# REFINEMENTS AND REVERSES OF TENSORIAL ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $H$ be a Hilbert space. In this paper we show among others that, if the operators $A_i \in B(H)$ satisfy the condition $\operatorname{Sp}(A_i) \subset [k, K] \subset (0, \infty)$, $i \in \{1, ..., m\}$ and $q_i \geq 0$, $i \in \{1, ..., m\}$ with $\sum_{i=1}^{m} q_i = 1$, then

$$\begin{aligned} 0 &\leq \frac{1}{2K}\left[\sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i^2 - \left(\sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i\right)^2\right] \\ &\leq \sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i - \bigotimes_{i=1}^{n} A_i^{q_i} \leq \frac{1}{2k}\left[\sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i^2 - \quad \sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i\right)^2\right], \end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of $A_i$ in position $i = 1, ..., n$ and with 1 in the other positions.

Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_t)$, $\operatorname{Sp}(B_t) \subseteq [m, M] \subset (0, \infty)$ for each $t \in \Omega$. Then for $\nu \in [0, 1]$ we also have the integral inequalities for Hadamard product

$$\begin{aligned} 0 &\leq \frac{1}{2M}\left[\left(\int_\Omega \frac{A_t^2 + B_t^2}{2} d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t d\mu(t)\right) \circ \left(\int_\Omega B_t d\mu(t)\right)\right] \\ &\leq \left(\int_\Omega [(1-\nu) A_t + \nu B_t] d\mu(t)\right) \circ 1 \\ &- \left(\int_\Omega A_t^{1-\nu} d\mu(t)\right) \circ \left(\int_\Omega B_t^{\nu} d\mu(t)\right) \\ &\leq \frac{1}{2m}\left[\left(\int_\Omega \frac{A_t^2 + B_t^2}{2} d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t d\mu(t)\right) \circ \left(\int_\Omega B_t d\mu(t)\right)\right]. \end{aligned}$$

## 1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a $k$-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of $A_i$ is contained in $I_i$ for $i = 1, ..., k$. We say that such a $k$-tuple is in the domain of $f$. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of $A_i$ for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

---

1991 *Mathematics Subject Classification.* 47A63; 47A99.

*Key words and phrases.* Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.



*RGMIA Res. Rep. Coll.* **25** (2022), Art. 93, 18 pp. Received 21/09/22

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever $f$ can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of $k$ functions each depending on only one variable.

It is know that, *if $f$ is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if $f$ is continuous on $[0, \infty)$, then [9, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of $A$ and $B$, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) \, dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function $f$ on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [12] obtained the following *Callebaut type inequalities* for tensorial product

$$(A \# B) \otimes (A \# B) \leq \frac{1}{2} \left[ (A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \tag{1.4}$$
$$\leq \frac{1}{2} (A \otimes B + B \otimes A)$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

In 2007, S. Wada [12] obtained the following *Callebaut type inequalities* for tensorial product

$$(A \# B) \otimes (A \# B) \leq \frac{1}{2} \left[ (A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \tag{1.5}$$
$$\leq \frac{1}{2} (A \otimes B + B \otimes A)$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of $A$ and $B$ in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space $H$.

It is known that, see [8], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.6}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If $f$ *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [9, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.7}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [10] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices $A$ and $B$.

Motivated by the above results, in this paper we show among others that, if the operators $A_i \in B(H)$ satisfy the condition $\operatorname{Sp}(A_i) \subset [k, K] \subset (0, \infty)$, $i \in \{1, ..., m\}$ and $q_i \geq 0$, $i \in \{1, ..., m\}$ with $\sum_{i=1}^m q_i = 1$, then

$$\begin{aligned}
0 &\leq \frac{1}{2K}\left[\sum_{i=1}^m q_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^m q_i \widehat{\mathbf{A}_i}\right)^2\right] \\
&\leq \sum_{i=1}^m q_i \widehat{\mathbf{A}_i} - \bigotimes_{i=1}^n A_i^{q_i} \leq \frac{1}{2k}\left[\sum_{i=1}^m q_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^m q_i \widehat{\mathbf{A}_i}\right)^2\right],
\end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of $A_i$ in position $i = 1, ..., n$ and with 1 in the other positions.

Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$be a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_t)$, $\operatorname{Sp}(B_t) \subseteq [m, M] \subset (0, \infty)$ for each $t \in \Omega$. Then for $\nu \in [0, 1]$ we also have the integral inequalities for Hadamard product

$$\begin{aligned}
0 &\leq \frac{1}{2M}\left[\left(\int_\Omega \frac{A_t^2 + B_t^2}{2} d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t d\mu(t)\right) \circ \left(\int_\Omega B_t d\mu(t)\right)\right] \\
&\leq \left(\int_\Omega [(1-\nu) A_t + \nu B_t] d\mu(t)\right) \circ 1 \\
&- \left(\int_\Omega A_t^{1-\nu} d\mu(t)\right) \circ \left(\int_\Omega B_t^{\nu} d\mu(t)\right) \\
&\leq \frac{1}{2m}\left[\left(\int_\Omega \frac{A_t^2 + B_t^2}{2} d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t d\mu(t)\right) \circ \left(\int_\Omega B_t d\mu(t)\right)\right].
\end{aligned}$$

## 2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{2.1}$$

that holds for any $A$, $B$, $C$, $D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2 .$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers $m$, $n$ we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

By induction over $m$, we derive

$$(A_1 \otimes A_2 \otimes ... \otimes A_m)^n = A_1^n \otimes A_2^n \otimes ... \otimes A_m^n \text{ for natural } n \geq 0 \tag{2.6}$$

and

$$\begin{aligned} &A_1 \otimes A_2 \otimes ... \otimes A_m \\ &= (A_1 \otimes 1 \otimes ... \otimes 1)(1 \otimes A_2 \otimes ... \otimes 1) ... (1 \otimes 1 \otimes ... \otimes A_m) \end{aligned} \tag{2.7}$$

and the $m$ operators $(A_1 \otimes 1 \otimes ... \otimes 1)$, $(1 \otimes A_2 \otimes ... \otimes 1)$, ... and $(1 \otimes 1 \otimes ... \otimes A_m)$ are commutative between them.

We define for $A_i$, $B_i \in B(H)$, $i \in \{1, ..., n\}$, $\bigotimes_{i=1}^{n} B_i := B_1 \otimes ... \otimes B_n$,

$$\hat{\mathbf{A}}_i := 1 \otimes ... \otimes A_i \otimes ... \otimes 1, \ i = 2, ..., n-1,$$

and

$$\hat{\mathbf{A}}_1 := A_1 \otimes 1 \otimes ... \otimes 1 \text{ while } \hat{\mathbf{A}}_n := 1 \otimes ... \otimes 1 \otimes A_n.$$

Basically $\hat{\mathbf{A}}_i$ is defined as a tensorial product of $A_i$ in position $i = 1, ..., n$ and with 1 in the other positions.

**Theorem 1.** *Assume $A_i$, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$, $i \in \{1, ..., m\}$ and with the spectral resolutions*

$$A_i = \int_{I_i} t_i dE(t_i), \ i \in \{1, ..., m\}.$$

*Let $f_i$, $i \in \{1, ..., m\}$ be continuous on $I_i$ and $\varphi$ continuous on an interval $K$ that contains the product of the intervals $f(I_1) ... f(I_m)$, then*

$$\varphi\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \int_{I_1} ... \int_{I_m} \varphi\left(\prod_{i=1}^{m} f_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.8}$$

*We also have*

$$\varphi\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right)=\boldsymbol{\varphi}\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right) \tag{2.9}$$

*for all* $i=1,...,m$.

*Proof.* By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t)=t^n$ with $n$ any natural number.

Then, by (1.1) and (2.6) we obtain

$$\begin{aligned}&\int_{I_1}...\int_{I_m}\left[f_1\left(t_1\right)...f_m\left(t_m\right)\right]^n dE\left(t_1\right)\otimes...\otimes dE\left(t_m\right)\\&=\int_{I_1}...\int_{I_m}\left[f_1\left(t_1\right)\right]^n...\left[f_m\left(t_m\right)\right]^n dE\left(t_1\right)\otimes...\otimes dE\left(t_m\right)\\&=\left[f\left(A_1\right)\right]^n\otimes...\otimes\left[f_m\left(A_m\right)\right]^n=\left[f_1\left(A_1\right)\otimes...\otimes f_m\left(A_m\right)\right]^n,\end{aligned}$$

which shows that the identity (2.6) is valid for the power function.

