistologies derive from S.a. metrics. Thus, in the relativist universe of events $W=\mathbb{R}\times\mathbb{R}^n$, $n = 1, 2, 3$, using causality and temporal norm, we construct the *hyperbolic perspectives* of an event e, of radius $r > 0$, by

$$H(e, r) = \{\, f\in \mathrm{K}[e] : \| f-e\|_t > r\}.$$

They form a base of a (u-) horistology, considered *intrinsic* of W. In particular, W may represent the set of hyperbolic numbers. Generally, if K is an order on X and $\mathfrak{I} = \{\rho_i : i\in I\}$ is a *directed* family of pseudo S.a. metrics defined on R, i.e.

$$\underset{i,j\in I}{\forall}\ \underset{k\in I}{\exists} \text{ such that } \rho_k\geq\max\{\rho_i,\rho_j\},$$

then the hyperbolic prospects

$$\mathsf{H}_{i,r} = \{(x, y)\in\mathsf{R} : \rho_i(x, y) > r\}$$

form an ideal base of a u-horistology, noted $\mathscr{H}_{\mathfrak{I}}$.

Another example of natural horistology concerns directed sets. If $(\mathfrak{D}, \leq)$ is a directed, as in topology, we extend $\leq$ to $\overline{\mathfrak{D}} = \mathfrak{D}\cup\{d^\infty\}$ by $a<d^\infty$ for all $a\in\mathfrak{D}$. Function $\chi^{\mathfrak{D}} : \overline{\mathfrak{D}}\to\mathscr{P}(\mathscr{P}(\overline{\mathfrak{D}}))$, of values

$$\chi^{\mathfrak{D}}(x) = \begin{cases}\{\varnothing\} & if\ x\in\mathfrak{D}\\ \{\mathscr{P}(\leftarrow,a) : a\in\mathfrak{D}\}\ if\ x=d^\infty\notin\mathfrak{D}\end{cases}$$

is a horistology of $\overline{\mathfrak{D}}$ (called *directed horistology*).

The horistological structures are strongly connected to order relations. Thus, if $(X, \mathscr{H})$ is a u-horistological world, then

$$\mathrm{K}(\mathscr{H}) = \cup\{\mathsf{P}\subset X^2 : \mathsf{P}\in\mathscr{H}\}$$

is a strict order on X, called $\mathscr{H}$ – *causality*. In a horistological world (X, χ),

$$\mathrm{K}(\chi) = \{(x, y)\in X^2 : \{y\}\in\chi(x)\}$$

is a strict order, called χ – *causality* (rather χ-*accessibility*). If χ derives from $\mathscr{H}$, then $\mathrm{K}(\chi) = \mathrm{K}(\mathscr{H})$.

A horistological universe (X, χ) is called *uniformly displayable* (cf. French *uniformisable*) if there exists a uniform horistology $\mathscr{H}$ on X such that $\chi = \chi_{\mathscr{H}}$. This is not always possible because instead of [UH_4] it is enough a weaker condition to generate a horistology, namely:

[½UH_4] $\underset{\mathsf{P}\in\mathscr{H}}{\forall}\ \underset{\mathsf{T}\in\mathscr{H}}{\exists}$ such that for $\underset{\mathsf{Q}\in\mathscr{H}}{\forall}$ we have $\mathsf{P}\circ\mathsf{Q}\subseteq\mathsf{T}$.

We say that $\mathscr{H}\subset\mathscr{P}(X^2)$ is a *semi-uniform* (briefly *s.u*-) *horistology* on X if it satisfies the conditions [UH_1] – [UH_3] and [½UH_4]. Every horistology χ on X is semi-uniformly displayable.

The problem of metrisability deals with another special notion: Relation R on X is *exhaustive* if $\cap\{\mathsf{R}^n : n\in\mathbb{N}^*\} = \varnothing$, where

$$\mathsf{R}^n \overset{def.}{=} \underbrace{\mathsf{R}\circ\mathsf{R}\circ\ldots\circ\mathsf{R}}_{n\ times}.$$

A u-horistology $\mathscr{H}$ on X is generated by a family of S.a. metrics if and only if it has a base consisting of transitive and exhaustive prospects.

Like topologies, which have equivalent definitions by interiority, adhesion, etc., horistological structures can also be defined by specific set operators. So, if (X, χ) is an arbitrary horistological world, then to each $A\in\mathscr{P}(X)$, we may attach the *premise* of A, defined by

$$\mathbf{p}(A) = \{x\in X : A\in\chi(x)\}.$$

We say that $\mathbf{p} : \mathscr{P}(X) \to \mathscr{P}(X)$ is the *premise operator* on (X, χ). To this operator there corresponds a binary relation

$$\mathrm{K}(\mathbf{p}) = \{(x, y)\in X^2 : x\in \mathbf{p}(\{y\})\}.$$

It is an *intrinsic strict order*, called $\mathbf{p}$-*causality*.

Similarly, we define the *co-premise operator* $\mathbf{q} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathbf{q}(B) = \{x\in X : B\cup P \neq X \text{ for all } P\in\chi(x)\},\ \forall\, B\in\mathscr{P}(X).$$

The premise operator $\mathbf{p}_\chi$, attached to χ, has the properties:

[p_1] $\mathrm{K}^{=}(\mathbf{p}_\chi) = \mathrm{K}(\mathbf{p}_\chi)\cup д$ is an order on X;

[p_2] $[A\subseteq B] \Rightarrow [\,\mathbf{p}_\chi(A)\supseteq\mathbf{p}_\chi(B)]$;

[p_3] $\mathbf{p}_\chi(A\cup B) = \mathbf{p}_\chi(A)\cap\mathbf{p}_\chi(B)$;

[p_4] $\mathbf{p}_\chi(A) = \mathbf{p}_\chi(\mathrm{K}^{=}(\mathbf{p}_\chi)[A])$.

Conversely, if X be an arbitrary non-void set and $\mathbf{p} : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfies the conditions [p_1] – [p_4], then $\chi_{\mathbf{p}} : X \to \mathscr{P}(\mathscr{P}(X))$, defined by

$$\chi_{\mathbf{p}}(x) = \{P\in\mathscr{P}(X) : x\in\mathbf{p}(P)\},$$

is a horistology on X. In addition, $\mathrm{K}(\mathbf{p}) = \mathrm{K}(\chi_{\mathbf{p}})$.

Functions χ and $\mathbf{p}$ define the same structure, in the sense that

$$\chi_{\mathbf{p}_\chi} = \chi \text{ and } \mathbf{p}_{\chi_{\mathbf{p}}} = \mathbf{p}.$$

A specific operation with horistological structures concerns restrictions to an arbitrary strict order, Λ:

(i) If $\mathscr{H}$ is a u-horistology on X, then family

$$\mathscr{H}|_{\Lambda} = \{\mathsf{P}\cap\Lambda : \mathsf{P}\in\mathscr{H}\}$$

is a u-horistology too, coarser than $\mathscr{H}$.

(ii) If χ is a horistology on X, then function $\chi|_{\Lambda}$, of values

$$\chi|_{\Lambda}(x) = \{P\cap \Lambda\,[x] : P\in\chi(x)\},$$

is a horistology too, coarser than χ.

(iii) If $\rho : \mathsf{R}\rightarrow\mathbb{R}_+$ is a S.a. (p-) metric, then the restricted S.a. (p-) metric $\rho|_{\Lambda^=}$ generates a (u-) horistology coarser than ρ.

Other operations refer to *infimum* and *supremum* of a family of structures: For example, if $\mathfrak{h} = \{\chi_k : k\in I\}$ is a non-void family of horistologies on X, then

$$\chi(x) = \{P\subset X : \exists P_k\in\chi_k(x) \text{ such that } P\subseteq \cap\{P_k : k\in I\}\}$$

is a horistology, namely $\chi = \inf \mathfrak{y}$, and $\mathrm{K}(\chi) = \cap\{\mathrm{K}(\chi_k) : k\in I\}$. The existence of an upper bound and supremum of a family of horistological structures is more problematic, hence the family of all (u-) horistologies on X isn't a lattice. However, the upper bounded sub-families have sup.

Finally, we may construct subspaces, product and quotients of horistologies:

If $(X, \mathscr{H}_X)$ is a u-horistological world and $Y\subseteq X$, then

$$\mathscr{H}_Y = \{\mathsf{P}\in\mathscr{P}(Y^2) : \exists\mathsf{Q}\in\mathscr{H}_X \text{ such that } \mathsf{P}\subseteq Y^2\cap\mathsf{Q}\}$$

is an u-horistology on Y. We say that $(Y, \mathscr{H}_Y)$ is a *u-horistological subspace* of $(X, \mathscr{H}_X)$. Unlike topology, in horistology there are *over-spaces*, i.e. if $Y\subseteq X$, then each (u-) horistology on Y is a (u-) horistology on X too.

If $(X, \mathscr{H}_X)$ and $(Y, \mathscr{H}_Y)$ are u-horistological worlds, then $Z = X \times Y$ is endowed with the u-horistology

$$\mathscr{H}_Z = \{\mathsf{P}\in\mathscr{P}(Z^2) : \exists\mathsf{Q}\in\mathscr{H}_X\ \exists\mathsf{S}\in\mathscr{H}_Y \text{ such that } \mathsf{P}\subseteq Q\times\mathsf{S}\},$$

which is the *product* u-horistology.

