SOME TENSORIAL HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if ψ is continuous convex on the interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then for all $\nu \in [0,1]$

$$((1-\nu) A \otimes 1 + \nu 1 \otimes B) \leq (1-\nu) \psi(A) \otimes 1 + \nu 1 \otimes \psi(B)$$

Moreover, we have the Hermite-Hadamard type inequalities

$$\begin{aligned} \left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) & \leq \int_0^1 ((1-\nu) A \otimes 1 + \nu 1 \otimes B) \, d\nu \\ & \leq \frac{(A) \otimes 1 + 1 \otimes \psi(B)}{2}. \end{aligned}$$

Several refinements and reverses of these inequalities are also provided.

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1) \, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [6] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

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and if f is continuous on $[0,\infty)$, then [8, p. 173]

$$f(A\otimes B)\geq(\leq)f(A)\otimes f(B)\text{ for all }A,\ B\geq 0. \tag{1.2}$$

This follows by observing that, if

$$A=\int_{[0,\infty)}tdE(t)\text{ and }B=\int_{[0,\infty)}sdF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)}f(st)\,dE(t)\otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A\text{ and }(A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [10] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B)&\leq\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\leq\frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.4}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Motivated by the above results, in this paper we show among others that, if ψ is continuous convex on the interval I and $A,\ B$ are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A),\ \operatorname{Sp}(B)\subset I$, then for all $\nu\in[0,1]$

$$\psi((1-\nu)A\otimes 1+\nu 1\otimes B)\leq(1-\nu)\psi(A)\otimes 1+\nu 1\otimes\psi(B)$$

Moreover, we have the Hermite-Hadamard type inequalities

$$\begin{aligned}\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)&\leq\int_0^1\psi((1-\nu)A\otimes 1+\nu 1\otimes B)\,d\nu\\&\leq\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}.\end{aligned}$$

Several refinements and reverses of these inequalities are also provided.

2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A,B,C,D\in B(H)$.

If we take $C=A$ and $D=B$, then we get

$$A^2\otimes B^2=(A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Theorem 1. *Assume A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ and $\operatorname{Sp}(B) \subset J$. Let f be continuous on I, g continuous on J and φ continuous on an interval K that contains the product of the intervals $f(I)g(J)$, then*

$$\varphi(f(A) \otimes g(B)) = \int_I \int_J \varphi(f(t)g(s))\, dE(t) \otimes dF(s), \tag{2.6}$$

where A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s).$$

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

Then, by (1.1) and (2.2) we have

$$\begin{aligned}\int_I \int_J [f(t)g(s)]^n dE(t) \otimes dF(s) &= \int_I \int_J [f(t)]^n [g(s)]^n dE(t) \otimes dF(s) \\ &= [f(A)]^n \otimes [g(B)]^n = [f(A) \otimes g(B)]^n,\end{aligned}$$

which shows that the identity (2.6) is valid for the power function.

This proves the statement. □

Corollary 1. *With the assumptions of Theorem 1 for f and A, we have*

$$\varphi(f(A) \otimes 1) = \int_I \int_J (\varphi \circ f)(t)\, dE(t) \otimes dF(s) = \varphi(f(A)) \otimes 1. \tag{2.7}$$

The proof follows by (2.6) for $g \equiv 1$.

Corollary 2. *Assume A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ and $\operatorname{Sp}(B) \subset J$. Let f be continuous and positive on I, g continuous and positive on J, then*

$$\ln(f(A) \otimes g(B)) = (\ln f(A)) \otimes 1 + 1 \otimes (\ln g(B)) \tag{2.8}$$

If r is a real number, then also

$$\begin{aligned}(f(A) \otimes g(B))^r &= (f(A))^r \otimes (g(B)^r) \\ &= ((f(A))^r \otimes 1)(1 \otimes (g(B)^r)).\end{aligned} \tag{2.9}$$

Proof. From (2.6) written for $\varphi = \ln$ we have

$$\begin{aligned}
&\ln\left(f\left(A\right)\otimes g\left(B\right)\right)\\
&=\int_I\int_J \ln\left(f\left(t\right)g\left(s\right)\right)dE\left(t\right)\otimes dF\left(s\right),\\
&=\int_I\int_J \left[\ln\left(f\left(t\right)\right)+\ln\left(g\left(s\right)\right)\right]dE\left(t\right)\otimes dF\left(s\right)\\
&=\int_I\int_J \ln\left(f\left(t\right)\right)dE\left(t\right)\otimes dF\left(s\right)+\int_I\int_J \ln\left(g\left(s\right)\right)dE\left(t\right)\otimes dF\left(s\right)\\
&=\left(\ln f\left(A\right)\right)\otimes 1+1\otimes\left(\ln g\left(B\right)\right),
\end{aligned}$$

which proves (2.8).

