A VARIANT OF RADON'S INEQUALITY FOR SEMINORMS

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Abstract. *In this paper some new Radon type inequalities for some seminorms on pseudo-Hilbert spaces and norms in Hilbert spaces are presented. Then several consequences and applications are given.*

Keywords: *pseudo-Hilbert spaces(Loynes spaces); Hilbert spaces; Radon's inequality; seminorms; norms.*

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1. INTRODUCTION

In pure and also applied mathematics, inequalities are a very important method for constructing qualitative and quantitative properties. Inequalities are historically viewed as a classical field of research. Applications of inequalities play significant roles in mathematics, physics, fractals, special functions, number theory and many other areas of research. The value of mathematical inequalities was very well established in past and inequalities like Jensen, Hardy, Cauchy-Schwarz, Hermite-Hadamard, Steffensen-Gruss, Radon, Popoviciu, Bergstrom and many other had an essential role in classical calculus. These inspired other researchers and in this way numerous novel results concerning inequalities have already been launched in the literature. Two classical book concerning such inequalities are the well-known book of E.F. Bechenbach and R. Bellman [1] and the book of D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink [2].

The classical Bergstrom's inequality was given in [3] and the Radon's inequality appeared first time in [4].

Last decades mathematicians have paid attention to the Bergstron's and Radon's inequalities due to its originality, symmetry and quality among mathematical inequalities. New generalizations, refinements, modifications and significant developments of Bergstrom's and Radon's inequalities have been presented in [5-13], see also the references therein for the interested reader.

The aim of this paper is to give new Radon type inequalities for seminorms q_p on Loynes spaces, for arbitrary seminorms of a family of seminorms which defines the topology of a linear space X and, as a consequence, also they are true on an arbitrary Hilbert space $\mathscr{H}$ with the classical norm $||.||$, using as a starting point Theorem 2.6., Theorem 2.7 and Theorem 2.8 form [13]. Several consequences and applications will be presented as well.

The organization of this paper is as follows: In Section 2 a short explanation of the concept and some associated work in this dirrection arepresented. In Section 3 the main outcomes are given in Theorem 3.1, Theorem 3.5, Theorem 3.7, Theorem 3.8 and Theorem 3.10 with the associated

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consequences and observations. Section 4 contains some conclusions and more dirrections for future research.

2. MATERIALS AND METHODS

2.1. MATERIALS

The concept of pseudo-Hilbert space was developed in time after publishing the papers of R. M. Loynes [14-15], these spaces being called pseudo-Hilbert spaces in [16] and Loynes spaces in [17] and [18]. By analogy the fundamental concepts properties and techniques of work in these spaces remain valid as in Hilbert spaces. In the following some known notions used in next sections will be presented, see [1-2],[17],[19].

Let **Z** be an admissible space in the Loynes sense. A topological linear space $\mathscr{H}$ is pre-Loynes **Z** space if the following conditions are satisfied: (L1) $\mathscr{H}$ is endowed with an **Z**- valued inner product (gramian), i.e. there is an application $(h,k) \in \mathscr{H} \times \mathscr{H} \rightarrow [\mathrm{h},\mathrm{k}] \in \boldsymbol{Z}$ with the properties: $[h,h] \geq 0$; $[h,h]=0$ implies $h=0$; $[h_1+h_2,h]=[h_1,h]+[h_2,h]$; $[\lambda h,k]=\lambda[h,k]$; $[h,k]^*=[k,h]$;

for all $h,k,h_1,h_2 \in \mathscr{H}$ and $\lambda \in \mathbb{C}$ and

(L2) The topology of $\mathscr{H}$ is the weakest locally convex topology on $\mathscr{H}$ for which the application $h \in \mathscr{H} \rightarrow [\mathrm{h},\mathrm{h}] \in \boldsymbol{Z}$ is continuous. If in addition, $\mathscr{H}$ is a complete space with this topology, then $\mathscr{H}$ is called Loynes **Z**-space. Every C^*- algebra with natural topology and involution is an example of admissible space. If **Z** is the above example of admissible space, then with inner product defined by $[z_1,z_2]=z_2^* z_1$ we get a Loynes **Z**-space.

The following results given in [19] introduce a seminorm on an arbitrary Loynes **Z**-space starting from a continuous and monotonous seminorm on **Z** and define a topology on $\mathscr{H}$ by using the corresponding topology on **Z**.

Lemma 2.1. If p is a continuous and monotous seminorm on **Z** then $q_p(h)=(p([h,h]))^{\frac{1}{2}}$ is a continuous seminorm on $\mathscr{H}$.

