TOMINAGA'S TYPE INTEGRAL INEQUALITIES FOR CONTINUOUS FIELDS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_\Omega 1d\mu(t) = 1$. In this paper we show among others that, if$(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0, 1]$ we have the tensorial inequality

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\
&\le (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right) \\
&\le S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right),
\end{aligned}$$

where $S(\cdot)$ is Specht's ratio. We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\
&\le \left(\int_\Omega [(1-\nu) A_\tau + \nu B_\tau] d\mu(\tau)\right) \circ 1 \\
&\le S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)
\end{aligned}$$

for all $\nu \in [0, 1]$.

1. Introduction

As is known to all, the famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \le (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1, \infty) \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.

RGMIA Res. Rep. Coll. **25** (2022), Art. 89, 19 pp. Received 21/09/22

Tominaga [10] had proved a multiplicative reverse Young inequality with the Specht's ratio [9] as follows:

$$(1.2)\qquad (1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}$$

for $a,\ b>0$ and $\nu\in[0,1]$.

He also obtained the following additive reverse

$$(1.3)\qquad (1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq L\,(a,b)\ln S\left(\frac{a}{b}\right)$$

for $a,\ b>0$ and $\nu\in[0,1]$, where $L\,(\cdot,\cdot)$ is the *logarithmic mean* defined by

$$L\,(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a}\text{ for }b\neq a,\\ \\ a\text{ if }b=a.\end{cases}$$

If $0<m\leq a,\ b\leq M$, then also [10]

$$(1.4)\qquad \left(a^{1-\nu}b^{\nu}\leq\right)(1-\nu)\,a+\nu b\leq S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}$$

and

$$(1.5)\qquad (0\leq)\,(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq aL\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)$$

for $\nu\in[0,1]$.

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i\,(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$(1.6)\qquad f\,(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f\,(\lambda_1,...,\lambda_k)\,dE_1\,(\lambda_1)\otimes...\otimes dE_k\,(\lambda_k)$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f\,(A_1,...,A_k)=f_1(A_1)\otimes...\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k)=f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f\,(st)\geq(\leq)\,f\,(s)\,f\,(t)\ \text{ for all }s,t\in[0,\infty)$$

and if f is continuous on $[0,\infty)$, then [7, p. 173]

$$(1.7)\qquad f\,(A\otimes B)\geq(\leq)\,f\,(A)\otimes f\,(B)\ \text{ for all }A,\ B\geq 0.$$

This follows by observing that, if

$$A=\int_{[0,\infty)} tdE\left(t\right) \text{ and } B=\int_{[0,\infty)} sdF\left(s\right)$$

are the spectral resolutions of A and B, then

$$f\left(A\otimes B\right)=\int_{[0,\infty)}\int_{[0,\infty)} f\left(st\right)dE\left(t\right)\otimes dF\left(s\right) \tag{1.8}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t\in[0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A \text{ and } \left(A\#B\right)\otimes\left(B\#A\right)=\left(A\otimes B\right)\#\left(B\otimes A\right).$$

In 2007, S. Wada [11] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}\left(A\#B\right)\otimes\left(A\#B\right) &\le \frac{1}{2}\left[\left(A\#_\alpha B\right)\otimes\left(A\#_{1-\alpha}B\right)+\left(A\#_{1-\alpha}B\right)\otimes\left(A\#_\alpha B\right)\right]\\ &\le \frac{1}{2}\left(A\otimes B+B\otimes A\right)\end{aligned} \tag{1.9}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\left\langle\left(A\circ B\right)e_j,e_j\right\rangle=\left\langle Ae_j,e_j\right\rangle\left\langle Be_j,e_j\right\rangle$$

for all $j\in\mathbb{N}$, where $\left\{e_j\right\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B=\mathcal{U}^*\left(A\otimes B\right)\mathcal{U} \tag{1.10}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [7, p. 173]

$$f\left(A\circ B\right)\ge(\le)\, f\left(A\right)\circ f\left(B\right) \text{ for all } A,\ B\ge 0. \tag{1.11}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\le\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\ge 0$$

and Fiedler inequality

$$A\circ A^{-1}\ge 1 \text{ for } A>0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\le\left(A^2\circ B^2\right)^{1/2} \text{ for } A,\ B\ge 0.$$

It has been shown in [8] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$. In [4] we showed among others that, for $\nu \in [0, 1]$,

$$\begin{aligned}(1.12)\qquad & \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\ & \le (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right) \\ & \le \exp\left[\frac{(M-m)^2}{4mM}\right]\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)\end{aligned}$$

and

$$\begin{aligned}(1.13)\qquad & 0 \le (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right) \\ & \quad - \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\ & \quad \le \frac{1}{4}(M-m)(\ln M - \ln m)\end{aligned}$$

for $\nu \in [0, 1]$.