This proves the identity (2.8)

By taking $f_j\equiv 1$ for $j=1,...,m$ and $j\neq i$ in (2.8) we get

$$\begin{aligned}\varphi\left(1\otimes...\otimes f_i\left(A_i\right)\otimes...\otimes 1\right)&=\int_{I_1}...\int_{I_m}\varphi\left(f_i\left(t_i\right)\right)dE\left(t_1\right)\otimes...\otimes dE\left(t_m\right)\\&=1\otimes...\otimes\varphi\left(f_i\left(A_i\right)\right)\otimes...\otimes 1,\end{aligned}$$

which proves (2.9). □

**Corollary 1.** *Assume* $A_i$, $i\in\{1,...,m\}$ *are selfadjoint operators with* $\operatorname{Sp}(A_i)\subset I_i$ *and* $f_i$, $i\in\{1,...,m\}$ *are continuous and positive on* $I_i$, *then*

$$\ln\left(\bigotimes_{i=1}^{m}f_i\left(A_i\right)\right)=\sum_{i=1}^{n}\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right). \tag{2.10}$$

*Also*

$$\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right)=\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right) \tag{2.11}$$

*for all* $i=1,...,m$.

*Proof.* Assume that

$$A_i=\int_I t_i dE\left(t_i\right)$$

are the spectral resolutions for $A_i$, $i=1,...,m$.

We have for $\varphi(u)=\ln u$, $u>0$, in (2.8) that

$$\begin{aligned}
&\ln\left(f_1(A)\otimes...\otimes f_m(A_m)\right)\\
&=\int_{I_1}...\int_{I_m}\ln\left(f_1(t_1)...f_m(t_m)\right)dE(t_1)\otimes...\otimes dE(t_m)\\
&=\int_{I_1}...\int_{I_m}\left[\ln f_1(t_1)+...+\ln f_m(t_m)\right]dE(t_1)\otimes...\otimes dE(t_m)\\
&=\int_{I_1}...\int_{I_m}\ln f_1(t_1)\,dE(t_1)\otimes...\otimes dE(t_m)+...\\
&+\int_{I_1}...\int_{I_m}\ln f_m(t_m)\,dE(t_1)\otimes...\otimes dE(t_m)\\
&=\left(\ln f_1(A_1)\right)\otimes 1\otimes...\otimes 1+...+1\otimes...\otimes 1\otimes\left(\ln f_m(A_m)\right)
\end{aligned}$$

and the identity (2.10) is proved. □

**Corollary 2.** *Assume $A_i$, $i\in\{1,...,m\}$ are selfadjoint operators with* $\operatorname{Sp}(A_i)\subset I_i$ *and $f_i$, $i\in\{1,...,m\}$ are continuous on $I_i$, then for $r>0$*

$$\left|\bigotimes_{i=1}^{m}f_i(A_i)\right|^r=\bigotimes_{i=1}^{m}\left|f_i(A_i)\right|^r \tag{2.12}$$

*and*

$$\left|\left(\widehat{\mathbf{f}_i(\mathbf{A}_i)}\right)\right|^r=\left|\widehat{\mathbf{f}_i(\mathbf{A}_i)}\right|^r \tag{2.13}$$

*for all $i=1,...,m$.*

*Proof.* From (2.8) we have for the function $\varphi(t)=|t|^r$ that

$$\begin{aligned}
&\left|f_1(A_1)\otimes...\otimes f_m(A_m)\right|^r\\
&=\int_{I_1}...\int_{I_m}\left|f_1(t_1)...f_m(t_m)\right|^r dE(t_1)\otimes...\otimes dE(t_m)\\
&=\int_{I_1}...\int_{I_m}\left|f_1(t_1)\right|^r...\left|f_m(t_m)\right|^r dE(t_1)\otimes...\otimes dE(t_m)\\
&=\left|f_1(A_1)\right|^r\otimes...\otimes\left|f_m(A_m)\right|^r,
\end{aligned}$$

which proves (2.12).