Proposition 2.1. If $\mathscr{H}$ is a pre-Loynes **Z**-space and Ƥ is a set of monotonous (increasing) seminorms defining the topology of **Z** then the topology of $\mathscr{H}$ is defined by the sufficient and directed set of seminorms $Q_{Ƥ}=\{q_p \mid p \in Ƥ\}$.

Let **Z** be an admissible space in the Loynes sense and $\mathscr{H}$, $\mathfrak{H}$ be two Loynes **Z**-spaces. The operator $T \in L(\mathscr{H},\mathfrak{H})$ is called gramian bounded if there is a constant $\mu>0$ so that in the sense of order of **Z**: $[Th,Th] \leq \mu[h,h], h \in \mathscr{H}$. The class of such operators is denoted by $B(\mathscr{H},\mathfrak{H})$. The introduced norm was $\left|\left|T\right|\right| = \inf\{\sqrt{\mu}, \mu>0, [Th,Th] \leq \mu[h,h], h \in \mathscr{H}\}$ and $B^*(\mathscr{H},\mathfrak{H})$= $= B(\mathscr{H},\mathfrak{H}) \cap L^*(\mathscr{H},\mathfrak{H})$ is a Banach space. Moreover, if $\mathscr{H}=\mathfrak{H}$ then $B^*(\mathscr{H})$ is a C^*- algebra. Hilbert spaces are particular cases of Loynes spaces.

2.2. METHODS

Some generalizations of Radon's inequality used in this paper will be stated below. First inequality is proven in [11] and is cited in [13], see inequality (1.4). The second and the third inequalities from below have been established in [13] in Theorem 2.8 and Theorem 3.11 and will be the starting point for the results from this paper.

Theorem 2.1. ([11]**)** If $n \in \mathbf{N}$, $x_k \geq 0$, $y_k > 0$ where $k \in \{1,2,\dots,n\}$ and $p \geq r \geq 0$,

$$\frac{x_1^{p+1}}{y_1^r} + \frac{x_2^{p+1}}{y_2^r} + \cdots + \frac{x_n^{p+1}}{y_n^r} \geq n^{r-p} \frac{(x_1 + x_2 + \cdots + x_n)^{p+1}}{(y_1 + y_2 + \cdots + y_n)^r}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$.

Theorem 2.2. ([13]**)** If $n \in \mathbf{N}$, $x_k \geq 0$, $y_k > 0$ where $k \in \{1,2,\dots,n\}$ and $p \geq 0$, $r \geq 1$, and $p + r - 2 \geq 0$, then

$$\frac{x_1^{p+r}}{y_1^p} + \frac{x_2^{p+r}}{y_2^p} + \cdots + \frac{x_n^{p+r}}{y_n^p} - \frac{(x_1 y_1^{r-1} + x_2 y_2^{r-1} + \cdots + x_n y_n^{r-1})^{p+r}}{(y_1^r + y_2^r + \cdots + y_n^r)^{p+r-1}} \geq$$

$$\geq \max_{1 \leq i < j \leq n} \left\{ \frac{x_i^{p+r}}{y_i^p} + \frac{x_j^{p+r}}{y_j^p} - \frac{(x_i y_i^{r-1} + x_j y_j^{r-1})^{p+r}}{(y_i^r + y_j^r)^{p+r-1}} \right\} \geq (\mathrm{p} + \mathrm{r} - 1) \max_{1 \leq i < j \leq n} \left\{ \frac{(y_i y_j)^{r-2} (x_i y_i^{r-1} + x_j y_j^{r-1})^{p+r-2} (x_i y_j - x_j y_i)^2}{(y_i^r + y_j^r)^{p+r-1}} \right\}.$$

Theorem 2.3. ([13]**)** If $n \in \mathbf{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1,2,\dots,n\}$ and $p \geq r \geq 0$ then

$$\frac{x_1^{p+1}}{y_1^r} + \frac{x_2^{p+1}}{y_2^r} + \cdots + \frac{x_n^{p+1}}{y_n^r} - n^{r-p} \frac{(x_1 + x_2 + \cdots + x_n)^{p+1}}{(y_1 + y_2 + \cdots + y_n)^r} \geq \max_{1 \leq i < j \leq n} \left\{ \frac{x_i^{p+1}}{y_i^r} + \frac{x_j^{p+1}}{y_j^r} - 2^{r-p} \frac{(x_i + x_j)^{p+1}}{(y_i + y_j)^r} \right\} \geq 0.$$

Two consequences of these inequalities established in [13] are stated below because we need them in next section.