We also obtained the following inequalities for the Hadamard product

$$\begin{aligned}(1.14)\qquad & \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\ & \le \int_\Omega ((1-\nu) A_\tau + \nu B_\tau)\, d\mu(\tau) \circ 1 \\ & \le \exp\left[\frac{(M-m)^2}{4mM}\right]\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)\end{aligned}$$

and

$$\begin{aligned}(1.15)\quad & 0 \le \int_\Omega ((1-\nu) A_\tau + \nu B_\tau)\, d\mu(\tau) \circ 1 - \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\ & \le \frac{1}{4}(M-m)(\ln M - \ln m)\end{aligned}$$

for $\nu \in [0, 1]$.

Motivated by the above results, in this paper we show among others that, if$(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0, 1]$ we have the tensorial inequality

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\
&\leq (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right) \\
&\leq S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right),
\end{aligned}$$

where $S(\cdot)$ is Specht's ratio. We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\
&\leq \left(\int_\Omega [(1-\nu) A_\tau + \nu B_\tau] d\mu(\tau)\right) \circ 1 \\
&\leq S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)
\end{aligned}$$

for all $\nu \in [0,1]$.

2. Main Results

In what follows we assume that $\int_\Omega 1 d\mu(t) = 1$.

Theorem 1. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0,\infty)$ for each $\tau \in \Omega$. Then for all $\nu \in [0,1]$ we have*

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\
&\leq (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right) \\
&\leq S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)
\end{aligned} \tag{2.1}$$

In particular,

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega A_\tau^{\nu} d\mu(\tau)\right) \\
&\leq (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega A_\tau d\mu(\tau)\right) \\
&\leq S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega A_\tau^{\nu} d\mu(\tau)\right).
\end{aligned} \tag{2.2}$$

Proof. From (1.4) we get

$$t^{1-\nu} s^{\nu} \leq (1-\nu) t + \nu s \leq S\left(\frac{M}{m}\right) t^{1-\nu} s^{\nu} \tag{2.3}$$

for all t, $s \in [m, M]$ and $\nu \in [0,1]$.

Assume that

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B. Now, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.3), then we get

$$\int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) \leq \int_m^M \int_m^M [(1-\nu)t + \nu s]\, dE(t) \otimes dF(s) \leq S\left(\frac{M}{m}\right) \int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s). \tag{2.4}$$

Since

$$\int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) = A^{1-\nu} \otimes B^{\nu}$$

and

$$\int_m^M \int_m^M [(1-\nu)t + \nu s]\, dE(t) \otimes dF(s) = (1-\nu) A \otimes 1 + \nu 1 \otimes B,$$

hence by (2.4) we get

$$A^{1-\nu} \otimes B^{\nu} \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq S\left(\frac{M}{m}\right) A^{1-\nu} \otimes B^{\nu} \tag{2.5}$$

for $\nu \in [0,1]$.

Now, from (2.5) we get

$$A_{\tau}^{1-\nu} \otimes B_{\gamma}^{\nu} \leq (1-\nu) A_{\tau} \otimes 1 + \nu 1 \otimes B_{\gamma} \leq S\left(\frac{M}{m}\right) A_{\tau}^{1-\nu} \otimes B_{\gamma}^{\nu} \tag{2.6}$$

for all τ, $\gamma \in \Omega$.

Now, if we take the integral $\int_{\Omega}$ over $d\mu(\tau)$, then we get

$$\int_{\Omega} \left(A_{\tau}^{1-\nu} \otimes B_{\gamma}^{\nu}\right) d\mu(\tau) \leq \int_{\Omega} [(1-\nu) A_{\tau} \otimes 1 + \nu 1 \otimes B_{\gamma}]\, d\mu(\tau) \leq S\left(\frac{M}{m}\right) \int_{\Omega} \left(A_{\tau}^{1-\nu} \otimes B_{\gamma}^{\nu}\right) d\mu(\tau). \tag{2.7}$$

Using the properties of the Bochner's integral and the tensorial product we have

$$\int_{\Omega} \left(A_{\tau}^{1-\nu} \otimes B_{\gamma}^{\nu}\right) d\mu(\tau) = \left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu}$$

and

$$\int_{\Omega} [(1-\nu) A_{\tau} \otimes 1 + \nu 1 \otimes B_{\gamma}]\, d\mu(\tau) = (1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes B_{\gamma}$$

for all $\gamma \in \Omega$.