Let E be equivalence on an arbitrary non-void set X. If the u-horistology $\mathscr{H}$ on X is stable relative to E, i.e. there exists an ideal base $\mathscr{B}$ of $\mathscr{H}$ such that at each $\mathsf{B}\in\mathscr{B}$ we have $\mathsf{E}\circ\mathsf{B}\circ\mathsf{E}\subseteq\mathsf{B}$, then the ideal $\mathscr{H}^\wedge$, generated by the ideal base $\mathscr{B}^\wedge = \{\mathsf{B}^\wedge : \mathsf{B}\in\mathscr{B}\}$, is a u-horistology on $X^\wedge = X / \mathsf{E}$.

We say that $(X^\wedge, \mathscr{H}^\wedge)$ is the *quotient* uniform horistological world.

II.3. Discrete Functions

If (X, χ_X) and (Y, χ_Y) are horistological universes, the question is what kind of functions $f : X \to Y$ do preserve the horistological properties. It is well known that the morphisms of the TOP category are the continuous functions. However, if we adopt a similar pattern for horistologies, e.g. $\ell \in Y$ is *h-limit* at $x_0 \in X$ if

$$\forall P \in \chi_Y(\ell)\ \ \exists Q \in \chi_X(x_0) \text{ such that } [f(x) \in P] \Rightarrow [x \in Q],$$

then we encounter difficulties even in algebraic operations. Thus, the bounded function model proves to be more appropriate. There, if $(X, \mathfrak{b}_X)$ and $(Y, \mathfrak{b}_Y)$ are bornologic spaces, function f is *bounded* if it carries bounded sets to bounded sets, i.e. $[\forall B \in \mathfrak{b}_X] \Rightarrow [f(B) \in \mathfrak{b}_Y]$.

Let (X, χ_X) and (Y, χ_Y) be horistological worlds, $x_0 \in X$, and $f : X \to Y$. We say that $g \in Y$ is a *germ* of f *at* x_0 (or better "*when x starts from* x_0" etc.) and we note $g = \underset{x_0 \to x}{germ}\, f(x)$, whenever $f(\chi_X(x_0)) \subseteq \chi_Y(g)$, i.e.

$$[P \in \chi_X(x_0)] \Rightarrow [(f(P) \in \chi_Y(g)] .$$

We note the set of all germs of f at x_0 by $Germ(f, x_0)$.

If, in addition, $f(x_0) \in Germ(f, x_0)$, we say that f is *discrete at* x_0. If $A \subseteq X$ and f is discrete at each $x_0 \in A$, then f is said to be *discrete on A*.

If $f : X \to Y$ is bijective and discrete on X, and f^{-1} is discrete on Y, then f is called *horistological isomorphism*, and we say that (X, χ_X) and (Y, χ_Y) are *isomorphic* horistological worlds.

In particular, if $(X, \mathsf{R}_X, \rho_X)$ and $(Y, \mathsf{R}_Y, \rho_Y)$ are S.a. (p-)metric worlds, then function $f : X \to Y$ is *discrete at* x_0 (relative to the horistologies χ_X and χ_Y, generated by ρ_X, respectively ρ_Y) if and only if

$$\forall \delta > 0\ \exists \varepsilon > 0 \text{ such that } [\rho_X(x_0, x) > \delta] \Rightarrow [\rho_Y(f(x_0), f(x)) > \varepsilon] .$$

The discreteness of a function can be characterized in terms of horistological operators: If p_X and p_Y are the corresponding premise operators, then $f : X \to Y$ is discrete on X if and only if $f(\mathsf{p}_X(A)) \subseteq \mathsf{p}_Y(f(A))$ holds at all $A \subseteq X$.

Discreteness of a function says more than monotony, i.e. the discreteness at x_0 (on X) implies the strict *local* (respectively *global*) monotony. Because (X, χ_X) and (Y, χ_Y) are ordered by $\mathrm{K}(\chi_X)$ and respectively $\mathrm{K}(\chi_Y)$, it follows that

$$[f \text{ discrete at } x_0 \in X \text{ and } (x_0, x) \in \mathrm{K}(\chi_X)] \Rightarrow [(f(x_0), f(x)) \in \mathrm{K}(\chi_Y)] \text{ and}$$

$$[f \text{ discrete on } X \text{ and } (x_1, x_2) \in \mathrm{K}(\chi_X)] \Rightarrow [(f(x_1), f(x_2)) \in \mathrm{K}(\chi_Y)] .$$

The discreteness of a function is not reducible to its monotony, i.e. there exist $\mathrm{K}(\chi_X)$ to $\mathrm{K}(\chi_Y)$ monotonous functions, which are nowhere discrete.

We may compose discrete functions acting between horistological worlds, e.g. (X, χ_X), (Y, χ_Y) and (Z, χ_Z). If $f : X \to Y$ is discrete at $x_0 \in X$ and $g : Y \to Z$ is discrete at $y_0 = f(x_0) \in Y$, then $g \circ f : X \to Z$ is discrete at x_0. The extension to global discreteness is obvious.

The horistological worlds form a mathematical category (noted HOR), for which the morphisms are discrete functions. HOR has no final object, hence it is not isomorphic with TOP.

Discrete functions may carry (induce) horistologies from a world to another: If (X, χ) is a horistological world, $Y \neq \varnothing$ is arbitrary, and function $f : X \to Y$ is injective, then $\chi^{\to} \overset{not.}{=} f^{\to}(\chi) : Y \to \mathscr{P}(\mathscr{P}(Y))$, where

$$\chi^{\to}(y) = \begin{cases} \{Q \subseteq f(P) : P \in \chi(x)\} & if\ y = f(x) \\ \{\varnothing\} & if\ y \notin f(X) \end{cases}$$

is a horistology on Y (called *direct image* of χ through f). In addition, $\chi^{\to}$ is the coarsest horistology of Y, which makes f discrete on X. Conversely, if χ is a horistology on Y, then $\chi^{\leftarrow} \overset{not.}{=} f^{\leftarrow}(\chi) : X \to \mathscr{P}(\mathscr{P}(X))$, of values

$$\chi^{\leftarrow}(x) = \{Q \subseteq f^{\leftarrow}(P) : P \in \chi(f(x))\},$$

is a horistology of X (called *inverse image* of χ through f). In addition, $\chi^{\leftarrow}$ is the finest horistology of X, which makes f discrete on X.

The isometries relative to S.a. metrics, e.g. the Lorentz transformations, are simple examples of discrete functions. The algebraic operations in $\mathbb{R}$, $\mathbb{H}$ and in relativist universes of events are discrete relative to the intrinsic horistologies. The indefinite inner product of $W = \mathbb{R} \times H$ is a discrete function on the set $K[0]$ relative to the intrinsic horistologies of W^2 and $\mathbb{R}$.

If $(X, \mathscr{H}_X)$ and $(Y, \mathscr{H}_Y)$ are u-horistological worlds, then $f : X \to Y$ is a *uniformly* (briefly *u-*) *discrete* on X if

$$[\forall \mathsf{P} \in \mathscr{H}_X] \Rightarrow [f_{II}(\mathsf{P}) \in \mathscr{H}_Y],$$

where $f_{II}(\mathsf{P}) = \{ f_{II}(a, b) = (f(a), f(b)) : (a, b) \in \mathsf{P}\}$. In particular, if $(X, \mathsf{R}_X, \rho_X)$ and $(Y, \mathsf{R}_Y, \rho_Y)$ are S.a. (p-)metric worlds, then f is *u-discrete on X* (relative to the u-horistologies $\mathscr{H}_X$ and $\mathscr{H}_Y$, generated by ρ_X and ρ_Y) if and only if

$$\forall \delta > 0\ \exists \varepsilon > 0 \text{ such that } [\rho_X(x_1, x_2) > \delta] \Rightarrow [\rho_Y(f(x_1), f(x_2)) > \varepsilon].$$

Obviously, u-discreteness on X implies discreteness at each $x_0 \in X$. We find a lot of properties of locally discrete functions: monotony relative to $K(\mathscr{H}_X)$ and $K(\mathscr{H}_Y)$, u-discreteness of the composition (which leads to category u-HOR), the possibility of carrying u-horistologies from a set to another etc.

II.4. Emergence

Emergence is the horistological correspondent of the *convergence* from topology. In topology, it is well known that the convergence of a net $\xi : \mathfrak{D} \to X$ reduces to the continuity at d^∞ of the extension $\overline{\xi} : \overline{\mathfrak{D}} = \mathfrak{D} \cup \{ d^\infty \} \to X$, relative to the intrinsic topology of $\overline{\mathfrak{D}}$. Using the intrinsic horistology of the same $\overline{\mathfrak{D}}$, the emergence of ξ represents the discreteness of $\overline{\xi}$ at d^∞. More exactly, an element $g \in X$ is said to be a *germ* of the net $\xi : \mathfrak{D} \to X$ if

$$\forall\, a \in \mathfrak{D} \;\; \exists P \in \chi(g) \text{ such that } [b < a] \Rightarrow [\xi(b) \in P] .$$

Equivalently, this means $\xi(\leftarrow, a) \overset{not.}{=} \{\xi(b) : b < a\} \in \chi(g)$ for all $a \in \mathfrak{D}$.

We note the set of all germs of ξ by $\mathsf{Germ}\ \xi$. If $\mathsf{Germ}\ \xi \neq \varnothing$, then we say that ξ is an *emergent* net; for each $g \in \mathsf{Germ}\ \xi$ we say "ξ *emerges* from g".