Application 2.4. ([13]**)** If $a, b, c > 0$, prove that

$$\frac{a^5}{b^2} + \frac{b^5}{c^2} + \frac{c^5}{a^2} - \frac{(ab^2+bc^2+ca^2)^5}{(a^3+b^3++c^3)^4} \geq max \left(\frac{a^5}{b^2} + \frac{b^5}{c^2} - \frac{(ab^2+bc^2)^5}{(b^3+c^3)^4}; \frac{b^5}{c^2} + \frac{c^5}{a^2} - \frac{(bc^2+ca^2)^5}{(c^3+a^3)^4}; \frac{c^5}{a^2} + \frac{a^5}{b^2} - \frac{(ca^2+ab^2)^5}{(a^3+b^3)^4} \right).$$

Application 2.5. ([13]**)** If $a, b, c > 0$, then the inequality

$$\frac{a^5}{b^2}+\frac{b^5}{c^2}+\frac{c^5}{a^2}-\frac{1}{9}(a+b+c)^3 \geq max\left(\frac{a^5}{b^2}+\frac{b^5}{c^2}-\frac{1}{4}\frac{(a+b)^5}{(b+c)^4};\frac{b^5}{c^2}+\frac{c^5}{a^2}-\frac{1}{4}\frac{(b+c)^5}{(c+a)^4};\frac{c^5}{a^2}+\frac{a^5}{b^2}-\frac{1}{4}\frac{(c+a)^5}{(a+b)^4}\right).$$

3. RESULTS

Using as a starting point Theorem 2.6., Theorem 2.7 and Theorem 2.8 from [13] some new inequalities for the seminorms q_{p_1} will be presented. Then it can be seen that these inequalities remain true for arbitrary seminorms of a family of seminorms which defines the topology of a linear space X and as a consequence also they are true on an arbitrary Hilbert space $\mathscr{H}$ with the classical norm $||.||$. In addition, several consequences and applications are provided in this section.

Theorem 3.1. For $n \in \mathbf{N},\ n \geq 2,\ a_k > 0, h_k \in \mathscr{H}$ so that $q_{p_1}(h_k) > 0, k \in \{1,2,\dots,n\}$, $p \geq 0,\ r \geq 1$ the following inequality holds:

$$\sum_{k=1}^{n}\frac{q_{p_1}^{p+r}(h_k)}{a_k^p}-\frac{q_{p_1}^{p+r}(\sum_{k=1}^{n}a_k^{r-1}h_k)}{(\sum_{k=1}^{n}a_k^r)^{p+r-1}} \geq \max_{1\leq i<j\leq n}\left\{\frac{q_{p_1}^{p+r}(h_i)}{a_i^p}+\frac{q_{p_1}^{p+r}(h_j)}{a_j^p}-\frac{q_{p_1}^{p+r}(a_i^{r-1}h_i+a_j^{r-1}h_j)}{(a_i^r+a_j^r)^{p+r-1}}\right\}, \quad (1)$$

where $q_{p_1}(h)=[p([h,h])]^{\frac{1}{2}}$.

Proof: It will be considered $x_i=\sum_{k=1}^{i}\frac{q_{p_1}^{p+r}(h_k)}{a_k^p}-\frac{q_{p_1}^{p+r}(\sum_{k=1}^{i}a_k^{r-1}h_k)}{(\sum_{k=1}^{i}a_k^r)^{p+r-1}},\ i \leq n$ and it will be proven that $(x_n)_n$ is an increasing sequence. Therefore the expression $x_{n+1}-x_n$ will be examinated below.

$$x_{n+1}-x_n=\sum_{k=1}^{n+1}\frac{q_{p_1}^{p+r}(h_k)}{a_k^p}-\frac{q_{p_1}^{p+r}\left(\sum_{k=1}^{n+1}a_k^{r-1}h_k\right)}{(\sum_{k=1}^{n+1}a_k^r)^{p+r-1}}-\sum_{k=1}^{n}\frac{q_{p_1}^{p+r}(h_k)}{a_k^p}+\frac{q_{p_1}^{p+r}\left(\sum_{k=1}^{n}a_k^{r-1}h_k\right)}{(\sum_{k=1}^{n}a_k^r)^{p+r-1}}=$$

$$=\frac{q_{p_1}^{p+r}(h_{n+1})}{a_{n+1}^p}+\frac{q_{p_1}^{p+r}(\sum_{k=1}^{n}a_k^{r-1}h_k)}{(\sum_{k=1}^{n}a_k^r)^{p+r-1}}-\frac{q_{p_1}^{p+r}(\sum_{k=1}^{n+1}a_k^{r-1}h_k)}{(\sum_{k=1}^{n+1}a_k^r)^{p+r-1}}.$$