Now, consider $\varphi(t) = t^r$, $t > 0$. Then by (2.6) we get

$$\begin{aligned}
\left(f\left(A\right)\otimes g\left(B\right)\right)^r &= \int_I\int_J \left(f\left(t\right)g\left(s\right)\right)^r dE\left(t\right)\otimes dF\left(s\right)\\
&=\int_I\int_J \left(f\left(t\right)\right)^r\left(g\left(s\right)\right)^r dE\left(t\right)\otimes dF\left(s\right)\\
&=\left(f\left(A\right)\right)^r\otimes\left(g\left(B\right)\right)^r,
\end{aligned}$$

which proves the first equality in (2.9). The second part follows by the property (2.1). □

Corollary 3. *Assume that* $A, B \geq 0$ *and* φ *is continuous on* $[0,\infty)$ *and* $\nu \in [0,1]$, *then*

$$\varphi\left(A^{1-\nu}\otimes B^{\nu}\right)=\int_0^\infty\int_0^\infty \varphi\left(t^{1-\nu}s^{\nu}\right)dE\left(t\right)\otimes dF\left(s\right).$$

In particular,

$$\varphi\left(A^{1/2}\otimes B^{1/2}\right)=\int_0^\infty\int_0^\infty \varphi\left(t^{1/2}s^{1/2}\right)dE\left(t\right)\otimes dF\left(s\right).$$

The proof follows by (2.6) by taking $f(t) = t^{1-\nu}$ and $g(s) = s^{\nu}$, $t, s \geq 0$.

Corollary 4. *Assume* A *and* B *are selfadjoint operators with* $\operatorname{Sp}(A) \subset I$ *and* $\operatorname{Sp}(B) \subset J$, *then for* $r > 0$,

$$|A\otimes B|^r = |A|^r\otimes|B|^r. \tag{2.10}$$

Proof. From (2.6) for the modulus function, we have

$$\begin{aligned}
|A\otimes B|^r &= \int_I\int_J |ts|^r dE\left(t\right)\otimes dF\left(s\right)\\
&= \int_I\int_J |t|^r|s|^r dE\left(t\right)\otimes dF\left(s\right)=|A|^r\otimes|B|^r,
\end{aligned}$$

which proves (2.10). □

The additive case is as follows:

Theorem 2. *Assume* A *and* B *are selfadjoint operators with* $\operatorname{Sp}(A) \subset I$ *and* $\operatorname{Sp}(B) \subset J$. *Let* h *be continuous on* I, k *continuous on* J *and* ψ *continuous on an interval* U *that contains the sum of the intervals* $h(I) + k(J)$, *then*

$$\psi\left(h\left(A\right)\otimes 1+1\otimes k\left(B\right)\right)=\int_I\int_J \psi\left(h\left(t\right)+k\left(s\right)\right)dE\left(t\right)\otimes dF\left(s\right), \tag{2.11}$$

where A and B have the spectral resolutions

$$A=\int_I tdE(t) \text{ and } B=\int_J sdF(s).$$

Proof. Let f and g continuous and positive such that $h(t)=\ln f(t)$, $t\in I$ and $k(s)=\ln g(s)$, $s\in J$. Then

$$\begin{aligned}(2.12)\qquad \psi(h(A)\otimes 1+1\otimes k(B)) &= \psi((\ln f(A))\otimes 1+1\otimes(\ln g(B)))\\ &= \psi(\ln(f(A)\otimes g(B))).\end{aligned}$$

Now if we apply identity (2.6) to the function $\varphi=\psi\circ\ln$, then we have

$$\begin{aligned}(2.13)\qquad \psi\circ\ln(f(A)\otimes g(B)) &= \int_I\int_J \psi\circ\ln(f(t)g(s))\,dE(t)\otimes dF(s)\\ &= \int_I\int_J \psi(\ln(f(t))+\ln g(s))\,dE(t)\otimes dF(s)\\ &= \int_I\int_J \psi(h(t)+k(s))\,dE(t)\otimes dF(s).\end{aligned}$$

By making use of (2.12) and (2.13) we derive (2.11). □

Corollary 5. *With the assumptions of Theorem 2 we have*

$$(2.14)\qquad |h(A)\otimes 1+1\otimes k(B)|=\int_I\int_J |h(t)+k(s)|\,dE(t)\otimes dF(s),$$

and the triangle inequality

$$|h(A)\otimes 1+1\otimes k(B)|\le |h(A)|\otimes 1+1\otimes|k(B)|.$$

The proof follows by taking $\psi(u)=|u|$, $u\in\mathbb{R}$.