The notion of emergence is a particular case of discreteness, i.e.

$$[g \in \mathsf{Germ}\ \xi] \Leftrightarrow [\overline{\xi} \textit{ is discrete at } d^\infty],$$

where function $\overline{\xi} : \overline{\mathfrak{D}} \to X$ has the values

$$\overline{\xi}(a) = \begin{cases} \xi(a) & if\ a \in \mathfrak{D} \\ g & if\ a = d^\infty \end{cases} .$$

In the case of sequences, emergence becomes a question of the order K(χ), but other cases, as the construction of the Darboux integral has a fully horistological feature. Let $\mathfrak{D}$ be the set of all partitions of $[a, b]$, ordered by the relation of inclusion and let $M_k = \max \{f(t) : t \in [t_{k-1}, t_k]\}$ and $m_k = \min \{f(t) : t \in [t_{k-1}, t_k]\}$. The superior Darboux integral sums

$$S_\delta(f) = \sum_{k=1}^{n} M_k (t_k - t_{k-1}),$$

form a net $\xi_f : \mathfrak{D} \to \mathbb{R}$, where $\xi_f(\delta) = S_\delta(f)$. Because ξ_f is decreasing, there exists $G = \inf \xi_f(\mathfrak{D})$, where $\xi_f(\mathfrak{D}) = \{S_\delta(f) : \delta \in \mathfrak{D}\}$. Relative to $\chi^{\mathbb{R}}$ and the usual order $\mathsf{R} = \le$ on $\mathbb{R}$, the net ξ_f is emergent and $\mathsf{Germ}\ \xi_f = \mathsf{R}^{-1}[G]$.

Similarly, the inferior Darboux integral sums

$$s_\delta(f) = \sum_{k=1}^{n} m_k (t_k - t_{k-1}),$$

form the net $\eta_f : \mathfrak{D} \to \mathbb{R}$, where $\eta_f(\delta) = s_\delta(f)$. Because η_f is increasing and bounded, there exists $g = \sup \eta_f(\mathfrak{D})$. If $\chi^{\mathbb{R}\frown}$ is the symmetric companion of the horistology $\chi^{\mathbb{R}}$, which plays the role of intrinsic horistology of $-\mathbb{R}$, then the net η_f is emergent and $\mathsf{Germ}\ \eta_f = \mathsf{R}[g]$.

In particular cases, e.g. f continuous, we obtain $G = g$. In horistological terms, this means to identify a unique germ, which is the Darboux integral,

$$\mathsf{Germ}\ \xi_f \cap \mathsf{Germ}\ \eta_f = \{ \int_a^b f(t)\,dt \}.$$

In topology, the uniqueness of a limit is solved by conditions on the space (T_2, Hausdorff), but in horistology, Germ ξ is always negatively conical relative to $K(\chi)$, hence normally contains plenty of germs. The method of dealing with pairs of nets sometimes leads to unique germs (like for Darboux integral from above), but generally, we may identify different types of germs, finally finding germs that form *singletons*.

Because $g \in$ Germ ξ may be a consequence of $\xi(\mathfrak{D}) \in \chi(g)$, if ξ is an emergent net but $\xi(\mathfrak{D}) \notin \chi(g)$, we say that $g \in$ Germ ξ is a *proper germ* of ξ. We note the set of all proper germs of ξ by p-Germ ξ .

Among the properties of the proper germs we remark:

(i) If $K^{-1}[x] \neq \varnothing$ for some $x \in$ Germ ξ, then p-Germ $\xi \subset$ Germ ξ ;

(ii) If $x, y \in$ p-Germ ξ , then $(x, y) \notin K \cup K^{-1}$;

(iii) If χ results by a S.a. metric $\rho : K \to \mathbb{R}_+$, and $x \in$ Germ ξ, then

$$[x \in \text{p-Germ } \xi] \Leftrightarrow [\, \inf \{\rho(x, \xi(a)) : a \in \mathfrak{D}\} = 0] ;$$

Simple examples show that proper germs don't solve the uniqueness problem, so we have to distinguish other cases of proper germs. In particular, we say that ξ *finitely emerges* from x, if

$$[\{\xi(n) : n \in I\} \in \chi(x)] \Leftrightarrow [\text{card } I \in \mathbb{N}] .$$

We note the set of all finite germs of ξ by f-Germ ξ .

In arbitrary horistological world and for every sequence $\xi : \mathbb{N} \to X$, we have

$$\text{f-Germ } \xi \subseteq \text{p-Germ } \xi ,$$

with equality for $K(\chi)$-decreasing sequences.

The reference to sequences (not to general nets) is a weakness of the finite germs, which justifies the interest for other types of germs. To define the next type of germs, we use the *sectional companion* of relation R, defined by

$$\tilde{R} \overset{not.}{=} \{(x, y) \in X^2 : R[x] \supseteq R[y] \text{ and } R^{-1}[x] \subseteq R^{-1}[y]\} .$$

Let $\xi : \mathfrak{D} \to X$ be an arbitrary net in the horistological world (X, χ), where we note $K(\chi) = K$ and $\Sigma = \Sigma(K) = \tilde{K} \setminus K$. If $x \in X$ satisfies the condition

$$x \in \text{p-Germ } \xi \subseteq \Sigma^{-1}[x].$$

then it is called an *emitting germ* (or *emitter*) of the net ξ. We note the set of all emitters of ξ by e-Germ ξ .

An equivalent definition of the emitters replaces condition p-Germ $\xi \subseteq \Sigma^{-1}[x]$ by p-Germ $\xi \subseteq \tilde{K}^{-1}[x]$, since for $x \in$ p-Germ ξ , the set $K^{-1}[x]$ does not contain other proper germs of ξ .

The uniqueness of the emitters in a horistological world (X, χ) depends on the relation $K(\chi)$. If $K(\chi)$ *distinguishes* the elements of X, i.e. $x = y$ whenever

$$K(\chi)[x] = K(\chi)[y] \text{ and } K^{-1}(\chi)[x] = K^{-1}(\chi)[y],$$

then card (e-Germ ξ) ≤ 1 holds for each net $\xi : \mathfrak{D} \to X$.

The proof is essentially based on property (*ii*) from above.

Another important topic on nets concerns the subnets. The topological notion of a subnet makes use of the continuity of the *intermediate* function between the directed sets. More exactly, let $\mathfrak{C}$ and $\mathfrak{D}$ be directed sets, and let us endow the extensions $\overline{\mathfrak{C}} = \mathfrak{C} \cup \{c^\infty\}$ and $\overline{\mathfrak{D}} = \mathfrak{D} \cup \{d^\infty\}$ by their intrinsic topologies $\tau^{\mathfrak{C}}$, respectively $\tau^{\mathfrak{D}}$. To see the meaning of the condition on $\psi : \mathfrak{C} \to \mathfrak{D}$, namely

$$\underset{d \in \mathfrak{D}}{\forall} \ \underset{c \in \mathfrak{C}}{\exists} \ \textit{such that } [b > c] \Rightarrow [\psi(b) > d], \tag{*}$$

we may extend ψ to $\overline{\psi} : \overline{\mathfrak{C}} \to \overline{\mathfrak{D}}$, such that $\overline{\psi}(c^\infty) = d^\infty$. Clearly, condition (*) expresses the continuity of $\overline{\psi}$ at c^∞ relative to the topologies $\tau^{\mathfrak{C}}$ and $\tau^{\mathfrak{D}}$.

To specify the topological character of the classical notion of *subnet*, we say that η is a *top-subnet* of ξ. Naturally, we may define a *horistological* type of subnets in horistological worlds: We say that $\eta : \mathfrak{C} \to X$ is a *hor-subnet* of ξ if there exists an intermediate function $\psi : \mathfrak{C} \to \mathfrak{D}$ such that $\eta = \xi \circ \psi$, and

$$\forall\, c \in \mathfrak{C} \ \exists d \in \mathfrak{D} \ \textit{such that } [\, b < c \,] \Rightarrow [\, \psi\,(b) < d] \,. \tag{**}$$

Obviously, function $\psi : \mathfrak{C} \to \mathfrak{D}$ satisfies the condition (**) if and only if its extension $\overline{\psi} : \overline{\mathfrak{C}} \to \overline{\mathfrak{D}}$ is discrete at c^∞, where

$$\overline{\psi}\,(a) = \begin{cases} \psi(a) & \textit{if } a \in \mathfrak{C} \\ d^\infty & \textit{if } a = c^\infty \,. \end{cases}$$

If $\eta : \mathfrak{C} \to X$ denotes a *hor-subnet* of the net $\xi : \mathfrak{D} \to X$, then:

(a) $\mathsf{Germ}\ \eta \supseteq \mathsf{Germ}\ \xi$;

(b) $\bigcap \{\mathsf{Germ}\ \eta : \eta$ *is hor-subnet of* $\xi\} = \mathsf{Germ}\ \xi$;

(c) If η_1 and η_2 are hor-subnets of ξ such that $\mathsf{Germ}\ \eta_1 \cap \mathsf{Germ}\ \eta_2 = \varnothing$ then ξ is not emergent.