The identity (2.13) follows in a similar way. □

**Corollary 3.** *Assume $A_i$, $i\in\{1,...,m\}$ are positive operators and $q_i\geq 0$, $i\in\{1,...,m\}$, then*

$$\varphi\left(\bigotimes_{i=1}^{m}A_i^{q_i}\right)=\int_{I_1}...\int_{I_m}\varphi\left(\prod_{i=1}^{m}t_i^{q_i}\right)dE(t_1)\otimes...\otimes dE(t_m). \tag{2.14}$$

We also have the additive result:

**Theorem 2.** *Assume $A_i$, $i\in\{1,...,m\}$ are selfadjoint operators with* $\operatorname{Sp}(A_i)\subset I_i$, *$i\in\{1,...,m\}$ and with the spectral resolutions*

$$A_i=\int_{I_i}t_i dE(t_i),\ i\in\{1,...,m\}.$$

*Let $g_i$, $i \in \{1, ..., m\}$ be continuous on $I_i$ and $\psi$ continuous on an interval $K$ that contains the sum of the intervals $g(I_1) + ... + g(I_m)$, then*

$$\psi\left(\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right) = \int_{I_1} ... \int_{I_m} \psi\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.15}$$

*Proof.* Let $f_i$, continuous, positive and such that $g_i(t_i) = \ln f_i(t_i)$, $t_i \in I_i$, $i \in \{1, ..., m\}$. Then

$$\sum_{i=1}^{m} g_i(t_i) = \sum_{i=1}^{m} \ln f_i(t_i) = \ln\left(\prod_{i=1}^{m} f_i(t_i)\right).$$

By (2.8) we get for $\varphi = \psi \circ \ln$ that

$$\begin{aligned}
&\int_{I_1} ... \int_{I_m} \psi\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} \psi\left(\ln\left(\prod_{i=1}^{m} f_i(t_i)\right)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} (\psi \circ \ln)\left(\prod_{i=1}^{m} f_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= (\psi \circ \ln)\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \psi\left(\ln\left(\bigotimes_{i=1}^{m} f_i(A_i)\right)\right).
\end{aligned}$$

By (2.10) we also have

$$\ln\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \sum_{i=1}^{n} \ln\left(\widehat{\mathbf{f}_i(\mathbf{A}_i)}\right) = \sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}$$

and the identity (2.15) is obtained. □

**Corollary 4.** *Assume that $A_i$, $i \in \{1, ..., m\}$ and $g_i$, $i \in \{1, ..., m\}$ are as in Theorem 2 and $r > 0$, then*

$$\left|\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right|^r = \int_{I_1} ... \int_{I_m} \left|\sum_{i=1}^{m} g_i(t_i)\right|^r dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.16}$$

*Also, if we take $\psi = \exp$, then we get*

$$\begin{aligned}
\exp\left(\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right) &= \int_{I_1} ... \int_{I_m} \exp\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \bigotimes_{i=1}^{m} \exp\left[g_i(A_i)\right].
\end{aligned} \tag{2.17}$$

The case of convex combination is as follows:

**Corollary 5.** *Assume $A_i$, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I$, $i \in \{1, ..., m\}$ and with the spectral resolutions*

$$A_i = \int_I t_i dE(t_i), \ i \in \{1, ..., m\}.$$

*If* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$, *and* $\psi$ *continuous on* $I$, *then*

$$\psi\left(\sum_{i=1}^{m} p_i \widehat{\mathbf{A}_i}\right) = \int_I \dots \int_I \psi\left(\sum_{i=1}^{n} p_i t_i\right) dE(t_1) \otimes \dots \otimes dE(t_m). \tag{2.18}$$

Follows by (2.4) for $g_i(t_i) = p_i t_i$, $i \in \{1, ..., m\}$.

## 3. Main Results

We have the following refinements and reverses for the tensorial arithmetic mean-geometric mean inequality:

**Theorem 3.** *Assume that* $A_i \geq 0$, $i \in \{1, ..., m\}$ *and* $q_i \geq 0$, $i \in \{1, ..., m\}$ *with* $\sum_{i=1}^{m} q_i = 1$, *then*

$$\begin{aligned} 0 &\leq \frac{1}{(m-1)} \min_{i \in \{1,\dots,m\}} \{q_i\} \left[ m \sum_{i=1}^{m} \widehat{\mathbf{A}_i} - \left( \sum_{i=1}^{m} \widehat{\mathbf{A}_i^{1/2}} \right)^2 \right] \\ &\leq \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i} - \bigotimes_{i=1}^{m} A_i^{q_i} \\ &\leq \max_{i \in \{1,\dots,m\}} \{q_i\} \left[ m \sum_{i=1}^{m} \widehat{\mathbf{A}_i} - \left( \sum_{i=1}^{m} \widehat{\mathbf{A}_i^{1/2}} \right)^2 \right]. \end{aligned} \tag{3.1}$$