Corollary 6. *With the assumption of Theorem 2 for h, k, A and B, we have*

$$(2.15)\qquad \exp(h(A)\otimes 1+1\otimes k(B))=(\exp h(A))\otimes(\exp k(B)).$$

Proof. From (2.11) for the exponential function, we have by (1.1) that

$$\begin{aligned}\exp(h(A)\otimes 1+1\otimes k(B)) &= \int_I\int_J \exp(h(t)+k(s))\,dE(t)\otimes dF(s)\\ &= \int_I\int_J \exp h(t)\exp k(s)\,dE(t)\otimes dF(s)\\ &= \exp h(A)\otimes\exp k(B)\end{aligned}$$

and the identity (2.15) is proved. □

Corollary 7. *Assume that ψ is continuous on the interval I and A, B are selfadjoint operators with spectra in I, then for all $\nu\in[0,1]$ we have the representation*

$$(2.16)\qquad \psi((1-\nu)A\otimes 1+\nu 1\otimes B)=\int_I\int_I \psi((1-\nu)t+\nu s)\,dE(t)\otimes dF(s),$$

where A and B have the spectral resolutions

$$A=\int_I tdE(t) \text{ and } B=\int_I sdF(s).$$

The proof follows by (2.11) for $h(t)=(1-\nu)t$ and $k(s)=\nu s$, for $\nu\in[0,1]$ and t, $s\in I$.

For the case of harmonic mean, we can state

Corollary 8. *Assume that φ is continuous on the interval $(0,\infty)$ and A, B are positive operators, then for all $\nu\in[0,1]$ we have the representation*

$$\varphi\left(\left[(1-\nu)A^{-1}\otimes 1+\nu 1\otimes B^{-1}\right]^{-1}\right) = \int_0^\infty\int_0^\infty \varphi\left[\left((1-\nu)t^{-1}+\nu s^{-1}\right)^{-1}\right]dE(t)\otimes dF(s), \tag{2.17}$$

where A and B have the spectral resolutions

$$A=\int_0^\infty t dE(t) \text{ and } B=\int_0^\infty s dF(s).$$

The proof follows from (2.15) by choosing $\psi=\varphi\circ\frac{1}{\ell}$, $\ell(t)=t$, $h(t)=(1-\nu)t^{-1}$ and $k(s)=\nu s^{-1}$, $t,s\in(0,\infty)$.

For $\varphi(t)=t$, we get

$$\left[(1-\nu)A^{-1}\otimes 1+\nu 1\otimes B^{-1}\right]^{-1} = \int_0^\infty\int_0^\infty \left((1-\nu)t^{-1}+\nu s^{-1}\right)^{-1}dE(t)\otimes dF(s), \tag{2.18}$$

for all $A>0$, $B>0$ and $\nu\in[0,1]$

We also have the following representations as well

$$|A\otimes 1\pm 1\otimes B|^r=\int_I\int_J |t\pm s|^r dE(t)\otimes dF(s), \tag{2.19}$$

where A and B are selfadjoint operators with $\operatorname{Sp}(A)\subset I$, $\operatorname{Sp}(B)\subset J$ and $r>0$.

3. Main Results

We investigate now some tensorial for convex and operator convex functions:

Theorem 3. *Assume that ψ is a continuous and convex (concave) function on the interval I and A and B are selfadjoint operators with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$, then for all $\nu\in[0,1]$*

$$\psi((1-\nu)A\otimes 1+\nu 1\otimes B)\le(\ge)(1-\nu)\psi(A)\otimes 1+\nu 1\otimes\psi(B) \tag{3.1}$$

and

$$\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right) \le(\ge)\frac{\psi((1-\nu)A\otimes 1+\nu 1\otimes B)+\psi(\nu A\otimes 1+(1-\nu)1\otimes B)}{2} \le(\ge)\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}. \tag{3.2}$$

Moreover, we have the Hermite-Hadamard type inequalities

$$\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)\le(\ge)\int_0^1\psi((1-\nu)A\otimes 1+\nu 1\otimes B)d\nu \le(\ge)\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}. \tag{3.3}$$