From (2.7) we then get

$$\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu} \leq (1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes B_{\gamma} \leq S\left(\frac{M}{m}\right)\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu} \tag{2.8}$$

for all $\gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\gamma)$, then we get

$$\begin{aligned}&\int_\Omega \left[\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes B_\gamma^\nu\right] d\mu(\gamma) \\ &\le \int_\Omega \left[(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes B_\gamma\right] d\mu(\gamma) \\ &\le S\left(\frac{M}{m}\right) \int_\Omega \left[\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes B_\gamma^\nu\right] d\mu(\gamma).\end{aligned} \tag{2.9}$$

Since

$$\begin{aligned}&\int_\Omega \left[\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes B_\gamma^\nu\right] d\mu(\gamma) \\ &= \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\gamma^\nu d\mu(\gamma)\right)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \left[(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes B_\gamma\right] d\mu(\gamma) \\ &= (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\gamma d\mu(\gamma)\right),\end{aligned}$$

hence by (2.9) we derive (2.1). □

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^\nu d\mu(\tau)\right) \\ &\le \int_\Omega ((1-\nu) A_\tau + \nu B_\tau) d\mu(\tau) \circ 1 \\ &\le S\left(\frac{M}{m}\right) \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^\nu d\mu(\tau)\right).\end{aligned} \tag{2.10}$$

In particular,

$$\begin{aligned}&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^\nu d\mu(\tau)\right) \\ &\le \int_\Omega A_\tau d\mu(\tau) \circ 1 \\ &\le S\left(\frac{M}{m}\right) \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^\nu d\mu(\gamma)\right).\end{aligned} \tag{2.11}$$

Proof. If we use the identity (1.10) and apply $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right of inequality (2.1), we get

$$\begin{aligned}&\mathcal{U}^* \left[\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\gamma^\nu d\mu(\gamma)\right)\right] \mathcal{U} \\ &\le \mathcal{U}^* \left[(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\gamma d\mu(\gamma)\right)\right] \mathcal{U} \\ &\le S\left(\frac{M}{m}\right) \mathcal{U}^* \left[\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\gamma^\nu d\mu(\gamma)\right)\right] \mathcal{U}.\end{aligned} \tag{2.12}$$

Since

$$\mathcal{U}^*\left[\left(\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\right)\otimes\left(\int_\Omega B_\gamma^\nu d\mu(\gamma)\right)\right]\mathcal{U}$$
$$=\left(\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\right)\circ\left(\int_\Omega B_\gamma^\nu d\mu(\gamma)\right)$$

and

$$\mathcal{U}^*\left[(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right)\otimes 1+\nu 1\otimes\left(\int_\Omega B_\gamma d\mu(\gamma)\right)\right]\mathcal{U}$$
$$=(1-\nu)\mathcal{U}^*\left[\left(\int_\Omega A_\tau d\mu(\tau)\right)\otimes 1\right]\mathcal{U}$$
$$+\nu\mathcal{U}^*\left[1\otimes\left(\int_\Omega B_\gamma d\mu(\gamma)\right)\right]\mathcal{U},$$

hence by (2.12), we derive (2.10). □

Theorem 2. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau)\subseteq[m,M]\subset(0,\infty)$ for each $\tau\in\Omega$. Then for all $\nu\in[0,1]$ we have*

$$\begin{aligned}0&\le(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right)\otimes 1+\nu 1\otimes\left(\int_\Omega B_\tau d\mu(\tau)\right)\\&\quad-\left(\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\right)\otimes\left(\int_\Omega B_\tau^\nu d\mu(\tau)\right)\\&\le L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau d\mu(\tau)\right)\otimes 1,\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}0&\le(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right)\otimes 1+\nu 1\otimes\left(\int_\Omega A_\tau d\mu(\tau)\right)\\&\quad-\left(\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\right)\otimes\left(\int_\Omega A_\tau^\nu d\mu(\tau)\right)\\&\le L\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau d\mu(\tau)\right)\otimes 1,\end{aligned}\tag{2.14}$$

Proof. From (1.5) we have

$$0\le(1-\nu)t+\nu s-t^{1-\nu}s^\nu\le tL\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)\tag{2.15}$$

for all t, $s\in[m,M]$ and $\nu\in[0,1]$.