*Proof.* We recall the following classical result due to H. Kober, see [11]:

$$\begin{aligned} 0 &\leq \frac{1}{2(m-1)} \min_{i \in \{1,\dots,m\}} \{q_i\} \sum_{i,j=1}^{m} \left(\sqrt{x_i} - \sqrt{x_j}\right)^2 \\ &\leq \sum_{i=1}^{m} q_i x_i - \prod_{i=1}^{m} x_i^{q_i} \leq \frac{1}{2} \max_{i \in \{1,\dots,m\}} \{q_i\} \sum_{i,j=1}^{m} \left(\sqrt{x_i} - \sqrt{x_j}\right)^2, \end{aligned} \tag{3.2}$$

where $x_i$, $q_i \geq 0$, $i \in \{1, ..., m\}$ with $\sum_{i=1}^{m} q_i = 1$.

Assume $A_i$, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I \subset [0, \infty)$, $i \in \{1, ..., m\}$ and with the spectral resolutions

$$A_i = \int_I t_i dE(t_i), \ i \in \{1, ..., m\}.$$

From (3.2) have

$$\begin{aligned} 0 &\leq \frac{1}{2(m-1)} \min_{i \in \{1,\dots,m\}} \{q_i\} \sum_{i,j=1}^{m} \left(\sqrt{t_i} - \sqrt{t_j}\right)^2 \\ &\leq \sum_{i=1}^{m} q_i t_i - \prod_{i=1}^{m} t_i^{q_i} \leq \frac{1}{2} \max_{i \in \{1,\dots,m\}} \{q_i\} \sum_{i,j=1}^{m} \left(\sqrt{t_i} - \sqrt{t_j}\right)^2 \end{aligned} \tag{3.3}$$

for all $t_i \in I$, $i \in \{1, ..., m\}$.

If we take the integral $\int_I ... \int_I$ over $dE(t_1) \otimes ... \otimes dE(t_m)$ in (3.3), then we get

$$
\begin{aligned}
0 &\le \frac{1}{2(m-1)} \min_{i\in\{1,...,m\}} \{q_i\} \\
&\times \int_I ... \int_I \sum_{i,j=1}^{m} (\sqrt{t_i} - \sqrt{t_j})^2 dE(t_1) \otimes ... \otimes dE(t_m) \\
&\le \sum_{i=1}^{m} q_i \int_I ... \int_I t_i dE(t_1) \otimes ... \otimes dE(t_m) \\
&- \int_I ... \int_I \prod_{i=1}^{m} t_i^{q_i} dE(t_1) \otimes ... \otimes dE(t_m) \\
&\le \frac{1}{2} \max_{i\in\{1,...,m\}} \{q_i\} \\
&\times \int_I ... \int_I \sum_{i,j=1}^{m} (\sqrt{t_i} - \sqrt{t_j})^2 dE(t_1) \otimes ... \otimes dE(t_m).
\end{aligned}
\tag{3.4}
$$

Observe that

$$
\begin{aligned}
\sum_{i,j=1}^{m} (\sqrt{t_i} - \sqrt{t_j})^2 &= \sum_{i,j=1}^{m} \left( t_i - 2t_i^{1/2} t_j^{1/2} + t_j \right) \\
&= 2\left[ m \sum_{i=1}^{m} t_i - \left( \sum_{i=1}^{m} t_i^{1/2} \right)^2 \right],
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_I ... \int_I \sum_{i,j=1}^{m} (\sqrt{t_i} - \sqrt{t_j})^2 dE(t_1) \otimes ... \otimes dE(t_m) \\
&= 2m \int_I ... \int_I \sum_{i=1}^{m} t_i dE(t_1) \otimes ... \otimes dE(t_m) \\
&- 2 \int_I ... \int_I \left( \sum_{i=1}^{m} t_i^{1/2} \right)^2 dE(t_1) \otimes ... \otimes dE(t_m) \\
&= 2\left[ m \sum_{i=1}^{m} \widehat{\mathbf{A}_i} - \left( \sum_{i=1}^{m} \widehat{\mathbf{A}_i^{1/2}} \right)^2 \right],
\end{aligned}
$$

where for the last term we employed equality (2.15) for $\psi(u) = u^2$ and $g_i(t_i) = t_i^{1/2}$, $i \in \{1, ..., m\}$.