Proof. Assume that A and B have the spectral resolutions

$$A=\int_I tdE(t) \text{ and } B=\int_I sdF(s).$$

By (2.16) and the convexity of ψ, we have successively

$$\begin{aligned}
(3.4)\qquad &\psi((1-\nu)A\otimes 1+\nu 1\otimes B)\\
&=\int_I\int_I \psi((1-\nu)t+\nu s)\,dE(t)\otimes dF(s)\\
&\le \int_I\int_I [(1-\nu)\psi(t)+\nu\psi(s)]\,dE(t)\otimes dF(s)\\
&=(1-\nu)\int_I\int_I \psi(t)\,dE(t)\otimes dF(s)+\nu\int_I\int_I \psi(s)\,dE(t)\otimes dF(s)\\
&=(1-\nu)\psi(A)\otimes 1+\nu 1\otimes\psi(B)
\end{aligned}$$

for all $\nu\in[0,1]$.

Now, if we take $1-\nu$ instead of ν in (3.1)

$$\psi(\nu A\otimes 1+(1-\nu)1\otimes B)\le \nu\psi(A)\otimes 1+(1-\nu)1\otimes\psi(B) \tag{3.5}$$

for all $\nu\in[0,1]$.

Now, for a convex function ψ defined on I we also have the double inequalities

$$\psi\left(\frac{t+s}{2}\right)\le\frac{\psi((1-\nu)t+\nu s)+\psi(\nu t+(1-\nu)s)}{2}\le\frac{\psi(t)+\psi(s)}{2}$$

for all t, $s\in I$ and $\nu\in[0,1]$.

If we take the double integral $\int_I\int_I$ over $dE(t)\otimes dF(s)$ in the inequality (3.5), then we get

$$\begin{aligned}
(3.6)\qquad &\int_I\int_I \psi\left(\frac{t+s}{2}\right)dE(t)\otimes dF(s)\\
&\le\frac{1}{2}\int_I\int_I[\psi((1-\nu)t+\nu s)+\psi(\nu t+(1-\nu)s)]\,dE(t)\otimes dF(s)\\
&\le\frac{1}{2}\int_I\int_I[\psi(t)+\psi(s)]\,dE(t)\otimes dF(s).
\end{aligned}$$

Since

$$\int_I\int_I \psi\left(\frac{t+s}{2}\right)dE(t)\otimes dF(s)=\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right),$$

$$\begin{aligned}
&\int_I\int_I[\psi((1-\nu)t+\nu s)+\psi(\nu t+(1-\nu)s)]\,dE(t)\otimes dF(s)\\
&=\psi((1-\nu)A\otimes 1+\nu 1\otimes B)+\psi(\nu A\otimes 1+(1-\nu)1\otimes B)
\end{aligned}$$

and

$$\int_I\int_I[\psi(t)+\psi(s)]\,dE(t)\otimes dF(s)=\psi(A)\otimes 1+1\otimes\psi(B),$$

hence by (3.6) we obtain (3.2).

Further, if we take the integral over $\nu \in [0,1]$ in (3.2), then we get

$$\begin{aligned}
(3.7)\qquad &\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)\\
&\leq \int_0^1 \frac{\psi\left(\left(1-\nu\right)A\otimes 1+\nu 1\otimes B\right)+\psi\left(\nu A\otimes 1+\left(1-\nu\right)1\otimes B\right)}{2}d\nu\\
&\leq \frac{\psi\left(A\right)\otimes 1+1\otimes\psi\left(B\right)}{2}.
\end{aligned}$$

Since

$$\int_0^1 \psi\left(\nu A\otimes 1+\left(1-\nu\right)1\otimes B\right)d\nu=\int_0^1 \psi\left(\left(1-\nu\right)A\otimes 1+\nu 1\otimes B\right)d\nu,$$

hence by (3.7) we derive (3.3). □

Theorem 4. *Assume that ψ is a continuous and convex function on the interval I and A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$, then for all $p,q\in(0,1)$,*

$$\begin{aligned}
(3.8)\qquad &\min\left\{\frac{p}{q},\frac{1-p}{q}\right\}\\
&\times\left[q\psi(A)\otimes 1+\left(1-q\right)1\otimes\psi(B)-\psi\left(qA\otimes 1+\left(1-q\right)1\otimes B\right)\right]\\
&\leq p\psi(A)\otimes 1+\left(1-p\right)1\otimes\psi(B)-\psi\left(pA\otimes 1+\left(1-p\right)1\otimes B\right)\\
&\leq \max\left\{\frac{p}{q},\frac{1-p}{q}\right\}\\
&\times\left[q\psi(A)\otimes 1+\left(1-q\right)1\otimes\psi(B)-\psi\left(qA\otimes 1+\left(1-q\right)1\otimes B\right)\right].
\end{aligned}$$