By using one of the basic properties of seminorms q_{p_1} we have,

$$q_{p_1}^{p+r}\left(\sum_{k=1}^{n+1}a_k^{r-1}h_k\right) \leq \left(q_{p_1}\left(\sum_{k=1}^{n}a_k^{r-1}h_k\right)+q_{p_1}(a_{n+1}^{r-1}h_{n+1}\right)^{p+r}$$

and from Radon's inequality applied for n=2, see [5], it will be obtained

$$\frac{q_{p_1}^{p+r}\left(\sum_{k=1}^{n+1}a_k^{r-1}h_k\right)}{(\sum_{k=1}^{n+1}a_k^r)^{p+r-1}} \leq \frac{\left(q_{p_1}\left(\sum_{k=1}^{n}a_k^{r-1}h_k\right)+q_{p_1}(a_{n+1}^{r-1}h_{n+1})\right)^{p+r}}{(\sum_{k=1}^{n}a_k^r+a_{n+1}^r)^{p+r-1}} \leq$$

$$\leq \frac{\left(q_{p_1}(\sum_{k=1}^{n} a_k^{r-1}h_k)\right)^{p+r}}{\left(\sum_{k=1}^{n} a_k^r\right)^{p+r-1}} + \frac{q_{p_1}^{p+r}(a_{n+1}^{r-1}h_{n+1})}{(a_{n+1}^r)^{p+r-1}}.$$

From here we see that the expression $x_{n+1} - x_n \geq 0$ if $\frac{q_{p_1}^{p+r}(a_{n+1}^{r-1}h_{n+1})}{(a_{n+1}^r)^{p+r-1}} \leq \frac{q_{p_1}^{p+r}(h_{n+1})}{a_{n+1}^p}$ which is obvious because $(a_{n+1}^{r-1})^{p+r}a_{n+1}^p = a_{n+1}^{r(p+r-1)}$.

Since $(x_n)_n$ is an increasing sequence we get, $x_{n+1} \geq x_n \geq \cdots \geq x_1$. But

$$x_1 = \frac{q_{p_1}^{p+r}(h_1)}{a_1^p} - \frac{q_{p_1}^{p+r}\left(a_1^{r-1}h_1\right)}{(a_1^r)^{p+r-1}} = q_{p_1}^{p+r}(h_1)\left(\frac{1}{a_1^p} - \frac{1}{a_1^{r(p+r-1)-(r-1)(p+r)}}\right) = 0.$$

In addition, it can be seen that $x_n \geq x_2 = \frac{q_{p_1}^{p+r}(h_1)}{a_1^p} + \frac{q_{p_1}^{p+r}(h_2)}{a_2^p} - \frac{q_{p_1}^{p+r}(a_1^{r-1}h_1 + a_2^{r-1}h_2)}{(a_1^r + a_2^r)^{p+r-1}}$ for all $n \in \mathbf{N}, \mathrm{n} \geq 2$.

By symmetry of x_n relatively to the variables a_i and h_i , $i, j \in \{1,2, \dots, n\}$ we have,

$x_n \geq \frac{q_{p_1}^{p+r}(h_i)}{a_i^p} + \frac{q_{p_1}^{p+r}(h_j)}{a_j^p} - \frac{q_{p_1}^{p+r}(a_i^{r-1}h_i + a_j^{r-1}h_j)}{(a_i^r + a_j^r)^{p+r-1}}$ for all

$n \in \mathbf{N}$, $\mathrm{n} \geq 2$, $i, j \in \{1,2, \dots, n\}$.

Corollary 3.2. Under previous conditions, the above inequality remains true for every arbitrary seminorm $p_1, p_1 : X \to R_+$ of a family of seminorms which defines the topology of the linear space X considered instead of seminorm q_{p_1} on $\mathscr{H}$. Thus we have,

$$\sum_{k=1}^{n} \frac{p_1^{p+r}(h_k)}{a_k^p} - \frac{p_1^{p+r}(\sum_{k=1}^{n} a_k^{r-1}h_k)}{(\sum_{k=1}^{n} a_k^r)^{p+r-1}} \geq \max_{1 \leq i < j \leq n} \left\{\frac{p_1^{p+r}(h_i)}{a_i^p} + \frac{p_1^{p+r}(h_j)}{a_j^p} - \frac{p_1^{p+r}(a_i^{r-1}h_i + a_j^{r-1}h_j)}{(a_i^r + a_j^r)^{p+r-1}}\right\}.$$