Also

$$
\sum_{i=1}^{m} q_i \int_I ... \int_I t_i dE(t_1) \otimes ... \otimes dE(t_m) = \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i}
$$

and

$$
\int_I ... \int_I \prod_{i=1}^{m} t_i^{q_i} dE(t_1) \otimes ... \otimes dE(t_m) = \bigotimes_{i=1}^{m} A_i^{q_i}
$$

and by (3.4) we derive (3.1). □

**Corollary 6.** *Assume that* $A,\ B \geq 0$ *and* $\nu \in [0,1]$, *then*

$$\begin{aligned} 0 &\leq \min\{\nu, 1-\nu\}\left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2}\right) \\ &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\leq \max\{\nu, 1-\nu\}\left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2}\right). \end{aligned} \tag{3.5}$$

It follows by (3.1) for $m = 2$, $q_1 = \nu$, $q_2 = 1 - \nu$, $A_1 = A$ and $A_2 = B$.

**Corollary 7.** *Assume that* $A,\ B \geq 0$ *and* $\nu \in [0,1]$, *then we have the following inequalities for the Hadamard product*

$$\begin{aligned} 0 &\leq 2\min\{\nu, 1-\nu\}\left(\frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2}\right) \\ &\leq (1-\nu) A \circ 1 + \nu 1 \circ B - A^{1-\nu} \circ B^{\nu} \\ &\leq 2\max\{\nu, 1-\nu\}\left(\frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2}\right). \end{aligned} \tag{3.6}$$

*Proof.* We have the representation

$$X \circ Y = \mathcal{U}^* (X \otimes Y) \mathcal{U}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take at the left of (3.5) $\mathcal{U}^*$ and at the right $\mathcal{U}$, then we get

$$\begin{aligned} 0 &\leq \min\{\nu, 1-\nu\}\mathcal{U}^*\left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2}\right)\mathcal{U} \\ &\leq \mathcal{U}^*\left[(1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}\right]\mathcal{U} \\ &\leq \max\{\nu, 1-\nu\}\mathcal{U}^*\left(A \otimes 1 + 1 \otimes B - 2A^{1/2} \otimes B^{1/2}\right)\mathcal{U}, \end{aligned} \tag{3.7}$$

which is equivalent to (3.6). □

We also have:

**Theorem 4.** *Assume that* $A_i \geq 0,\ i \in \{1, ..., m\}$ *and* $q_i \geq 0,\ i \in \{1, ..., m\}$ *with* $\sum_{i=1}^{m} q_i = 1$, *then*

$$\begin{aligned} 0 &\leq \frac{1}{1 - \min_{i \in \{1,\ldots,m\}}\{q_i\}}\left[\sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i - \left(\sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i^{1/2}}\right)^2\right] \\ &\leq \sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i - \bigotimes_{i=1}^{m} A_i^{q_i} \\ &\leq \frac{1}{\min_{i \in \{1,\ldots,m\}}\{q_i\}}\left[\sum_{i=1}^{m} q_i \widehat{\mathbf{A}}_i - \left(\sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i^{1/2}}\right)^2\right]. \end{aligned} \tag{3.8}$$

*Proof.* We recall the following classical result due to P. H. Diananda, see [5]:

$$0 \le \frac{1}{2\left(1-\min_{i\in\{1,...,m\}}\{q_i\}\right)}\sum_{i,j=1}^{m} q_i q_j (\sqrt{x_i}-\sqrt{x_j})^2 \le \sum_{i=1}^{m} q_i x_i - \prod_{i=1}^{m} x_i^{q_i} \le \frac{1}{2\min_{i\in\{1,...,m\}}\{q_i\}}\sum_{i,j=1}^{m} q_i q_j (\sqrt{x_i}-\sqrt{x_j})^2, \tag{3.9}$$

where $x_i$, $q_i \ge 0$, $i \in \{1,...,m\}$ with $\sum_{i=1}^{m} q_i = 1$.

From (3.9) we get

$$0 \le \frac{1}{2\left(1-\min_{i\in\{1,...,m\}}\{q_i\}\right)}\sum_{i,j=1}^{m} q_i q_j (\sqrt{t_i}-\sqrt{t_j})^2 \le \sum_{i=1}^{m} q_i t_i - \prod_{i=1}^{m} t_i^{q_i} \le \frac{1}{2\min_{i\in\{1,...,m\}}\{q_i\}}\sum_{i,j=1}^{m} q_i q_j (\sqrt{t_i}-\sqrt{t_j})^2, \tag{3.10}$$

for all $t_i \in I$, $i \in \{1,...,m\}$.