In particular,

$$\begin{aligned}
(3.9)\qquad &2\min\left\{p,1-p\right\}\left[\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}-\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right]\\
&\leq p\psi(A)\otimes 1+\left(1-p\right)1\otimes\psi(B)-\psi\left(pA\otimes 1+\left(1-p\right)1\otimes B\right)\\
&\leq 2\min\left\{p,1-p\right\}\left[\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}-\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right].
\end{aligned}$$

Proof. Recall the following result obtained by the author in 2006 [5] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}
(3.10)\qquad &\min_{j\in\{1,2,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left[\frac{1}{Q_n}\sum_{j=1}^n q_j\psi\left(x_j\right)-\psi\left(\frac{1}{Q_n}\sum_{j=1}^n q_jx_j\right)\right]\\
&\leq \frac{1}{P_n}\sum_{j=1}^n p_j\psi\left(x_j\right)-\psi\left(\frac{1}{P_n}\sum_{j=1}^n p_jx_j\right)\\
&\leq \max_{j\in\{1,2,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left[\frac{1}{Q_n}\sum_{j=1}^n q_j\psi\left(x_j\right)-\psi\left(\frac{1}{Q_n}\sum_{j=1}^n x_j\right)\right],
\end{aligned}$$

where $\psi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space $X, \{x_j\}_{j\in\{1,2,...,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,...,n\}}$, $\{q_j\}_{j\in\{1,2,...,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j$, $Q_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (3.10) that

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{q}\right\}\left[q\psi(x) + (1-q)\,\psi(y) - \psi\left[qx + (1-q)\,y\right]\right] \\ &\le p\psi\left(x\right) + \left(1-p\right)\psi\left(y\right) - \psi\left[px + \left(1-p\right)y\right] \\ &\le \max\left\{\frac{p}{q}, \frac{1-p}{q}\right\}\left[q\psi(x) + (1-q)\,\psi(y) - \psi\left[qx + (1-q)\,y\right]\right] \end{aligned} \tag{3.11}$$

for all $x, y \in C$ and $p, q \in (0, 1)$.

Assume that A and B have the spectral resolutions

$$A = \int_I t dE\left(t\right) \text{ and } B = \int_I s dF\left(s\right).$$

If we take the double integral $\int_I \int_I$ over $dE\left(t\right) \otimes dF\left(s\right)$ in the inequality (3.11), then we get

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{q}\right\} \\ &\times \int_I \int_I \left[q\psi(t) + (1-q)\,\psi(s) - \psi\left[qt + (1-q)\,s\right]\right] dE\left(t\right) \otimes dF\left(s\right) \\ &\le \int_I \int_I \left[p\psi\left(t\right) + \left(1-p\right)\psi\left(s\right) - \psi\left[pt + \left(1-p\right)s\right]\right] dE\left(t\right) \otimes dF\left(s\right) \\ &\le \max\left\{\frac{p}{q}, \frac{1-p}{q}\right\} \\ &\times \int_I \int_I \left[q\psi(t) + (1-q)\,\psi(s) - \psi\left[qt + (1-q)\,s\right]\right] dE\left(t\right) \otimes dF\left(s\right). \end{aligned} \tag{3.12}$$

Since

$$\begin{aligned} &\int_I \int_I \left[q\psi(t) + (1-q)\,\psi(s) - \psi\left[qt + (1-q)\,s\right]\right] dE\left(t\right) \otimes dF\left(s\right) \\ &= q \int_I \int_I \psi(t) dE\left(t\right) \otimes dF\left(s\right) + (1-q) \int_I \int_I \psi(s) dE\left(t\right) \otimes dF\left(s\right) \\ &- \int_I \int_I \psi\left[qt + (1-q)\,s\right] dE\left(t\right) \otimes dF\left(s\right) \\ &= q\psi(A) \otimes 1 + (1-q)\,1 \otimes \psi(B) - \psi\left(qA \otimes 1 + (1-q)\,1 \otimes B\right) \end{aligned}$$

and

$$\begin{aligned} &\int_I \int_I \left[p\psi\left(t\right) + \left(1-p\right)\psi\left(s\right) - \psi\left[pt + \left(1-p\right)s\right]\right] dE\left(t\right) \otimes dF\left(s\right) \\ &= p\psi(A) \otimes 1 + (1-p)\,1 \otimes \psi(B) - \psi\left(pA \otimes 1 + (1-p)\,1 \otimes B\right), \end{aligned}$$

hence by (3.12) we derive (3.8). □

Theorem 5. *Assume that ψ is a continuous and convex function on the interval I and A and B are selfadjoint operators with* $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, *then for all*