Remark 3.3. If $a_k > 0, h_k \in \mathscr{H}$ so that $q_{p_1}(h_k) > 0, k \in \{1,2, \dots, n\}$, $p \geq 0$, $n \in \mathbf{N}, \mathrm{m} \in \mathbf{N}, \mathrm{m} \geq \mathrm{p} \geq 0$, then we have,

$$n^{p-m} \frac{q_{p_1}^{m+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k^r)^p} \leq \sum_{k=1}^{n} \frac{q_{p_1}^{m+1}(h_k)}{a_k^p}, \qquad (2)$$

where the seminorm q_{p_1} is given by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: It is used inequality (1.4) from [13] applied for $x_k = q_{p_1}(h_k), k \in \{1,2, \dots, n\}$, and the generalized triangle inequality for the seminorm q_{p_1}, $q_{p_1}(\sum_{k=1}^{n} h_k) \leq \sum_{k=1}^{n} q_{p_1}(h_k)$, obtaining:

$$\sum_{k=1}^{n} \frac{q_{p_1}^{m+1}(h_k)}{a_k^p} \geq n^{p-m} \frac{\left(\sum_{k=1}^{n} q_{p_1}(h_k)\right)^{m+1}}{(\sum_{k=1}^{n} a_k)^p} \geq n^{p-m} \frac{q_{p_1}^{m+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k)^p}.$$

Remark 3.4. If $n \in \mathbf{N}$, $n \geq 2$, $h_k \in \mathscr{H}$ so that $q_{p_1}(h_k) > 0, k \in \{1,2, \dots, n\}$, $p \geq 0$, $r \geq 1$ then we have the inequality:

$$\sum_{k=1}^{n} q_{p_1}^{p+r}(h_k) - \frac{q_{p_1}^{p+r}(\sum_{k=1}^{n} h_k)}{n^{p+r-1}} \geq \max_{1 \leq i < j \leq n} \left\{q_{p_1}^{p+r}(h_i) + q_{p_1}^{p+r}(h_j) - \frac{q_{p_1}^{p+r}(h_i + h_j)}{2^{p+r-1}}\right\}, \qquad (3)$$

where q_{p_1} is given by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: We put in Theorem 3.1 $a_k = 1,\ k \in \{1,2,\dots,n\}$ and the inequality (3) will be obtained.

Theorem 3.5. If $\mathscr{H}$ is a Hilbert space, $x_k \in \mathscr{H},\ k \in \{1,2,\dots,n\}$, under previous conditions from Theorem 3.1 the following inequality takes place:

$$\sum_{k=1}^{n} \frac{||x_k||^{p+r}}{a_k^p} - \frac{||\sum_{k=1}^{n} a_k^{r-1} x_k||^{p+r}}{(\sum_{k=1}^{n} a_k^r)^{p+r-1}} \ge \max_{1\le i<j\le n} \left\{ \frac{||x_i||^{p+r}}{a_i^p} + \frac{||x_j||^{p+r}}{a_j^p} - \frac{||a_i^{r-1}x_i + a_j^{r-1}x_j||^{p+r}}{(a_i^r + a_j^r)^{p+r-1}} \right\}. \quad (4)$$

Proof: Like in the proof of Theorem 3.1, we take $d_i = \sum_{k=1}^{i} \frac{||x_k||^{p+r}}{a_k^p} - \frac{||\sum_{k=1}^{i} a_k^{r-1} x_k||^{p+r}}{(\sum_{k=1}^{i} a_k^r)^{p+r-1}},\ i \le n$ and it is the same method.

Application 3.6. For $n \in \mathbf{N},\ n \ge 2,\ a_k > 0\ ,k \in \{1,2,\dots,n\},\ p \ge 0,\ r \ge 1$, next inequality holds:

$$\sum_{k=1}^{n} \frac{1}{a_k^p} - \frac{(\sum_{k=1}^{n} a_k^{r-1})^{p+r}}{(\sum_{k=1}^{n} a_k^r)^{p+r-1}} \ge \max_{1\le i<j\le n} \left\{ \frac{1}{a_i^p} + \frac{1}{a_j^p} - \frac{(a_i^{r-1} + a_j^{r-1})^{p+r}}{(a_i^r + a_j^r)^{p+r-1}} \right\}. \quad (5)$$

Proof: In inequality (4) it will be considered $x_k = 1, k \in \{1,2,\dots,n\}$, $\mathscr{H}$ being the set of real number with ussualy norm $||.||$, or in Theorem 2.8 from [13] it will be considered in inequality (2.12) $x_k = 1, k \in \{1,2,\dots,n\}$.