Let

$$A=\int_m^M tdE(t)\text{ and }B=\int_m^M sdF(s)$$

be the spectral resolutions of A and B. Now, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.15), then we get

$$\begin{aligned} 0 &\le \int_m^M \int_m^M \left[(1-\nu)t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \int_m^M \int_m^M t dE(t) \otimes dF(s), \end{aligned}$$

which gives

$$\begin{aligned} 0 &\le (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A \otimes 1, \end{aligned} \tag{2.16}$$

for all $\nu \in [0,1]$.

Now, from (2.16) we get

$$\begin{aligned} 0 &\le (1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma - A_\tau^{1-\nu} \otimes B_\gamma^{\nu} \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) A_\tau \otimes 1, \end{aligned}$$

for all $\tau, \gamma \in \Omega$.

Now if we use a similar argument to the one in the proof of Theorem 1, we deduce the desired result (2.13). □

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} 0 &\le \int_\Omega ((1-\nu) A_\tau + \nu B_\tau) d\mu(\tau) \circ 1 \\ &- \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \left(\int_\Omega A_\tau d\mu(\tau)\right) \circ 1. \end{aligned} \tag{2.17}$$

In particular,

$$\begin{aligned} 0 &\le \int_\Omega A_\tau d\mu(\tau) \circ 1 - \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^{\nu} d\mu(\tau)\right) \\ &\le L\left(1, \frac{M}{m}\right) \ln S\left(\frac{M}{m}\right) \left(\int_\Omega A_\tau d\mu(\tau)\right) \circ 1. \end{aligned} \tag{2.18}$$

We also have:

Theorem 3. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0,\infty)$ for each $\tau \in \Omega$. Then for all $\nu \in [0,1]$ we have*

$$\begin{aligned} 1 &\le (1-\nu) \left(\int_\Omega A_\tau^{\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{-\nu} d\mu(\tau)\right) \\ &+ \nu \left(\int_\Omega A_\tau^{\nu-1} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{1-\nu} d\mu(\tau)\right) \\ &\le S\left(\frac{M}{m}\right). \end{aligned} \tag{2.19}$$

In particular,

$$1\le(1-\nu)\left(\int_{\Omega}A_{\tau}^{\nu}d\mu(\tau)\right)\otimes\left(\int_{\Omega}A_{\tau}^{-\nu}d\mu(\tau)\right)+\nu\left(\int_{\Omega}A_{\tau}^{\nu-1}d\mu(\tau)\right)\otimes\left(\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\right)\le S\left(\frac{M}{m}\right). \tag{2.20}$$

Proof. We also have, by dividing in both sides of (2.3) that

$$1\le(1-\nu)t^{\nu}s^{-\nu}+\nu t^{\nu-1}s^{1-\nu}\le S\left(\frac{M}{m}\right) \tag{2.21}$$

for all $t,\ s\in[m,M]$.

Let

$$A=\int_{m}^{M}tdE(t)\text{ and }B=\int_{m}^{M}sdF(s)$$

be the spectral resolutions of A and B. Now, if we take the double integral $\int_{m}^{M}\int_{m}^{M}$ over $dE(t)\otimes dF(s)$ in (2.21), then we get

$$\begin{aligned}&\int_{m}^{M}\int_{m}^{M}dE(t)\otimes dF(s)\\&\le\int_{m}^{M}\int_{m}^{M}\left[(1-\nu)t^{\nu}s^{-\nu}+\nu t^{\nu-1}s^{1-\nu}\right]dE(t)\otimes dF(s)\\&\le S\left(\frac{M}{m}\right)\int_{m}^{M}\int_{m}^{M}dE(t)\otimes dF(s)\end{aligned}$$

which is equivalent to

$$1\le(1-\nu)A^{\nu}\otimes B^{-\nu}+\nu A^{\nu-1}\otimes B^{1-\nu}\le S\left(\frac{M}{m}\right). \tag{2.22}$$

This implies that

$$1\le(1-\nu)A_{\tau}^{\nu}\otimes B_{\gamma}^{-\nu}+\nu A_{\tau}^{\nu-1}\otimes B_{\gamma}^{1-\nu}\le S\left(\frac{M}{m}\right)$$

for all $\tau,\ \gamma\in\Omega$.