$q\in(0,1)$,

$$0\leq\frac{1}{2}\left[q\psi(A)\otimes 1+(1-q)\,1\otimes\psi(B)-\psi\left(qA\otimes 1+(1-q)\,1\otimes B\right)\right] \tag{3.13}$$
$$\leq\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}-\int_0^1\psi\left(tA\otimes 1+(1-t)\,1\otimes B\right)$$
$$\leq\frac{q^2-q+1}{2q\,(1-q)}\times\left[q\psi(A)\otimes 1+(1-q)\,1\otimes\psi(B)-\psi\left(qA\otimes 1+(1-q)\,1\otimes B\right)\right].$$

Proof. From (3.8) we get for $t,q\in(0,1)$ that

$$\min\left\{\frac{t}{q},\frac{1-t}{q}\right\}\times\left[q\psi(A)\otimes 1+(1-q)\,1\otimes\psi(B)-\psi\left(qA\otimes 1+(1-q)\,1\otimes B\right)\right] \tag{3.14}$$
$$\leq t\psi(A)\otimes 1+(1-t)\,1\otimes\psi(B)-\psi\left(tA\otimes 1+(1-t)\,1\otimes B\right)$$
$$\leq\max\left\{\frac{t}{q},\frac{1-t}{q}\right\}\times\left[q\psi(A)\otimes 1+(1-q)\,1\otimes\psi(B)-\psi\left(qA\otimes 1+(1-q)\,1\otimes B\right)\right].$$

If we integrate over $t\in[0,1]$ the inequality (3.14), then we get

$$\int_0^1\min\left\{\frac{t}{q},\frac{1-t}{q}\right\}dt\times\left[q\psi(A)\otimes 1+(1-q)\,1\otimes\psi(B)-\psi\left(qA\otimes 1+(1-q)\,1\otimes B\right)\right] \tag{3.15}$$
$$\leq\frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}-\int_0^1\psi\left(tA\otimes 1+(1-t)\,1\otimes B\right)$$
$$\leq\int_0^1\max\left\{\frac{t}{q},\frac{1-t}{q}\right\}dt\times\left[q\psi(A)\otimes 1+(1-q)\,1\otimes\psi(B)-\psi\left(qA\otimes 1+(1-q)\,1\otimes B\right)\right],$$

for $q\in(0,1)$.

Observe that

$$\frac{t}{q}-\frac{1-t}{1-q}=\frac{t-q}{q\,(1-q)}$$

showing that

$$\min\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}=\begin{cases}\frac{t}{q} \text{ if } 0\leq t\leq q\leq 1\\[2ex] \frac{1-t}{1-q} \text{ if } 0\leq q\leq t\leq 1\end{cases}$$

and

$$\max\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}=\begin{cases}\frac{1-t}{1-q} \text{ if } 0\le t\le q\le 1\\ \\ \frac{t}{q} \text{ if } 0\le q\le t\le 1.\end{cases}$$

Then

$$\begin{aligned}\int_0^1 \min\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}dt &= \int_0^q \frac{t}{q}dt+\int_q^1 \frac{1-t}{1-q}dt\\ &= \frac{q^2}{2q}+\frac{1}{1-q}\left(1-q-\left(\frac{1-q^2}{2}\right)\right)=\frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \max\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}dt &= \int_0^q \frac{1-t}{1-q}dt+\int_q^1 \frac{t}{q}dt\\ &= \frac{1}{1-q}\left(q-\frac{q^2}{2}\right)+\frac{1-q^2}{2q}\\ &= \frac{q^2-q+1}{2q\,(1-q)}\end{aligned}$$

and by (3.15) we obtain the desired result (3.13). □

Theorem 6. *Assume that* ψ *is a continuous and convex function on the interval* I *and* A *and* B *are selfadjoint operators with* $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$*, then for all* $\lambda\in(0,1)$,

$$\begin{aligned}(3.16)\qquad &\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)\\ &\le (1-\lambda)\,\psi\left[\frac{(1-\lambda)\,A\otimes 1+(1+\lambda)\,1\otimes B}{2}\right]\\ &+\lambda\psi\left[\frac{(2-\lambda)\,A\otimes 1+\lambda 1\otimes B}{2}\right]\\ &\le \int_0^1 \left(\psi\left((1-u)\,A\otimes 1+u1\otimes B\right)\right)du\\ &\le \frac{1}{2}\left[\psi\left((1-\lambda)\,A\otimes 1+\lambda 1\otimes B\right)+(1-\lambda)\,1\otimes\psi(B)+\lambda\psi(A)\otimes 1\right]\\ &\le \frac{\psi(A)\otimes 1+1\otimes\psi(B)}{2}.\end{aligned}$$