Theorem 3.7. Let $m, n \in \mathbf{N}, n > m$ be two positive integer numbers and $a_k > 0,\ h_k \in \mathscr{H}$ so that $q_{p_1}(h_k) > 0, k \in \{1,2,\dots,n\},\ p \ge\ r \ge 0$. The following inequality takes place:

$$\sum_{k=m+1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} + m^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{m} h_k)}{(\sum_{k=1}^{m} a_k)^r} \ge n^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k)^r}, \quad (6)$$

where the seminorm q_{p_1} is defined by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: The left term of inequality (6) will be denoted by N and can be written as below,

$$N = \sum_{k=m+1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} + m^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{m} h_k)}{(\sum_{k=1}^{m} a_k)^r} = \sum_{k=m+1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} + m \frac{[\frac{1}{m} q_{p_1}(\sum_{k=1}^{m} h_k)]^{p+1}}{\left(\frac{1}{m}\sum_{k=1}^{m} a_k\right)^r} \ge$$

$$\ge n^{r-p} \frac{\left(\sum_{k=m+1}^{n} q_{p_1}(h_k) + \frac{m}{m} q_{p_1}(\sum_{k=1}^{m} h_k)\right)^{p+1}}{\left(\sum_{k=m+1}^{n} a_k + \frac{m}{m}\sum_{k=1}^{m} a_k\right)^r}$$

by using the inequality (1.4) from [13] given for $x_k = q_{p_1}(h_k), k \in \{m+1,\dots,n\}$, and $x_k = \frac{1}{m} q_{p_1}(\sum_{k=1}^{m} h_k), k \in \{1,2,\dots,m\}$. From definition of seminorms it is known that $\sum_{k=m+1}^{n} q_{p_1}(h_k) + q_{p_1}(\sum_{k=1}^{m} h_k) \ge q_{p_1}(\sum_{k=1}^{n} h_k)$ and then we get,

$$N = \sum_{k=m+1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} + m^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{m} h_k)}{(\sum_{k=1}^{m} a_k)^r} \ge n^{r-p} \frac{\left(q_{p_1}(\sum_{k=1}^{n} h_k)\right)^{p+1}}{(\sum_{k=1}^{n} a_k)^r}.$$

Theorem 3.8. If $n \in \mathbf{N},\ n \ge 2,\ a_k > 0,\ h_k \in \mathscr{H}$ so that $q_{p_1}(h_k) > 0, k \in \{1,2,\dots,n\}$, $p \ge\ r \ge 0$ then the following inequality holds,

$$\sum_{k=1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} - n^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k)^r} \ge \max_{1\le i<j\le n} \left\{ \frac{q_{p_1}^{p+1}(h_i)}{a_i^r} + \frac{q_{p_1}^{p+1}(h_j)}{a_j^r} - 2^{r-p} \frac{q_{p_1}^{p+1}(h_i + h_j)}{(a_i + a_j)^r} \right\} \ge 0\ , \quad (7)$$

where the seminorm q_{p_1} is defined by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: It is necessary to consider the sequence $b_i = \sum_{k=1}^{i} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} - i^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{i} h_k)}{(\sum_{k=1}^{i} a_k)^r}$, $i \leq n$ and we will prove that $(b_n)_n$ is increasing.

For that in inequality (7) it will be added in each member the sum, $\sum_{k=1}^{m} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r}$, obtaining:

$$\sum_{k=m+1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} + m^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{m} h_k)}{(\sum_{k=1}^{m} a_k)^r} + \sum_{k=1}^{m} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} \geq n^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k)^r} + \sum_{k=1}^{m} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r},$$

or

$$\sum_{k=1}^{n} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} - n^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k)^r} \geq \sum_{k=1}^{m} \frac{q_{p_1}^{p+1}(h_k)}{a_k^r} - m^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{m} h_k)}{(\sum_{k=1}^{m} a_k)^r},$$

that is $b_n \geq b_m$. Therefore $(b_n)_n$ is increasing and

$b_n \geq b_{n-1} \geq \cdots \geq b_2 \geq b_1 = \frac{q_{p_1}^{p+1}(h_1)}{a_1^r} - \frac{q_{p_1}^{p+1}(h_1)}{a_1^r} = 0$. By the same reason as in Theorem 3.1 we get from symmetry of b_n relatively to the variables a_i and h_i, $i, j \in \{1,2,\ldots,n\}$ the inequality (7).