Now if we use a similar argument to the one in the proof of Theorem 1, we deduce the desired result (2.13). □

Corollary 3. *With the assumptions of Theorem 1 we have*

$$1\le(1-\nu)\left(\int_{\Omega}A_{\tau}^{\nu}d\mu(\tau)\right)\circ\left(\int_{\Omega}B_{\tau}^{-\nu}d\mu(\tau)\right)+\nu\left(\int_{\Omega}A_{\tau}^{\nu-1}d\mu(\tau)\right)\circ\left(\int_{\Omega}B_{\tau}^{1-\nu}d\mu(\tau)\right)\le S\left(\frac{M}{m}\right). \tag{2.23}$$

In particular,

$$1 \leq (1-\nu)\left(\int_\Omega A_\tau^\nu d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^{-\nu} d\mu(\tau)\right) + \nu\left(\int_\Omega A_\tau^{\nu-1} d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \leq S\left(\frac{M}{m}\right). \tag{2.24}$$

For $\nu = 1/2$, we also have

$$1 \leq \left(\int_\Omega A_\tau^{1/2} d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^{-1/2} d\mu(\tau)\right) \leq S\left(\frac{M}{m}\right). \tag{2.25}$$

3. Related Results

We can state some related results as follows:

Theorem 4. *Assume that f, g are continuous and nonnegative on the interval I and there exists $0 \leq \gamma < \Gamma$ such that*

$$\gamma \leq \frac{f(t)}{g(t)} \leq \Gamma \text{ for all } t \in I,$$

then for $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau) \subset I$ for each $\tau \in \Omega$,

$$\begin{aligned}
&\left(\int_\Omega f^{2(1-\nu)}(A_\tau) g^{2\nu}(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega f^{2\nu}(B_\tau) g^{2(1-\nu)}(B_\tau) d\mu(\tau)\right) \\
&\leq (1-\nu)\left(\int_\Omega f^2(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega g^2(B_\tau) d\mu(\tau)\right) \\
&+ \nu\left(\int_\Omega g^2(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega f^2(B_\tau) d\mu(\tau)\right) \\
&\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\int_\Omega f^{2(1-\nu)}(A_\tau) g^{2\nu}(A_\tau) d\mu(\tau)\right) \\
&\otimes \left(\int_\Omega f^{2\nu}(B_\tau) g^{2(1-\nu)}(B_\tau) d\mu(\tau)\right)
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
0 &\leq (1-\nu)\left(\int_\Omega f^2(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega g^2(B_\tau) d\mu(\tau)\right) \\
&+ \nu\left(\int_\Omega g^2(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega f^2(B_\tau) d\mu(\tau)\right) \\
&- \left(\int_\Omega f^{2(1-\nu)}(A_\tau) g^{2\nu}(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega f^{2\nu}(B_\tau) g^{2(1-\nu)}(B_\tau) d\mu(\tau)\right) \\
&\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \\
&\times \left(\int_\Omega f^2(A_\tau) d\mu(\tau)\right) \otimes \left(\int_\Omega g^2(B_\tau) d\mu(\tau)\right).
\end{aligned} \tag{3.2}$$

for all $\nu \in [0,1]$.

Proof. For any $t,\ s\in I$ we have

$$\gamma^2\le\frac{f^2(t)}{g^2(t)},\frac{f^2(s)}{g^2(s)}\le\Gamma^2.$$

If we use the inequality (1.4) for

$$a=\frac{f^2(t)}{g^2(t)},\ b=\frac{f^2(s)}{g^2(s)},$$

then we get

$$\begin{aligned}\left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu}&\le(1-\nu)\frac{f^2(t)}{g^2(t)}+\nu\frac{f^2(s)}{g^2(s)}\\&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu}\end{aligned}\tag{3.3}$$

for any $t,\ s\in I$.

Now, if we multiply (3.3) by $g^2(t)\,g^2(s)>0$, then we get

$$\begin{aligned}&f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\\&\le(1-\nu)\,f^2(t)\,g^2(s)+\nu g^2(t)\,f^2(s)\\&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\end{aligned}\tag{3.4}$$

for any $t,\ s\in I$.

Assume that

$$A=\int_I t\,dE(t)\ \text{ and }\ B=\int_I s\,dF(s)$$

are the spectral resolutions of A and B.

Further on, if we take the double integral $\int_I\int_I$ over $dE(t)\otimes dF(s)$ in (3.4), then we get

$$\begin{aligned}&\int_I\int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dE(t)\otimes dF(s)\\&\le(1-\nu)\int_I\int_I f^2(t)\,g^2(s)\,dE(t)\otimes dF(s)\\&+\nu\int_I\int_I g^2(t)\,f^2(s)\,dE(t)\otimes dF(s)\\&\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\int_I\int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dE(t)\otimes dF(s).\end{aligned}\tag{3.5}$$