In particular,

$$\begin{aligned}
&\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right) \\
&\leq \frac{1}{2}\psi\left(\frac{A\otimes 1+3\otimes B}{4}\right)+\psi\left(\frac{A\otimes 3+1\otimes B}{4}\right) \\
&\leq \int_0^1 \psi\left(\left(1-u\right)A\otimes 1+u1\otimes B\right)du \\
&\leq \frac{1}{2}\left[\psi\left(\frac{A\otimes 1+1\otimes B}{2}\right)+\frac{\psi\left(A\right)\otimes 1+1\otimes\psi\left(B\right)}{2}\right] \\
&\leq \frac{\psi\left(A\right)\otimes 1+1\otimes\psi\left(B\right)}{2}.
\end{aligned} \tag{3.17}$$

Proof. In 2002, Barnett et al. [4] obtained the following refinement of Hermite-Hadamard inequality:

Let $\psi : I \to \mathbb{R}$ be a convex function on the interval I, then for any t, $s \in I$ and for any $\lambda \in [0,1]$ we have the inequalities

$$\begin{aligned}
&\psi\left(\frac{t+s}{2}\right) \\
&\leq \left(1-\lambda\right)\psi\left[\frac{\left(1-\lambda\right)t+\left(1+\lambda\right)s}{2}\right]+\lambda\psi\left[\frac{\left(2-\lambda\right)t+\lambda s}{2}\right] \\
&\leq \int_0^1 \psi\left(\left(1-u\right)t+us\right)du \\
&\leq \frac{1}{2}\left[\psi\left(\left(1-\lambda\right)t+\lambda s\right)+\left(1-\lambda\right)\psi\left(s\right)+\lambda\psi\left(t\right)\right] \\
&\leq \frac{\psi\left(t\right)+\psi\left(s\right)}{2}.
\end{aligned} \tag{3.18}$$

Assume that A and B have the spectral resolutions

$$A=\int_I tdE\left(t\right) \text{ and } B=\int_I sdF\left(s\right).$$

If we take the double integral $\int_I\int_I$ over $dE\left(t\right)\otimes dF\left(s\right)$ in the inequality (3.18), then we get

$$\begin{aligned}
&\int_I\int_I \psi\left(\frac{t+s}{2}\right)dE\left(t\right)\otimes dF\left(s\right) \\
&\leq \left(1-\lambda\right)\int_I\int_I \psi\left[\frac{\left(1-\lambda\right)t+\left(1+\lambda\right)s}{2}\right]dE\left(t\right)\otimes dF\left(s\right) \\
&+\lambda\int_I\int_I \psi\left[\frac{\left(2-\lambda\right)t+\lambda s}{2}\right]dE\left(t\right)\otimes dF\left(s\right) \\
&\leq \int_I\int_I\left(\int_0^1 \psi\left(\left(1-u\right)t+us\right)du\right)dE\left(t\right)\otimes dF\left(s\right) \\
&\leq \frac{1}{2}\int_I\int_I\left[\psi\left(\left(1-\lambda\right)t+\lambda s\right)+\left(1-\lambda\right)\psi\left(s\right)+\lambda\psi\left(t\right)\right]dE\left(t\right)\otimes dF\left(s\right) \\
&\leq \int_I\int_I \frac{\psi\left(t\right)+\psi\left(s\right)}{2}dE\left(t\right)\otimes dF\left(s\right).
\end{aligned} \tag{3.19}$$