It is well known how to define a topology by specifying a class of convergent nets (or filters). The horistological structures can be similarly defined by classes of emergent nets (or ideals), but we have to use adapted notions: If $(\mathfrak{D}, \leq)$ is a directed set and $a \in \mathfrak{D}$, then $(\leftarrow, a) = \{d \in \mathfrak{D} : d \leq a\}$ is directed by the same relation $\leq$. If $\xi : \mathfrak{D} \to X$ is a net in X, then $\xi|_{(\leftarrow, a]}$, which is the *restriction* of ξ to $(\leftarrow, a]$ is a hor-subnet of ξ. If (X, χ) is a horistological world, then

$$\mathsf{Nets}\ X = \cup\{X^{\mathfrak{D}} : \mathfrak{D} = \textit{directed set}\}$$

represents *the set of all nets* $\xi : \mathfrak{D} \to X$. Similarly, if $A \subseteq X$ is a horistological sub-world of X, then $\mathsf{Nets}\ A$ consists of all nets with $\xi(d) \in A$ for all $d \in \mathfrak{D}$. In particular, $\mathsf{Nets}\ \{x\}$ consists of all constant nets $\xi : \mathfrak{D} \to \{x\}$. The *operator of emergent nets* in (X, χ) is a function $\mathfrak{N} : X \to \mathscr{P}(\mathsf{Nets}\ X)$, of values

$$\mathfrak{N}(x) = \{\xi \in \mathsf{Nets}\ X : x \in \mathsf{Germ}\ \xi\}$$

at each $x \in X$. Similar operators act on each horistological sub-world $A \subseteq X$, in the sense that $\mathfrak{N}_A : A \to \mathscr{P}(\mathsf{Nets}\ A)$ takes the values

$$\mathfrak{N}_A\,(a) = \{\xi \in \mathsf{Nets}\ A : a \in (\mathsf{Germ}\ \xi) \cap A\}.$$

If we view $\mathsf{Germ} : \mathsf{Nets}\ X \to \mathscr{P}(X)$ as an operator, then $\mathfrak{N} = \mathsf{Germ}^{\leftarrow}$.

Let $\mathfrak{D}$ together with $\mathfrak{E}_d$, for each $d\in\mathfrak{D}$, be directed sets; let $\xi:\mathfrak{D}\to X$, and for each $d\in\mathfrak{D}$, let $\xi_d: \mathfrak{E}_d \to\{y_d\}$ be nets in X. Finally, let $\eta:\mathfrak{D}\to X$ be a net of values $\eta(d) = y_d$ at each $d\in\mathfrak{D}$. We say that the triplet (ξ, ξ_d, η) forms a *standard bundle of nets*.

The operator of emergence of nets has the following properties:

[en_1] $[\text{Nets } \{x\}]\cap[\mathfrak{N}(x)] = \varnothing,\ \forall x\in X$;

[en_2] $[\xi\in\mathfrak{N}(x)] \Leftrightarrow [\xi|_{(\leftarrow,a]}\in\mathfrak{N}(x),\ \forall a\in\mathfrak{D}]$;

[en_3] For arbitrary $A, B \subseteq X$, we have

$$[\text{Nets } A \subseteq \mathfrak{N}(x) \textit{ and } \text{Nets } B \subseteq \mathfrak{N}(x)] \Rightarrow [\text{Nets } (A \cup B)\subseteq \mathfrak{N}(x)];$$

[en_4] If (ξ, ξ_d, η) is a standard bundle of nets, then

$$[\xi\in\mathfrak{N}(x) \textit{ and } \xi_d \in\mathfrak{N}(\xi(d))] \Rightarrow [\eta\in\mathfrak{N}(x)] .$$

In addition, the proper order of χ allows an expression in terms of $\mathfrak{N}$,

$$\mathrm{K}(\chi) = \{(x, y) \in X^2 : \text{Nets } \{y\} \subseteq \mathfrak{N}(x)\}.$$

We may define abstract operators of emergence on arbitrary non-void sets. A function $\mathfrak{N} : X \to \mathscr{P}(\text{Nets } X)$ is an *abstract operator of emergence of nets* in X if it satisfies the conditions [en_1] – [en_4]. If so, the couple $(X, \mathfrak{N})$ forms a *world of emergence of nets*. In addition, we say that relation

$$\mathrm{K}(\mathfrak{N}) = \{(x, y) \in X^2 : \text{Nets } \{y\} \subseteq \mathfrak{N}(x)\}$$

is the *proper* (*strict*) *order* of $\mathfrak{N}$.

If $(X, \mathfrak{N})$ is a world of emergence of nets, then function $\chi_{\mathfrak{N}} : X \to \mathscr{P}(\mathscr{P}(X))$, of values

$$\chi_{\mathfrak{N}}(x) = \{P \in \mathscr{P}(X) : \text{Nets } P \subseteq \mathfrak{N}(x)\},$$

is a horistology of X. In addition, $\mathrm{K}(\mathfrak{N}) = \mathrm{K}(\chi_{\mathfrak{N}})$.

The ideals of perspectives and the operator of emergent nets are equivalent methods of defining a horistological structure, i.e.

$$\chi_{\mathfrak{N}_\chi} = \chi \textit{ and } \mathfrak{N}_{\chi_{\mathfrak{N}}} = \mathfrak{N}.$$

In a horistological world (X, χ), we may study emergence in terms of ideals. An ideal $\mathscr{I}\subseteq \mathscr{P}(X)$ is *emergent* from $x\in X$ if $\mathscr{I}\subseteq\chi(x)$. If so, we note $\mathscr{I}\leftarrow x$ and we define the *operator of emergent ideals* by

$$\mathfrak{I}(x) = \{\mathscr{I}\subseteq \mathscr{P}(X) : \mathscr{I}\leftarrow x\}.$$

Knowing an abstract operator of emergent ideals in an arbitrary nonvoid set X is equivalent to defining a horistology on this set.

II.5. Discrete Sets

In topology there exist several types of sets – open, closed etc. – with enough properties to recover the entire structure. In horistology there are other specific types of sets – discrete, admittance etc. – which have a similar role. In order to highlight them, besides K = K(χ), we have to operate with another strict order, Λ, on X (usually, $\Lambda \subseteq$ K): In particular, we refer to the *restriction* of χ to Λ, (noted $\chi|_\Lambda$ in Section II.3), for which K($\chi_{|\Lambda}$) = K(χ)$\cap\Lambda$.

Let M be a subset of X. We say that an element $x \in M$ is Λ-*detachable* from M (alternatively, M is Λ-*discrete* at x etc.) if

$$M \cap \Lambda[x] \in \chi(x).$$

The set of all Λ-detachable points of M forms the Λ-*discrete part* of M, noted $\partial_\Lambda(M)$. If each point of M is Λ-detachable, i.e. $\partial_\Lambda(M) = M$, then we consider that M is Λ-*discrete*. Function $\partial_\Lambda : \mathscr{P}(X) \to \mathscr{P}(X)$, which extracts the Λ-discrete part $\partial_\Lambda(M)$ of each subset $M \in \mathscr{P}(X)$, is called operator of Λ-*discreteness*. In the case Λ = K, we may omit mentioning Λ, and simply speak of *detachability* and *discreteness*. Alternatively, we may interpret the Λ-discreteness as discreteness relative to $\chi_{|\Lambda}$.

Among the remarkable properties of the operator ∂_Λ we mention:

[d$_0$] $\partial_\Lambda(M) \subseteq M$ for all $M \in \mathscr{P}(X)$;

[d$_1$] card $M \in \mathbb{N} \Rightarrow \partial_\Lambda(M) = M$;

[d$_2$] $L \subseteq M \Rightarrow L \cap \partial_\Lambda(M) \subseteq \partial_\Lambda(L)$;

[d$_3$] $\partial_\Lambda(M) \cap \partial_\Lambda(L) \subseteq \partial_\Lambda(M \cup L)$;

[d$_4$] $x \in \partial_\Lambda(M) \Leftrightarrow x \in M \cap \partial_\Lambda\left(\{x\} \cup \Lambda^{=}[M \cap \Lambda[x]]\right)$;

[d$_5$] $x \in \partial_\Lambda(\{x\} \cup M) \Leftrightarrow x \in \partial_\Lambda\left(\{x\} \cup \Lambda^{=}[M]\right)$, $\forall\ x \in X$ and $M \subseteq \Lambda[x]$;

[d$_6$] [$\Lambda \supseteq \Pi =$ *strict order*] $\Rightarrow \partial_\Lambda(M) \subseteq \partial_\Pi(M)$;

[d$_7$] [$\partial_\Lambda(M) = M$ & $\Lambda \supseteq \Pi =$ *strict order*] $\Rightarrow \partial_\Pi(M) = M$;

[d$_8$] [$\partial_\Lambda(M) = M$ & $L \subseteq M$] $\Rightarrow \partial_\Lambda(L) = L$;

[d$_9$] $\partial_\Lambda(\partial_\Lambda(M)) = \partial_\Lambda(M)$.

In particular, property [d$_1$] shows that horistological discreteness covers the classical notion of locally finitude. Selecting several properties of ∂_Λ, we may define an abstract *operator of discreteness* on arbitrary non-void set X. More exactly, function $\partial : \mathscr{P}(X) \to \mathscr{P}(X)$ is an *operator of discreteness* if:

[∂_1] card $M \in \mathbb{N} \Rightarrow \partial(M) = M$;

[∂_2] $L \subseteq M \Rightarrow L \cap \partial(M) \subseteq \partial(L)$;

[∂_3] $\partial(M) \cap \partial(L) \subseteq \partial(M \cup L)$.

In addition, if Λ is a strict order on X, such that the equivalence

$[\partial_4]$ $x \in \partial(M) \Leftrightarrow x \in M \cap \partial\big(\{x\} \cup \Lambda^{=}[M \cap \Lambda[x]]\big)$

holds for all $M \in \mathscr{P}(X)$, then Λ is said to be *compatible* with ∂.