Since

$$\begin{aligned}&\int_I\int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dE(t)\otimes dF(s)\\&=\int_I f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,dE(t)\otimes\int_I f^{2\nu}(s)\,g^{2(1-\nu)}(s)\,dF(s)\\&=\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)\otimes\left(f^{2\nu}(B)\,g^{2(1-\nu)}(B)\right),\end{aligned}$$

$$\int_I \int_I f^2(t) g^2(s) dE(t) \otimes dF(s) = \int_I f^2(t) dE(t) \otimes \int_I g^2(s) dF(s)$$
$$= f^2(A) \otimes g^2(B),$$

and

$$\int_I \int_I g^2(t) f^2(s) dE(t) \otimes dF(s) = \int_I g^2(t) dE(t) \otimes \int_I f^2(s) dF(s)$$
$$= g^2(A) \otimes f^2(B),$$

hence by (3.5) we get

$$\begin{aligned} &\left(f^{2(1-\nu)}(A) g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B) g^{2(1-\nu)}(B)\right) \\ &\leq (1-\nu) f^2(A) \otimes g^2(B) + \nu g^2(A) \otimes f^2(B) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(f^{2(1-\nu)}(A) g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B) g^{2(1-\nu)}(B)\right) \end{aligned} \tag{3.6}$$

for all $\nu \in [0,1]$.

From (1.5) we obtain

$$\begin{aligned} &(0 \leq)\,(1-\nu) \frac{f^2(t)}{g^2(t)} + \nu \frac{f^2(s)}{g^2(s)} - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu} \left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \\ &\leq \frac{f^2(t)}{g^2(t)} L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \end{aligned}$$

for any $t,\ s \in I$.

If we multiply by $g^2(t) g^2(s) > 0$ then we get

$$\begin{aligned} &(1-\nu) f^2(t) g^2(s) + \nu g^2(t) f^2(s) - f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) \\ &\leq f^2(t) g^2(s) L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \end{aligned} \tag{3.7}$$

for any $t,\ s \in I$.

If we take the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (3.7) and use a similar argument as above, we deduce

$$\begin{aligned} 0 &\leq (1-\nu) f^2(A) \otimes g^2(B) + \nu g^2(A) \otimes f^2(B) \\ &- \left(f^{2(1-\nu)}(A) g^{2\nu}(A)\right) \otimes \left(f^{2\nu}(B) g^{2(1-\nu)}(B)\right) \\ &\leq L\left(1, \left(\frac{\Gamma}{\gamma}\right)^2\right) \ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) f^2(A) \otimes g^2(B) \end{aligned} \tag{3.8}$$

for all $\nu \in [0,1]$.

Now, from (3.6) we have

$$\begin{aligned} &\left(f^{2(1-\nu)}(A_\tau) g^{2\nu}(A_\tau)\right) \otimes \left(f^{2\nu}(B_\gamma) g^{2(1-\nu)}(B_\gamma)\right) \\ &\leq (1-\nu) f^2(A_\tau) \otimes g^2(B_\gamma) + \nu g^2(A_\tau) \otimes f^2(B_\gamma) \\ &\leq S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right) \left(f^{2(1-\nu)}(A_\tau) g^{2\nu}(A_\tau)\right) \otimes \left(f^{2\nu}(B_\gamma) g^{2(1-\nu)}(B_\gamma)\right) \end{aligned} \tag{3.9}$$

for all $\tau,\ \gamma \in \Omega$.

Now, if we take the integral $\int_\Omega$ over $d\mu(\tau)$ and then the integral $\int_\Omega$ over $d\mu(\gamma)$ and perform the required calculations as in the proof of Theorem 1 we derive the desired result (3.1).

The inequality (3.2) follows in the same way by employing the inequality (3.9). □

Corollary 4. *With the assumption of Theorem 4, we have the following inequalities for the Hadamard product*

$$\begin{aligned}(3.10)\quad &\left(\int_\Omega f^{2(1-\nu)}(A_\tau)\, g^{2\nu}(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega f^{2\nu}(B_\tau)\, g^{2(1-\nu)}(B_\tau)\, d\mu(\tau)\right)\\ &\le (1-\nu)\left(\int_\Omega f^2(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega g^2(B_\tau)\, d\mu(\tau)\right)\\ &+\nu\left(\int_\Omega g^2(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega f^2(B_\tau)\, d\mu(\tau)\right)\\ &\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\int_\Omega f^{2(1-\nu)}(A_\tau)\, g^{2\nu}(A_\tau)\, d\mu(\tau)\right)\\ &\circ\left(\int_\Omega f^{2\nu}(B_\tau)\, g^{2(1-\nu)}(B_\tau)\, d\mu(\tau)\right)\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad 0 &\le (1-\nu)\left(\int_\Omega f^2(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega g^2(B_\tau)\, d\mu(\tau)\right)\\ &+\nu\left(\int_\Omega g^2(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega f^2(B_\tau)\, d\mu(\tau)\right)\\ &-\left(\int_\Omega f^{2(1-\nu)}(A_\tau)\, g^{2\nu}(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega f^{2\nu}(B_\tau)\, g^{2(1-\nu)}(B_\tau)\, d\mu(\tau)\right)\\ &\le L\left(1,\left(\frac{\Gamma}{\gamma}\right)^2\right)\ln S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\\ &\times\left(\int_\Omega f^2(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega g^2(B_\tau)\, d\mu(\tau)\right).\end{aligned}$$