Since

$$\int_I\int_I \psi\left[\frac{(1-\lambda)\,t+(1+\lambda)\,s}{2}\right]dE\,(t)\otimes dF\,(s)$$
$$=\psi\left[\frac{(1-\lambda)\,A\otimes 1+(1+\lambda)\,1\otimes B}{2}\right],$$

$$\int_I\int_I \psi\left[\frac{(2-\lambda)\,t+\lambda s}{2}\right]dE\,(t)\otimes dF\,(s)$$
$$=\psi\left[\frac{(2-\lambda)\,A\otimes 1+\lambda 1\otimes B}{2}\right]$$

and, by using Fubini's theorem,

$$\int_I\int_I\left(\int_0^1\psi\left((1-u)\,t+us\right)du\right)dE\,(t)\otimes dF\,(s)$$
$$=\int_0^1\left(\int_I\int_I\psi\left((1-u)\,t+us\right)dE\,(t)\otimes dF\,(s)\right)du$$
$$=\int_0^1\left(\psi\left((1-u)\,A\otimes 1+u1\otimes B\right)\right)du,$$

hence by employing the corresponding calculations from the proof of Theorem 3 and the inequality (3.19), we derive the desired result (3.16). □

4. Some Examples

Consider the convex (concave) function $f(t)=t^r$, $r\in(-\infty,0)\cup[1,\infty)$ ($r\in(0,1)$) defined on $(0,\infty)$. If we use Theorem 3 we have for $A,B>0$

$$\left(\frac{A\otimes 1+1\otimes B}{2}\right)^r \tag{4.1}$$
$$\leq(\geq)\,\frac{\left((1-\nu)\,A\otimes 1+\nu 1\otimes B\right)^r+\left(\nu A\otimes 1+(1-\nu)\,1\otimes B\right)^r}{2}$$
$$\leq(\geq)\,\frac{A^r\otimes 1+1\otimes B^r}{2}$$

for all $\nu\in[0,1]$.

Moreover, we have the Hermite-Hadamard type inequalities

$$\left(\frac{A\otimes 1+1\otimes B}{2}\right)^r\leq(\geq)\int_0^1\left((1-\nu)\,A\otimes 1+\nu 1\otimes B\right)^r d\nu \tag{4.2}$$
$$\leq(\geq)\,\frac{A^r\otimes 1+1\otimes B^r}{2}.$$

The function $f(t)=\exp(\alpha t)$, $\alpha\neq 0$, is convex on $\mathbb{R}$ and by Theorem 3, we have for any selfadjoint operators A and B, that

$$\exp\left[\alpha\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right]\leq\frac{1}{2}\left\{\exp\left[\alpha\left((1-\nu)\,A\otimes 1+\nu 1\otimes B\right)\right]\right. \tag{4.3}$$
$$\left.+\exp\left[\alpha\left(\nu A\otimes 1+(1-\nu)\,1\otimes B\right)\right]\right\}$$
$$\leq\frac{\exp(\alpha A)\otimes 1+1\otimes\exp(\alpha B)}{2}.$$

Moreover, we have the Hermite-Hadamard type inequalities

$$\exp\left[\alpha\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right]\leq\int_0^1\exp\left[\alpha\left(\left(1-\nu\right)A\otimes 1+\nu 1\otimes B\right)\right]d\nu \leq\frac{\exp\left(\alpha A\right)\otimes 1+1\otimes\exp\left(\alpha B\right)}{2}. \tag{4.4}$$

The function $f(t)=\ln t$ is concave and by Theorem 3 we get

$$\begin{aligned}\ln\left(\frac{A\otimes 1+1\otimes B}{2}\right) &\geq\frac{\ln\left(\left(1-\nu\right)A\otimes 1+\nu 1\otimes B\right)+\ln\left(\nu A\otimes 1+\left(1-\nu\right)1\otimes B\right)}{2}\\ &\geq\frac{\left(\ln A\right)\otimes 1+1\otimes\left(\ln B\right)}{2}.\end{aligned} \tag{4.5}$$

Moreover, we have the Hermite-Hadamard type inequalities

$$\ln\left(\frac{A\otimes 1+1\otimes B}{2}\right)\geq\int_0^1\ln\left(\left(1-\nu\right)A\otimes 1+\nu 1\otimes B\right)d\nu \geq\frac{\left(\ln A\right)\otimes 1+1\otimes\left(\ln B\right)}{2}. \tag{4.6}$$

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] N. S. Barnett, P. Cerone and S. S. Dragomir, Some new inequalities for Hermite-Hadamard divergence in information theory. in *Stochastic Analysis and Applications.* Vol. **3**, 7–19, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint RGMIA Res. Rep. Coll. 5 (2002), No. 4, Art. 8, 11 pp. [Online https://rgmia.org/papers/v5n4/NIHHDIT.pdf]

[5] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[6] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[7] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[8] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[9] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[10] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

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