In this case, the triplet (X, ∂, Λ), where $[\partial_1]$ - $[\partial_4]$ hold, is called world of *discreteness*. As before, we say that $x \in \partial(M)$ is *detachable* from M, $\partial(M)$ is the *discrete part* of M, and $M \in \mathscr{P}(X)$ is a *discrete* set if $\partial(M) = M$.

If (X, ∂, Λ) is a world of discreteness, then function

$$\chi_{(\partial,\Lambda)} : X \to \mathscr{P}(\mathscr{P}(X)),$$

of values

$$\chi_{(\partial,\Lambda)}(x) = \{P \subseteq \Lambda[x] : x \in \partial(\{x\} \cup P)\},$$

is a horistology on X. In addition, the proper order of $\chi_{(\partial,\Lambda)}$ is $\mathrm{K}(\chi_{(\partial,\Lambda)}) = \Lambda$. On the other hand, if ∂_Λ is the operator of discreteness in the horistological space $(X, \chi_{(\partial,\Lambda)})$, then $\partial_\Lambda = \partial$.

Conversely, if (X, χ) is a horistological world, K be the proper order of χ and ∂ is the operator of discreteness in (X, χ), relative to K, then K is compatible with ∂, and for the horistology $\chi_{(\partial,\mathrm{K})}$ we have $\chi_{(\partial,\mathrm{K})} = \chi$.

The discrete sets and discrete functions have simple and natural connections: If (X_1,χ_1) and (X_2,χ_2) be horistological worlds, let function $f : X_1 \to X_2$ be 1:1 and *discrete* on X_1. If a point $x \in M$ is detachable from $M \in \mathscr{P}(X_1)$, then $f(x)$ is Λ-detachable from $f(M)$ in X_2, where $\Lambda = f_{II}(\mathrm{K}(\chi_1))$.

This property highlights a special type of functions that act between worlds of discreteness $(X_1, \partial_1, \Lambda_1)$ and $(X_2, \partial_2, \Lambda_2)$. If function $f : X_1 \to X_2$ satisfies the condition $f(\partial_1(M)) \subseteq \partial_2(f(M))$ for all $M \subseteq X_1$, then f is called *detachability preserving* function.

Alternatively, if the implication $M = \partial_1(M) \Rightarrow f(M) = \partial_2(f(M))$ holds at each $M \subseteq X_1$, then f is named *discreteness preserving* function.

The discrete functions are discreteness preserving. More exactly: Let (X_1,χ_1) and (X_2,χ_2) be horistological worlds, and let function $f : X_1 \to X_2$ be 1:1 and *discrete* on X_1. If ∂_1 and ∂_2 are the discreteness operators on (X_1,χ_1) and $(X_2,\chi_{2|\Lambda})$, where $\Lambda = f_{II}(\mathrm{K}(\chi_1))$, then function f is *discreteness preserving*.

The discreteness preserving functions are specific to worlds of events since the causal automorphisms of the Minkowskian space-time $\mathbb{R}^4$ preserve the discreteness of the sets relative to the intrinsic horistology. In fact, as Zeeman's Theorem says, the group of causal automorphisms consists of Lorentz transformations, translations and dilations, all of them being discrete functions. In addition, because $f_{II}(\mathrm{K}(\chi)) = \mathrm{K}(\chi)$ holds for each causal automorphism f, we may conclude that these functions really do preserve discreteness.

Conversely, let $f : X_1 \to X_2$ be a strictly monotonic function relative to the orders $\mathrm{K}(\chi_1)$ and $\mathrm{K}(\chi_2)$ of the horistological worlds (X_1, χ_1) and (X_2, χ_2). If f preserves detachability, then it is discrete on X_1.

If, in addition $f_{II}(\mathrm{K}(\chi_1)) = \mathrm{K}(\chi_2)$, then

$$[\, f \textit{ is discrete on } X_1] \Leftrightarrow [\, f \textit{ preserves detachability}].$$

In point-set topology, some properties concerning a point x and a set M refer to $x \in M$ (e.g. interior), while other accept $x \notin M$ (adherent). In horistology, we have a similar situation: the detachability from M refers to points $x \in M$, but the premise points $x \in \mathbf{p}(M)$ are always outside of M. Relative to discreteness, we may remark that condition $M \cap \Lambda[x] \in \chi(x)$, which defines the detachability of x, is applicable to $x \notin M$ too. If satisfied, it shows that M "*admits*" adding x to its discrete part $\partial_\Lambda(M)$, i.e. $\partial_\Lambda(\{x\} \cup M) = \{x\} \cup \partial_\Lambda(M)$. Thus, if Λ is an order on the horistological world (X, χ), such that $\Lambda \subseteq \mathrm{K}$, and M is a subset of X, then a point $x \in X \setminus M$ is Λ-*admitted* by M (alternatively, M *admits* x etc.) if

$$M \cap \Lambda[x] \in \chi(x).$$

The set of all points Λ-admitted by M forms the Λ-*admittance* of M, noted $\alpha_\Lambda(M)$. Function $\alpha_\Lambda : \mathscr{P}(X) \to \mathscr{P}(X)$, which attaches the Λ-admittance $\alpha_\Lambda(M)$ to each subset $M \in \mathscr{P}(X)$, by the formula

$$\alpha_\Lambda(M) = \{x \in X : M \cap \Lambda[x] \in \chi(x)\},$$

is called *operator of* Λ-*admittance*. In the case $\Lambda = \mathrm{K}$, we may omit mentioning Λ, and simply speak of *admittance*. Alternatively, we may interpret the Λ-admittance as admittance relative to $\chi_{|\Lambda}$.

We may select the most significant properties of the Λ-*admittance* to define an *abstract operator of admittance* in arbitrary nonvoid sets. More exactly, if X is a non-void set and function $\alpha : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfy the conditions:

[α_1] card $M \in \mathbb{N} \Rightarrow \alpha(M) = X \setminus M$;

[α_2] $L \supseteq M \Rightarrow \alpha(L) \subseteq \alpha(M)$;

[α_3] $\alpha(M \cup L) = \alpha(M) \cap \alpha(L)$ for all $L, M \in \mathscr{P}(X)$,

then a is called *operator of pre-admittance*. If Λ is a strict order on X such that

[α_4] $x \in \alpha(M) \Leftrightarrow x \in \alpha\big(\{x\} \cup \Lambda^{=}[M \cap \Lambda[x]]\big)$ for all $M \in \mathscr{P}(X)$,

then we say that α is *compatible* with Λ. If so, the triplet (X, α, Λ), which satisfies the conditions [α_1] - [α_4], represents a *world of admittance*. Function α is an *abstract operator of admittance*.

Like (∂, Λ) from above, (α, Λ) generates a horistology too: If (X, α, Λ) is a world of admittance, then function

$$\chi_{(\alpha,\Lambda)} : X \to \mathscr{P}(\mathscr{P}(X)),$$

of values

$$\chi_{(\alpha,\Lambda)}(x) = \{P \subseteq \Lambda[x] : x \in \alpha(P)\},$$

is a horistology on X. In addition, the proper order of $\chi_{(\alpha,\Lambda)}$ is $\mathrm{K}(\chi_{(\alpha,\Lambda)}) = \Lambda$.

Consequently, if α is the operator of admittance in (X, χ), relative to the proper order K, of the horistological world (X, χ), then K is compatible with α, and $\chi_{(\alpha,\mathrm{K})} = \chi$.

Similarly, let (X, α, Λ) be a world of admittance and $\chi_{(\alpha,\Lambda)}$ be the horistology attached to (α, Λ). If α_Λ represents the operator of admittance in the horistological world $(X, \chi_{(\alpha,\Lambda)})$, then it is compatible with Λ, and $\alpha_\Lambda = \alpha$.

To conclude, the *families of perspectives* and the *operators of admittance* represent equivalent ways of defining horistologies.

There are connections between the discreteness of a function and admittance:

Let (X_1,χ_1) and (X_2,χ_2) be horistological worlds, and let a function $f : X_1 \to X_2$ be 1:1 and *discrete* on X_1. If an element $x \in X_1$ is admitted by a set $M \in \mathscr{P}(X_1)$, then $f(x)$ is Λ-admitted by $f(M)$ in X_2, where

$$\Lambda = f_{II}(\mathrm{K}(\chi_1)).$$

Consequently, if α_1 and α_2 are operators of admittance in X_1 and X_2, then f *preserves the admittance*, i.e. $f(\alpha_1(M)) \subseteq \alpha_2(f(M))$ holds at each $M \subseteq X_1$.

The converse relation holds: Let (X_1,χ_1) and (X_2,χ_2) be horistological worlds and let α_1 and α_2 be the corresponding operators of admittance. If a function $f : X_1 \to X_2$ is strictly monotonic relative to $\mathrm{K}(\chi_1)$ and $\mathrm{K}(\chi_2)$, then:

$$[\,f \textit{ preserves admittance}\,] \Rightarrow [\,f \textit{ is discrete on } X_1\,].$$

Combining these properties we obtain a stronger connection: Let (X_1,χ_1) and (X_2,χ_2) be horistological worlds and let the function $f : X_1 \to X_2$ be 1:1 and strictly monotonic relative to $\mathrm{K}(\chi_1)$ and $\mathrm{K}(\chi_2)$. If $f_{II}(\mathrm{K}(\chi_1)) = \mathrm{K}(\chi_2)$, then

$$[\,f \textit{ preserves admittance}\,] \Leftrightarrow [\,f \textit{ is discrete on } X_1\,].$$

Other connections between horistological notions like *perspectives*, set-to-set *operators* (premise, discreteness, admittance, etc.), *emergence* and properties of the functions (*discreteness*, *detachability* etc.) produce a lot of mathematical statements. For example, M is a discrete set in the horistological world (X, χ) if and only if the property

$$\textit{every net in } M \cap \mathrm{K}[x] \textit{ is emergent from } x$$

holds at each $x \in M$.