for all $\nu\in[0,1]$.

Remark 1. *If we take* $B_\tau = A_\tau$ *for* $\tau\in\Omega$ *in Corollary 4, then we get*

$$\begin{aligned}(3.12)\quad &\left(\int_\Omega f^{2(1-\nu)}(A_\tau)\, g^{2\nu}(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega f^{2\nu}(A_\tau)\, g^{2(1-\nu)}(A_\tau)\, d\mu(\tau)\right)\\ &\le\left(\int_\Omega f^2(A_\tau)\, d\mu(\tau)\right)\circ\left(\int_\Omega g^2(A_\tau)\, d\mu(\tau)\right)\\ &\le S\left(\left(\frac{\Gamma}{\gamma}\right)^2\right)\left(\int_\Omega f^{2(1-\nu)}(A_\tau)\, g^{2\nu}(A_\tau)\, d\mu(\tau)\right)\\ &\circ\left(\int_\Omega f^{2\nu}(A_\tau)\, g^{2(1-\nu)}(A_\tau)\, d\mu(\tau)\right)\end{aligned}$$

and

$$0 \leq \left(\int_\Omega f^2 (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega g^2 (A_\tau) \, d\mu (\tau) \right) \tag{3.13}$$
$$- \left(\int_\Omega f^{2(1-\nu)} (A_\tau) \, g^{2\nu} (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega f^{2\nu} (A_\tau) \, g^{2(1-\nu)} (A_\tau) \, d\mu (\tau) \right)$$
$$\leq L \left(1, \left(\frac{\Gamma}{\gamma} \right)^2 \right) \ln S \left(\left(\frac{\Gamma}{\gamma} \right)^2 \right)$$
$$\times \left(\int_\Omega f^2 (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega g^2 (A_\tau) \, d\mu (\tau) \right).$$

Moreover, if we take $\nu = 1/2$ in (3.12) and (3.13), then we get

$$\left(\int_\Omega f (A_\tau) \, g (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega f (A_\tau) \, g (A_\tau) \, d\mu (\tau) \right) \tag{3.14}$$
$$\leq \left(\int_\Omega f^2 (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega g^2 (A_\tau) \, d\mu (\tau) \right)$$
$$\leq S \left(\left(\frac{\Gamma}{\gamma} \right)^2 \right) \left(\int_\Omega f (A_\tau) \, g (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega f (A_\tau) \, g (A_\tau) \, d\mu (\tau) \right)$$

and

$$0 \leq \left(\int_\Omega f^2 (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega g^2 (A_\tau) \, d\mu (\tau) \right) \tag{3.15}$$
$$- \left(\int_\Omega f (A_\tau) \, g (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega f (A_\tau) \, g (A_\tau) \, d\mu (\tau) \right)$$
$$\leq L \left(1, \left(\frac{\Gamma}{\gamma} \right)^2 \right) \ln S \left(\left(\frac{\Gamma}{\gamma} \right)^2 \right)$$
$$\times \left(\int_\Omega f^2 (A_\tau) \, d\mu (\tau) \right) \circ \left(\int_\Omega g^2 (A_\tau) \, d\mu (\tau) \right).$$

4. Some Examples

Consider the functions $f(t) = t^p$ and $g(t) = t^q$ for $t > 0$ and $p, q \neq 0$. Then

$$\frac{f(t)}{g(t)} = t^{p-q}, \text{ for } t > 0.$$

Therefore

$$m^{p-q} \leq \frac{f(t)}{g(t)} \leq M^{p-q} \text{ for } t \in [m, M] \text{ and } p > q$$

and

$$M^{p-q} \leq \frac{f(t)}{g(t)} \leq m^{p-q} \text{ for } t \in [m, M] \text{ and } p < q.$$