Corollary 3.9. For every arbitrary seminorm p_1, $p_1 : X \to R_+$ of a family of seminorms which defines the topology of the linear space X, under conditions of Theorem 3.8, the following inequality is true:

$$\sum_{k=1}^{n} \frac{p_1^{p+1}(h_k)}{a_k^r} - n^{r-p} \frac{p_1^{p+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} a_k)^r} \geq \max_{1 \leq i < j \leq n} \left\{ \frac{p_1^{p+1}(h_i)}{a_i^r} + \frac{p_1^{p+1}(h_j)}{a_j^r} - 2^{r-p} \frac{p_1^{p+1}(h_i + h_j)}{(a_i + a_j)^r} \right\} \geq 0. \quad (8)$$

Proof: The proof will be as in Theorem 3.8.

Theorem 3.10. If $\mathscr{H}$ is a Hilbert space space, $x_k \in \mathscr{H}$, $k \in \{1,2,\ldots,n\}$, under previous conditions from Theorem 3.8 the following inequality takes place:

$$\sum_{k=1}^{n} \frac{||x_k||^{p+1}}{a_k^r} - n^{r-p} \frac{||\sum_{k=1}^{n} x_k||^{p+1}}{(\sum_{k=1}^{n} a_k)^r} \geq \max_{1 \leq i < j \leq n} \left\{ \frac{||x_i||^{p+1}}{a_i^r} + \frac{||x_j||^{p+1}}{a_j^r} - 2^{r-p} \frac{||x_i + x_j||^{p+1}}{(a_i + a_j)^r} \right\}. \quad (9)$$

Proof: The proof will be as in Theorem 3.8.

Remark 3.11. If $n \in \mathbf{N}$, $n \geq 2$, $h_k \in \mathscr{H}$, $q_{p_1}(h_k) > 0$, $k \in \{1,2,\ldots,n\}$, $p \geq r \geq 0$ then the following inequality holds,

$$\sum_{k=1}^{n} q_{p_1}^{p-r+1}(h_k) - n^{r-p} \frac{q_{p_1}^{p+1}(\sum_{k=1}^{n} h_k)}{(\sum_{k=1}^{n} q_{p_1}(h_k))^r} \geq \max_{1 \leq i < j \leq n} \left\{ q_{p_1}^{p-r+1}(h_i) + q_{p_1}^{p-r+1}(h_j) - 2^{r-p} \frac{q_{p_1}^{p+1}(h_i + h_j)}{(q_{p_1}(h_i) + q_{p_1}(h_j))^r} \right\} \geq 0, \quad (10)$$

where q_{p_1} is defined by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: It results from Theorem 3.8 when $a_k = q_{p_1}(h_k) > 0, k \in \{1,2,\ldots,n\}$.

Remark 3.12. If $n \in \mathbf{N}$, $n \geq 2$, $h_k \in \mathscr{H}$, $q_{p_1}(h_k) > 0$, $k \in \{1,2,\dots,n\}$, $r \in [0,1]$ then next inequality is true:

$$\sum_{k=1}^{n} q_{p_1}^{2-r}(h_k) - n^{r-1} \frac{q_{p_1}^2\left(\sum_{k=1}^{n} h_k\right)}{\left(\sum_{k=1}^{n} q_{p_1}(h_k)\right)^r} \geq \max_{1 \leq i < j \leq n} \left\{ q_{p_1}^{2-r}(h_i) + q_{p_1}^{2-r}(h_j) - 2^{r-1} \frac{q_{p_1}^2(h_i + h_j)}{\left(q_{p_1}(h_i) + q_{p_1}(h_j)\right)^r} \right\} \geq 0, \tag{11}$$

when the seminorm q_{p_1} is defined by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: It results from Remark 3.11 when $p = 1$.

Remark 3.13. If $n \in \mathbf{N}$, $n \geq 2$, $h_k \in \mathscr{H}$, $q_{p_1}(h_k) > 0$, $k \in \{1,2,\dots,n\}$, $p \geq r \geq 0$ then next inequality is true:

$$\sum_{k=1}^{n} q_{p_1}^{p+1}(h_k) - \frac{1}{n^p} q_{p_1}^{p+1}\left(\sum_{k=1}^{n} h_k\right) \geq \max_{1 \leq i < j \leq n} \left\{ q_{p_1}^{p+1}(h_i) + q_{p_1}^{p+1}(h_j) - \frac{1}{2^p} q_{p_1}^{p+1}(h_i + h_j) \right\} \geq 0, \tag{12}$$

where q_{p_1} is defined by $q_{p_1}(h) = [p([h,h])]^{\frac{1}{2}}$.

Proof: It results from Theorem 3.8 if we put $a_k = 1$, $k \in \{1,2,\dots,n\}$.

An analogue of Application 3.1 from [13] is given below.