II.6. Living Systems

Even if we study the living systems by classical means – differential and integral equations, variational methods etc. – their evolution hides elements of structural discreteness.

In the simplest case, the mathematical model reduces to a single equation,

$$y'(x) = \varphi(x, y(x)).$$

The standard problem, say [*P*], consists in finding a solution of this equation, which satisfies the Cauchy condition $y(x_0) = y^0$. An important theorem of existence and uniqueness of a solution asks the Lipschitz condition of φ, hence it is continuous; in particular, φ is bounded by *M*. Problem [*P*] is equivalent to solving the integral equation

$$y(x) = y^0 + \int_{x_0}^{x} \varphi(\tau, y(\tau))d\tau \ ,$$

where we may recognize $x = \theta = time$, $y' = p$, $M = ¢$ and Ξ = the universe of events of the form $\varepsilon = (\theta, p)$. Consequently, $|p| < ¢$ and

$$\frac{1}{¢}\int_{\theta_0}^{\theta} \varphi(\tau, y(\tau))d\tau = ð([\varepsilon_0, \varphi, \varepsilon])$$

is the difficulty of going through this interval of life. Transposing the problem in the universe *W*, we see that solving problem [*P*] reduces to knowing the difficulty $đ(t) = ð([\varepsilon_0, \varphi, \varepsilon(\theta)])$, like in Section I.4. The substitution $t = \theta - \theta_0$ converts the integral equation into a solution of [*P*], namely

$$y(t) - y^0 = ¢\ đ(t).$$

It is widely accepted that Nature obeys the Least Action Principle, where

$$\mathbb{A} = \int_{e_0}^{e_1} L\ dt,$$

is the *action* and *L* is called *Lagrangean*. The Theory of Variations says that each function that minimizes $\mathbb{A}$ shall satisfy the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0.$$

Naturally, all living systems respect the Least Action Principle, but to use it, we have to find the correct Lagrange function *L*. Based on the analogy between proper time and hope, we suggest the formula

$$\mathbb{A} = -\alpha \int_{e_0}^{e_1} dħ = \frac{-\alpha}{¢} \int_{t_0}^{t_1} \sqrt{¢^2 - p^2}\, \mathrm{d}t \ ,$$

where, like in realativity, $\alpha > 0$. Consequently, the Lagrange function is

$$L = \sqrt{¢^2 - p^2}.$$

If we identify *x* with difficulty, then $\dot{x} = p$ is a generalized speed, hence the Euler-Lagrange equation reduces to $\frac{\partial L}{\partial p}$ = constant in time. Consequently,

$$p(¢^2 - p^2)^{-1/2} = k,$$

hence maintaining a constant *p*-deviation shows that the inertial Relativist Living Systems respect the Least Action Principle.

II.7. Discrete Instability

It is well known that stability expresses the continuity of the "*initial state – evolution*" function of a dynamical system. Now, we analyze the opposite property, usually called *instability*. Instead of a logical negation of continuity, we refer to the dual property, which is the *horistological discreteness* relative to horistological structures on the set W of states, respectively on the global state set ST, where the evolution functions take values, $\mathsf{f} : W \to \mathsf{ST} = W^T$. We note them χ^W, respectively χ^{ST}.

The initial state $w_0 \in W$ of system $\mathfrak{S}$ is called $\chi^W - \chi^{\mathsf{ST}}$ *discretely instable* if the "*initial state – evolution*" function $\mathsf{f} : W \to \mathsf{ST}$ is discrete at w_0, i.e.

$$[P \in \chi^W (w_0)] \Rightarrow [(\mathsf{f}(P) \in \chi^{\mathsf{ST}} (\mathsf{f}(w_0))] .$$

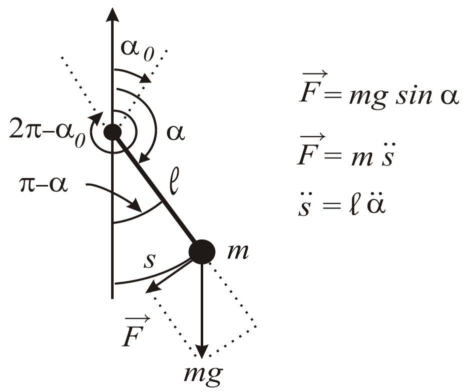


Fig.1.

The *mathematical pendulum* is a simple example for the *stability – instability* dichotomy (as sketched in Fig.1). Its equation is

$$\ddot{\alpha} - \omega^2 \sin \alpha = 0,$$

where $\omega^2 = g / \ell$ and α is the angle of the rod and the local vertical. Relative to the usual topologies of W and W^T, $\alpha_1(0) = 0$ is instable, while $\alpha_2(0) = \pi$ is stable initial state of the pendulum. In addition, $\alpha_1(0)$ is *discretely instable* relative to the horistology $\chi^{\mathbb{R}}$, respectively to the horistology $\chi^{|\cdot|}$ of W^T.

Another example is a time invariant linear system described by the equation

$$x' = A x + B u ,$$

where A and B are constant matrices and u is the input. It has the solution

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau .$$

Its null solution is internally stable if and only if all proper values of the matrix A have negative real parts and those with null real part are simple. Because only the increasing exponentials represent discrete components of x, the discrete instability holds if and only if *all* proper values of A have positive real parts.

II.8. Unification by Discreteness

The presence of the super-additive norms and metrics, hence of a horistology in the background of a theory, is a sign of its structural discreteness. Perhaps the most significant unification by structural discreteness is that of the relativist and quantum physics.

The most important aspects in favor of a structural discreteness of the universes of events derive from the following facts:

- The intrinsic norms (and metrics) are super-additive;
- The Lorentz transformations are discrete functions;
- The horistological discreteness of the universe of events fits reality.

In particular, the relativist theories that reduce discreteness to finiteness have the same character, since the finite sets are always hor-discrete. Theories on Lorentz transformations, e.g. Zeeman's analysis of causality, or Nottale's Scale Theory, also join structural discreteness.

To quantize the relativist proper time, we may restrict the S.a. norm $\|\cdot\|_t$ to

$$K_{\hbar} = \{(e_1, e_2) \in \mathfrak{E}^2 : t_2 - t_1 > \sqrt{\hbar^2 + \|x_2 - x_1\|^2}\ .$$

The main arguments supporting the idea of a structural discreteness of the quantum physics rise from the presence of some super-additive norms and metrics, which allow a natural quantization. Quantum physics operates with quantized S.a. metrics with at least two purposes, namely to evaluate quantized physical quantities, respectively to express the principles of uncertainty.

The errors Δx and Δp in evaluating the position and impulse of a particle obey the Heisenberg's inequality $|\Delta x|\ |\Delta p| > \hbar$. Obviously, this relation involves the quantized super-additive norm $\mathfrak{h} : \Pi_{\hbar}^{=}[0, 0] \to \mathbb{R}_+$, of values

$$\mathfrak{h}(x, y) = \begin{cases} 0 & if\ (x, y) = (0,0) \\ \sqrt{xy} & if\ (x, y) \in \Pi_{\hbar}[(0,0)] \end{cases}$$

where

$$\Pi_{\hbar} = \{((x, y), (u, v)) \in \Pi : (u - x)\ (v - y) > \hbar\ \}.$$

Thus, in the resulting horistology $\chi_{\hbar}$, we recognize the Heisenberg's principle of uncertainty, since each hyperbolic perspective has a radius greater than $\sqrt{\hbar}$.

The Heisenberg's principle allows formulations in terms of discreteness of the error function, which attaches to each measurement the pair of errors ($|\Delta x|$, $|\Delta p|$). All what we need is a horistology on the set of all possible measurements and the product horistology of $\mathbb{R} \times \mathbb{R}$.

II.9. Horistological $\mathbb{R}$

In the classical framework, there exist two ways to obtain $\mathbb{R}$ from $\mathbb{Q}$, namely:
1. Order completion, based on Dedekind *cuts*, and
2. Topological completion, started by Cantor, which involves *fundamental* (Cauchy) *sequences*, continuous fractions, decimal approximations, etc.

Now, we may remark a third variant of constricting $\mathbb{R}$, which operates with the intrinsic horistological structure of $\mathbb{Q}$. If $\mathsf{R} = <$ denotes the strict order of $\mathbb{Q}$, then $\nmid \cdot \nmid : \mathsf{R}[0] \to \mathbb{Q}_+$ of values $\nmid x \nmid = x$, is a S.a. norm on $\mathbb{Q}$. The resulting S.a. metric $\rho : \mathsf{R} \to \mathbb{Q}_+$ generates the hyperbolic prospects

$$\mathsf{H}_r = \{(x, y) \in \mathbb{Q}^2 : y - x > r\}$$

where $r > 0$, and finally, the intrinsic u-horistology of $\mathbb{Q}$, namely

$$\mathscr{H}^{\mathbb{Q}} = (\mathsf{H} \subset \mathbb{Q}^2 : \exists r > 0 \textit{ such that } \mathsf{H} \subset \mathsf{H}_r\} .$$

As usually, $\mathscr{H}^{\mathbb{Q}}$ generates the intrinsic horistology of $\mathbb{Q}$, noted $\chi^{\mathbb{Q}}$.