Now, assume that $(A_\tau)_{\tau \in \Omega}$ and $(B_\tau)_{\tau \in \Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subset [m, M]$ for each $\tau \in \Omega$. From Theorem 2 we get for $p > q$ that

$$\begin{aligned}
(4.1)\qquad & \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\
& \le (1-\nu) \int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau) \\
& + \nu \int_\Omega A_\tau^{2q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2p} d\mu(\tau) \\
& \le S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau)
\end{aligned}$$

and

$$\begin{aligned}
(4.2)\qquad & 0 \le (1-\nu) \int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau) \\
& + \nu \int_\Omega A_\tau^{2q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2p} d\mu(\tau) \\
& - \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\
& \le L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \\
& \times \int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau).
\end{aligned}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned}
(4.3)\qquad & \int_\Omega A_\tau^{p+q} d\mu(\tau) \otimes \int_\Omega B_\tau^{p+q} d\mu(\tau) \\
& \le \frac{1}{2}\left(\int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau) + \int_\Omega A_\tau^{2q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2p} d\mu(\tau)\right) \\
& \le S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{p+q} d\mu(\tau) \otimes \int_\Omega B_\tau^{p+q} d\mu(\tau)
\end{aligned}$$

and

$$\begin{aligned}
(4.4)\qquad & 0 \le \frac{1}{2}\left(\int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau) + \int_\Omega A_\tau^{2q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2p} d\mu(\tau)\right) \\
& - \int_\Omega A_\tau^{p+q} d\mu(\tau) \otimes \int_\Omega B_\tau^{p+q} d\mu(\tau) \\
& \le L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \\
& \times \int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau).
\end{aligned}$$

We also have the inequalities for the Hadamard product

$$
\begin{aligned}
(4.5)\qquad & \int_{\Omega} A_{\tau}^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2\nu p+2(1-\nu)q} d\mu(\tau) \\
& \leq (1-\nu) \int_{\Omega} A_{\tau}^{2p} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2q} d\mu(\tau) \\
& + \nu \int_{\Omega} A_{\tau}^{2q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2p} d\mu(\tau) \\
& \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_{\Omega} A_{\tau}^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2\nu p+2(1-\nu)q} d\mu(\tau)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.6)\qquad & 0 \leq (1-\nu) \int_{\Omega} A_{\tau}^{2p} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2q} d\mu(\tau) \\
& + \nu \int_{\Omega} A_{\tau}^{2q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2p} d\mu(\tau) \\
& - \int_{\Omega} A_{\tau}^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2\nu p+2(1-\nu)q} d\mu(\tau) \\
& \leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \\
& \times \int_{\Omega} A_{\tau}^{2p} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2q} d\mu(\tau).
\end{aligned}
$$

In particular, for $\nu = 1/2$,

$$
\begin{aligned}
(4.7)\qquad & \int_{\Omega} A_{\tau}^{p+q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{p+q} d\mu(\tau) \\
& \leq \frac{1}{2}\left(\int_{\Omega} A_{\tau}^{2p} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2q} d\mu(\tau) + \int_{\Omega} A_{\tau}^{2q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2p} d\mu(\tau)\right) \\
& \leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_{\Omega} A_{\tau}^{p+q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{p+q} d\mu(\tau)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.8)\qquad & 0 \leq \frac{1}{2}\left(\int_{\Omega} A_{\tau}^{2p} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2q} d\mu(\tau) + \int_{\Omega} A_{\tau}^{2q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2p} d\mu(\tau)\right) \\
& - \int_{\Omega} A_{\tau}^{p+q} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{p+q} d\mu(\tau) \\
& \leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \\
& \times \int_{\Omega} A_{\tau}^{2p} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{2q} d\mu(\tau).
\end{aligned}
$$

Moreover, if we take $B_\tau = A_\tau$, $\tau \in \Omega$ in (4.5)-(4.8), then we get

$$\begin{aligned} &\int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega A_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\ &\leq \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau) \\ &\leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega A_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} 0 &\leq \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau) \\ &\quad - \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega A_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\ &\leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \\ &\quad \times \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau). \end{aligned} \tag{4.10}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned} &\int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega A_\tau^{p+q} d\mu(\tau) \\ &\leq \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau) \\ &\leq S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega A_\tau^{p+q} d\mu(\tau) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} 0 &\leq \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau) - \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega A_\tau^{p+q} d\mu(\tau) \\ &\leq L\left(1, \left(\frac{M}{m}\right)^{2(p-q)}\right) \ln S\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \\ &\quad \times \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau). \end{aligned} \tag{4.12}$$

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