Application 3.14. If $h_k \in \mathscr{H}$, $q_{p_1}(h_k) > 0$, $k \in \overline{1,3}$, then it holds,

$$\frac{q_{p_1}^5(h_1)}{q_{p_1}^2(h_2)} + \frac{q_{p_1}^5(h_2)}{q_{p_1}^2(h_3)} + \frac{q_{p_1}^5(h_3)}{q_{p_1}^2(h_1)} - \frac{q_{p_1}^5(h_1 q_{p_1}^2(h_2) + h_2 q_{p_1}^2(h_3) + h_3 q_{p_1}^2(h_1))}{(q_{p_1}^3(h_2) + q_{p_1}^3(h_3) + q_{p_1}^3(h_1))^4} \geq$$

$$\geq max \left\{ \frac{q_{p_1}^5(h_1)}{q_{p_1}^2(h_2)} + \frac{q_{p_1}^5(h_2)}{q_{p_1}^2(h_3)} - \frac{q_{p_1}^5\left(h_1 q_{p_1}^2(h_2) + h_2 q_{p_1}^2(h_3)\right)}{\left(q_{p_1}^3(h_2) + q_{p_1}^3(h_3)\right)^4}; \frac{q_{p_1}^5(h_2)}{q_{p_1}^2(h_3)} + \frac{q_{p_1}^5(h_3)}{q_{p_1}^2(h_1)} - \frac{q_{p_1}^5\left(h_2 q_{p_1}^2(h_3) + h_3 q_{p_1}^2(h_1)\right)}{\left(q_{p_1}^3(h_3) + q_{p_1}^3(h_1)\right)^4}; \frac{q_{p_1}^5(h_3)}{q_{p_1}^2(h_1)} + \frac{q_{p_1}^5(h_1)}{q_{p_1}^2(h_2)} - \frac{q_{p_1}^5\left(h_3 q_{p_1}^2(h_1) + h_1 q_{p_1}^2(h_2)\right)}{\left(q_{p_1}^3(h_1) + q_{p_1}^3(h_2)\right)^4} \right\}. \tag{13}$$

Proof: It results from Theorem 3.1 when $n = 3$, $r = 3$, $p = 2$ and $a_k = q_{p_1}(h_k)$, $k \in \overline{1,3}$.

Another analogue of Application 3.2 from [13] is given below.

Application 3.15. If $h_k \in \mathscr{H}$, $q_{p_1}(h_k) > 0$, $k \in \overline{1,3}$, then we have,

$$\frac{q_{p_1}^5(h_1)}{q_{p_1}^2(h_2)} + \frac{q_{p_1}^5(h_2)}{q_{p_1}^2(h_3)} + \frac{q_{p_1}^5(h_3)}{q_{p_1}^2(h_1)} - \frac{1}{9} \frac{q_{p_1}^5(h_1 + h_2 + h_3)}{(q_{p_1}(h_2) + q_{p_1}(h_3) + q_{p_1}(h_1))^2} \geq$$

$$\geq max \left\{ \frac{q_{p_1}^5(h_1)}{q_{p_1}^2(h_2)} + \frac{q_{p_1}^5(h_2)}{q_{p_1}^2(h_3)} - \frac{1}{4} \frac{q_{p_1}^5(h_1 + h_2)}{\left(q_{p_1}(h_2) + q_{p_1}(h_3)\right)^4}; \frac{q_{p_1}^5(h_2)}{q_{p_1}^2(h_3)} + \frac{q_{p_1}^5(h_3)}{q_{p_1}^2(h_1)} - \frac{1}{4} \frac{q_{p_1}^5(h_2 + h_3)}{\left(q_{p_1}(h_3) + q_{p_1}(h_1)\right)^4}; \frac{q_{p_1}^5(h_3)}{q_{p_1}^2(h_1)} + \frac{q_{p_1}^5(h_1)}{q_{p_1}^2(h_2)} - \frac{1}{4} \frac{q_{p_1}^5(h_3 + h_1)}{\left(q_{p_1}(h_1) + q_{p_1}(h_2)\right)^4} \right\}. \tag{14}$$

Proof: It results from Theorem 3.8 when $n = 3$, $r = 2$, $p = 4$ and $a_k = q_{p_1}(h_k)$, $k \in \overline{1,3}$.

4. DISCUSSION AND CONCLUSIONS

The main findings of this study are designed to prove new generalizations of Radon and Bergstrom's inequalities for seminorms and norms on pseudo-Hilbert, Hilbert and any normed spaces utilizing the method from the newly discovered refinements of Radon's inequalities given in the literature in recent years. The correlation between the results presented here and analogue results from literature is also considered. Furthermore, several consequences and applications were presented to illustrate the outcome of the research.

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