The horistological completion of $\mathbb{Q}$ shall follow the topological model, based on fundamental nets. Because in topology "a net ξ is *fundamental* if and only if it is uniformly continuous on $\mathfrak{D}$", we have to adapt the construction of $\mathbb{R}$ to the uniform horistological structures of $\mathbb{Q}$ and $\mathfrak{D}$. Thus, to obtain a natural s-u-horistology on a directed set $(\mathfrak{D}, \leq)$, to each $d \in \mathfrak{D}$, we attach the set

$$\mathsf{H}_d = \{(m, n) \in \mathfrak{D}^2 : n < m < d\},$$

such that $\mathscr{B}^{\mathfrak{D}}_{\mathsf{hor}} = \{\mathsf{H}_d : d \in \mathfrak{D}\}$ forms an ideal base. It generates a u-horistology on $\mathfrak{D}$, noted $\mathscr{H}^{\mathfrak{D}}$.

If $(X, \mathscr{H})$ is a (s-) u-horistological world and $(\mathfrak{D}, \leq)$ is a directed set, we say that $\xi : \mathfrak{D} \to X$ is a *horistologically* (briefly *hor-*) *fundamental net* if

$$\forall d \in \mathfrak{D} \ \exists \mathsf{H} \in \mathscr{H} \text{ such that } [n < m < d] \Rightarrow [\xi_{II}(m, n) \in \mathsf{H}].$$

Obviously, the net ξ is hor-fundamental if and only if ξ is u-discrete relative to the intrinsic s-u-horistology $\mathscr{H}^{\mathfrak{D}}$ of $\mathfrak{D}$, i.e. $\forall d \in \mathfrak{D} \ \exists \mathsf{H} \in \mathscr{H}$ such that

$$[(m, n) \in \mathsf{H}_d] \Rightarrow [\xi_{II}(m, n) \in \mathsf{H}],$$

or, equivalently, $\xi_{II}(\mathscr{H}^{\mathfrak{D}}) \subseteq \mathscr{H}$. In particular, if (X, R, ρ) be a S.a. (p-) metric world, then a net $\xi : \mathfrak{D} \to X$ is hor-fundamental if and only if

$$\forall d \in \mathfrak{D} \ \exists \varepsilon > 0 \text{ such that } [n < m < d] \Rightarrow [\rho(\xi(m), \xi(n)) > \varepsilon],$$

where tacitly, $(\xi(m), \xi(n)) \in \mathsf{R}$.

For example, if a sequence $\xi : \mathbb{N} \to X$ is decreasing relative to the usual strict order of $\mathbb{N}$ and $\mathrm{K}(\mathscr{H})$, then ξ is hor-fundamental. If a hor-fundamental sequence ξ in $\mathbb{R}$ has a lower bound, then it has a (single) proper germ, which also is an emitter of ξ. However, in $\mathbb{Q}$ it is possible to have $\mathsf{e\text{-}Germ}\ \xi = \varnothing$, as a consequence of its incompleteness.

The completion of a u-horistological world $(X, \mathscr{H})$ by hor-fundamental sequences operates with several specific notions: A sequence $\xi : \mathbb{N}\to X$ is called *emission* in X if it is hor-fundamental and lower K-bounded. We note the set of all emission in X by $€(X)$, or simply $€$. We say that $(X, \mathscr{H})$ is *horistologically complete* if each emission ξ has an emitter, i.e. e-Germ $\xi \neq \varnothing$ for all $\xi\in €(X)$.

Let ξ and η be two emissions in $(X, \mathscr{H})$ and $\mathrm{K}(\mathscr{H}) = \mathrm{K}$. If

$$\forall\, m\in\mathbb{N} \;\; \exists\, n(m)\in\mathbb{N} \textit{ such that } (\xi(n(m)), \eta(m))\in \mathrm{K},$$

then we say that ξ *precedes* η, and we note $\xi \precsim \eta$. If $\xi \precsim \eta$ and $\eta \precsim \xi$, then we note $\xi \approx \eta$. If $\xi \precsim \eta$ without $\eta \precsim \xi$, then we say ξ *strictly precedes* η, and we note $\xi \prec \eta$, which means

$$\exists\, m\in\mathbb{N} \textit{ such that } [\forall\, n\in\mathbb{N}] \Rightarrow (\xi(n), \eta(m))\notin \mathrm{K}.$$

We may organize $€(X)$ as a u-horistological world: If for each prospect $\mathsf{H}\in\mathscr{H}$ we note

$$\mathsf{H}^{€} = \{(\xi, \eta)\in €^{2} : (\xi(n), \eta(n))\in \mathsf{H} \textit{ for all } n\in\mathbb{N}\},$$

then the family

$$\mathscr{H}^{€} = \{\mathbf{P}\in \mathscr{P}(€^{2}) : \exists\, \mathsf{P}\in\mathscr{H} \text{ such that } \mathbf{P}\subseteq \mathsf{P}^{€}\}$$

is a u-horistology on $€$. In addition, the (strict) proper order of $\mathscr{H}^{€}$ is

$$\mathrm{K}(\mathscr{H}^{€}) = \{(\xi, \eta)\in €^{2} : (\xi(n), \eta(n))\in \mathrm{K}(\mathscr{H}) \textit{ for all } n\in\mathbb{N}\}.$$

Each emission $\xi\in €(\mathbb{Q})$ generates a $\approx$ – equivalence class

$$\xi^{\wedge} = \{\eta\in €(\mathbb{Q}) : \xi \approx \eta\},$$

called *hor-real number*. The set of all hor-real numbers, noted

$$€^{\wedge} = €(\mathbb{Q}) / \approx \; = \{ \xi^{\wedge} : \xi\in €(\mathbb{Q})\},$$

forms the *horistological* $\mathbb{R}$ (briefly *hor*-$\mathbb{R}$).

Using some representatives $\xi^{/} \in \xi^{\wedge}$ and $\eta^{/} \in \eta^{\wedge}$, we can introduce algebraic operations and an order relation between hor-real numbers. Thus, it follows that hor-$\mathbb{R}$ is a totally and completely ordered field. In addition, the resulting hor-$\mathbb{R}$ is isomorphic to other copies of $\mathbb{R}$ (obtained by topological, order, or other completions of $\mathbb{Q}$), since two commutative fields, endowed with complete and total orders, are always isomorphic.

II.10. $\mathcal{L}_p$'s duals with $p < 1$

Paper [CB] starts with examples of S.a. norms and S.a. normed linear spaces, in particular L_p norms for p in $(-\infty, 0)$ or (0, 1). Then it presents representations of the "*dual*" space for some of these examples, consisting *strictly plus* linear functionals $f : X \to \mathbb{R}$, which satisfy the reverse of the inequality (of continuity)

$$f(x) \leq k\|x\|,$$

with some k in $\mathbb{R}_+$ for all x in X. (See Theorems 1-5 and details below.)

The simplest example is the plane $X = \mathbb{R}^2$ of relativist events (hyperbolic numbers, Banach space with J-metric etc.) ordered by the cone

$$C_2 = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 > |a_2|\},$$

of future events (positive hyperbolic numbers etc.). Function $p_2 : C_2 \to (0, \infty)$, of values $p_2(a_1, a_2) = sqrt(a_1^2 - a_2^2)$ satisfies the conditions ($(0, 0) \notin C_2$!)

$$p_2(\lambda a) = \lambda\, p_2(a) \qquad (1)$$

and

$$p_2(a + b) \geq p_2(a) + p_2(b) \qquad (2)$$

for any a and b in C_2 and λ in $(0,\infty)$, hence it is a S.a. norm (like $a_1 - |a_2|$ etc.).

The triplet $(\mathbb{R}^2, C_2, p_2)$ is a S.a. normed linear space, which generates other universes of events, happening in a normed linear space $(X, \|\cdot\|)$ over $\mathbb{R}$, namely $E = \mathbb{R} \times X$ (previously noted W). This universe is ordered by the cone

$$C = \{(t, x) \in E\colon (t, \|x\|) \in C_2\}$$

and the S.a. norm $p : C \to (0,\infty)$ has the values $p(t, x) = p_2(t, \|x\|)$. So, (E, C, p) will be a S.a. normed linear space.

Other examples are the spaces of s-measurable functions. Let $(M, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{L}^s$, for $s \in (0, 1]$ denote the functions $f : M \to \mathbb{R}$ which are measurable, with $\int |f|^s < \infty$ and let L^s be the equivalence classes of functions in $\mathcal{L}^s$ equal a.e. Then function $\|\cdot\|_s$, of values

$$\|f\|_s = \left(\int |f|^s\right)^{1/s},$$

restricted to

$$C(L^s) = \{ f \in L^s : f \geq 0, f \neq 0 \}$$

or to

$$C(\mathcal{L}^s) = \{ f \in \mathcal{L}^s : f \geq 0, f \neq 0 \},$$

is a S.a. norm because it satisfies (1) and (2) for all $f, g \in C(L^s)$ or $C(\mathcal{L}^s)$, and $\lambda \in (0,\infty)$.

Similarly, let $s \in (-\infty, 0)$, and let

$$\mathcal{C}_s = \{g\colon M \to (0,\infty) : g \textit{ measurable}, \int g^s \in (0,\infty)\}.$$

Define $\|g\|_s = (\int |g|^s)^{1/s}$ for $g \in \mathcal{C}_s$. Let C_s denote the equivalent classes of elements of $\mathcal{C}_s$ which are equal a.e., giving $\|g\|_s$ well defined on C_s. In the case $\mu(M) = 1$, $\|g\|_{-1}$ is the harmonic mean and it is well known that (1) and (2) hold. We will assume $(M, \mathcal{B}, \mu)$ is σ- finite so that C_s is non-empty.

Another example refers to positive lower bounded measurable functions. Let

$$\mathcal{C}_{-\infty} = \{f : M \to (0,\infty) : f \textit{ measurable and } \exists m > 0 \text{ with } f \geq m \text{ a.e}\}.$$

Function $\|\cdot\|_{-\infty} : \mathcal{C}_{-\infty} \to (0, \infty)$, of values

$$\|f\|_{-\infty} = \sup\{m > 0 : f \geq m \textit{ a.e.}\},$$

is a S.a. norm (compare to lower bounded continuous functions on a compact). Let $C_{-\infty}$ denote the equivalence classes of elements of $\mathcal{C}_{-\infty}$, which are equal a.e., giving $\|f\|_{-\infty}$ well defined on $C_{-\infty}$.

Finally, let $(M, \mathcal{B}, \mu)$ be a probability space, and let C be equivalence classes of measurable functions f from M to $(0,\infty)$ with $\int |\log f| < \infty$. For x in C, let

$$\|x\| = \exp \int \log x,$$

which is a S.a. norm. In fact, (1) uses $\int d\mu = 1$, while (2) follows from the fact that $\{x : \|x\| \geq 1\}$ is a convex set.

In particular, if $M = \mathbb{N}_n$ and $\mu(i) = n^{-1}$ for all i, then $\|x\|$ is the classical geometric mean, while if $M = \mathbb{N}$, then $\|x\| = \prod_{i=1}^{\infty} x_i^{\mu(i)}$.

Let $(X, C, \|\cdot\|)$ be a S.a. normed linear space. Let function $A : C \to (0,\infty)$ satisfy $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$ for all x and y in C and λ, μ in $(0,\infty)$. We say A is a *strictly plus functional* if there is $\varepsilon > 0$ such that $p(x) \geq 1$ implies $Ax \geq \varepsilon$. We may add strictly plus functionals and multiply by $\lambda \in (0,\infty)$ according to

$$(\lambda A + \mu B)\, x = \lambda Ax + \mu Bx$$

for $x \in C$. So the strictly plus functionals form a cone C^*_{sp} in the vector space of functions from C to $\mathbb{R}$, and we note its linear span by X^*_{sp}. For $A \in C^*_{sp}$ we let

$$\|A\|_{sp} = \inf\{Ax : \|x\| \geq 1\}.$$

We will identify a strictly plus functional with its linear extension to X since the cones satisfy $C - C = X$, respectively $C^*_{sp} - C^*_{sp} = X^*_{sp}$.

Similarly to the polar of a S.a. norm, $(X_{sp}^*, C_{sp}^*, \|\cdot\|_{sp})$ is a S.a. normed linear space, called *strictly plus dual*. The elements f of C_{sp}^* are uniformly discrete in the sense that for $\delta > 0$ there is $\varepsilon > 0$ such that for x, y in X with $x \in y + C$,

$$p(x-y) > \delta \;\Rightarrow\; f(x) - f(y) > \varepsilon .$$

Let (X_1, C_1, p_1) and (X_2, C_2, p_2) be two S.a. normed linear spaces and let operator $A : X_1 \to X_2$ be a linear bijection. If $A(C_1) = C_2$ and $p_2(Ax) = p_1(x)$ for each x in C_1, then we say these spaces are *isometrically isomorphic*, or simply identify them and write $(X_1, C_1, p_1) = (X_2, C_2, p_2)$, or $X_1 = X_2$.

To find the strictly plus dual of (E, C, p), let $(\mathbb{R}^{2*}, C_2^*, p_2^*)$ be the strictly plus dual of the space $(\mathbb{R}^2, C_2, p_2)$, with basis $\{e_1^*, e_2^*\}$ defined by $e_1^*(e_k) = \delta_{1k}$ and $e_2^*(e_k) = -\delta_{2k}$ for $k = 1$ and for $k = 2$. Let $(X^*, \|\cdot\|)$ denote the dual of $(X, \|\cdot\|)$, and let $G = \mathbb{R} \oplus X^*$ with time-cone

$$C^* = \{(t, f) \in G\colon\; te_1^* + \|f\| e_2^* \in C_2^*\}$$

and $p^*\colon C^* \to (0,\infty)$ defined by $p^*(t, f) = p_2^*(t, \|f\|)$. So, we obtain:

Theorem 1. (G, C^*, p^*) *is a S.a. normed linear space, isometrically isomorphic with* $(E_{sp}^*, C_{sp}^*, \|\cdot\|_{sp})$.

Let $s \in (0, 1)$ be fixed and let, as usually, l_s denote sequences x of real numbers with $\|x\|_s = (\sum |x_n|^s)^{1/s} < \infty$, ordered by the cone

$$C = \{x \geq 0 : \|x\|_s > 0\}.$$

Then $(l_s, C, \|\cdot\|_s)$ is a S.a. normed linear space, but in contrast to the finite dimensional case, there are no strictly plus linear functionals on l_s. It follows that if $L^s(M, \mathcal{B}, \mu)$ is infinite dimensional, then there are no strictly plus functionals on it. In fact, if $G : l_s \to \mathbb{R}$ is linear and strictly plus, then there exists a sequence g with $g_n > 0$ for all n and $G(y) = \sum_{n=1}^{\infty} g_n y_n$ for all y. Consequently,

Theorem 2. *For* $s \in (0,1)$, *there are no strictly plus linear functionals* on l_s.

In the case of l_s with s < 0, let s' be given by $\frac{1}{s} + \frac{1}{s'} = 1$, and let $(l_{s'})_0$ denote the finite sequences in $l_{s'}$. Let $C(l_s)$, $C(l_{s'})_0$ and $C(l_s)_{sp}^*$ denote the time-cones:

- $C(l_s)$ denotes the sequences x with $x_n > 0$ for all n and $\sum (1/x_n)^{-s} < \infty$. And l_s itself stands for $C(l_s) - C(l_s)$, i.e. the set of all sequences.
- $C(l_{s'})_0$ consists of sequences x, with $x_n > 0$ for some n, $x_n \geq 0$ for all n, and $x_n = 0$ for all but finitely many n, and

- $C(l_s)^*_{sp}$ denotes the strictly plus functionals on l_s .

Theorem 3. $(l_{s'})_0 = (l_s)^*_{sp}$.

To study duality in finite dimensional spaces, the following result is useful:

Proposition: *Suppose* $(M, \mathcal{B}, \mu)$ *is a* σ *– finite measure space and function* $g : M \to (0,\infty)$ *is measurable. Suppose there exists* $k \in (0,\infty)$ *such that for all* $f \geq 0$ *in* $L^s(M)$, *where* $s \in (0,1)$, *we have*

$$\int_M fg \, d\mu \geq k \| f \|_s \ .$$

Then $1/g \in L^{-s'}(M)$ where 1/ s + 1/ s $'$ = 1, and $\|g\|_{s'} \geq k$.

Theorem 4. Let s$\in$(0, 1) and let l_s^n denote $\mathbb{R}^n$ with s.a. norm $\|x\|_s$ on the cone $C_s = \{x \geq 0: x_i > 0$ for some $i\}$. Let $s^{-1} + (s')^{-1} = 1$, and let $l_{s'}^n$ denote $\mathbb{R}^n$ with s.a. norm $\|\cdot\|_{s'}$ on the cone $C_{s'} = \{x \geq 0: x_i > 0$ for all $i\}$. Then

(i) $(l_s^n)^*_{sp} = l_{s'}^n$, and

(ii) $(l_{s'}^n)^*_{sp} = l_s^n$.

If the S.a. norm given by the geometric mean, $\mathbb{R}^n$ equals its strictly plus dual.

Theorem 5. *Let* $X = \mathbb{R}^n$ be ordered by the cone $C = \{x \in \mathbb{R}^n$: $x_i > 0$ *for all* $i\}$, *and let* $p(x) = (\prod x_i)^{1/n}$. *Then* $X^*_{sp} = X$ *under the isomorphism* $\Phi : X \to X^*_{sp}$ *given by*

$$\Phi(x)(y) = \tfrac{1}{n} \sum x_i y_i$$

for all $y \in X$.

The following result (also in [CB]) resembles the Hahn-Banach Theorem, but refers to S.a. normed linear spaces:

Theorem (Hahn-Banach). *Let* (X, C, p) *be a S.a. normed linear space. Let* S *be a linear subspace and suppose* $S \cap C$ *is nonempty and* $f : S \cap C \to \mathbb{R}$ *satisfies, for* x, y *in* $S \cap C$ *and* $\mu, \lambda > 0$,

(i) $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$, *and*

(ii) $f(x) \geq p(x)$.

If we suppose that for all $y \in C$ *there is* $t \in S \cap C$ *such that* $t \in y + C$, *then there is* $F : C \to \mathbb{R}$ *extending* f *and satisfying* (*i*) *and* (*ii*).