If we take the integral $\int_I ... \int_I$ over $dE(t_1) \otimes ... \otimes dE(t_m)$ in (3.10), then we get

$$\begin{aligned} 0 &\le \frac{1}{2\left(1-\min_{i\in\{1,...,m\}}\{q_i\}\right)} \\ &\times \int_I ... \int_I \sum_{i,j=1}^{m} q_i q_j (\sqrt{t_i}-\sqrt{t_j})^2 dE(t_1) \otimes ... \otimes dE(t_m) \\ &\le \sum_{i=1}^{m} q_i \int_I ... \int_I \sum_{i,j=1}^{m} t_i dE(t_1) \otimes ... \otimes dE(t_m) \\ &- \int_I ... \int_I \prod_{i=1}^{m} t_i^{q_i} dE(t_1) \otimes ... \otimes dE(t_m) \\ &\le \frac{1}{2\min_{i\in\{1,...,m\}}\{q_i\}} \\ &\times \int_I ... \int_I \sum_{i,j=1}^{m} q_i q_j (\sqrt{t_i}-\sqrt{t_j})^2 dE(t_1) \otimes ... \otimes dE(t_m). \end{aligned} \tag{3.11}$$

Observe that

$$\sum_{i,j=1}^{m} q_i q_j (\sqrt{t_i} - \sqrt{t_j})^2 = \sum_{i,j=1}^{m} q_i q_j \left( t_i - 2t_i^{1/2} t_j^{1/2} + t_j \right)$$
$$= \sum_{i,j=1}^{m} q_i q_j t_i - 2\sum_{i,j=1}^{m} q_i q_j t_i^{1/2} t_j^{1/2} + \sum_{i,j=1}^{m} q_i q_j t_j$$
$$= 2\left[ \sum_{i=1}^{n} q_i t_i - \left( \sum_{i=1}^{n} q_i t_i^{1/2} \right)^2 \right],$$

which gives that

$$\int_I \dots \int_I \sum_{i,j=1}^{m} q_i q_j (\sqrt{t_i} - \sqrt{t_j})^2 dE(t_1) \otimes \dots \otimes dE(t_m)$$
$$= 2\sum_{i=1}^{n} q_i \int_I \dots \int_I t_i dE(t_1) \otimes \dots \otimes dE(t_m)$$
$$- 2\int_I \dots \int_I \left( \sum_{i=1}^{n} q_i t_i^{1/2} \right)^2 dE(t_1) \otimes \dots \otimes dE(t_m)$$
$$= 2\left[ \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i} - \left( \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i^{1/2}} \right)^2 \right],$$

where for the last term we employed equality (2.15) for $\psi(u) = u^2$ and $g_i(t_i) = q_i t_i^{1/2}$, $i \in \{1, ..., m\}$.

By employing the inequality (3.11) we deduce the desired result (3.8). □

For $m = 2$, $q_1 = \nu$ and $q_2 = 1 - \nu$ observe that

$$\frac{1}{1 - \min\{\nu, 1-\nu\}} = \frac{1}{\max\{\nu, 1-\nu\}} = \min\{\nu, 1-\nu\},$$

which shows that the case $m = 2$ in Theorem 4 gives the same particular case as of Theorem 2.

**Theorem 5.** *Assume $A_i$, $i \in \{1, ..., m\}$ are selfadjoint operators with* $\operatorname{Sp}(A_i) \subset [k, K] \subset (0, \infty)$, $i \in \{1, ..., m\}$ *and* $q_i \geq 0$, $i \in \{1, ..., m\}$ *with* $\sum_{i=1}^{m} q_i = 1$, *then*

$$0 \leq \frac{1}{2K}\left[ \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i^2} - \left( \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i} \right)^2 \right] \tag{3.12}$$
$$\leq \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i} - \bigotimes_{i=1}^{m} A_i^{q_i} \leq \frac{1}{2k}\left[ \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i^2} - \left( \sum_{i=1}^{m} q_i \widehat{\mathbf{A}_i} \right)^2 \right].$$

*Proof.* In 1978, Cartwright and Field [4] obtained the following refinement and reverse for the difference between the arithmetic mean and geometric mean

$$0 \leq \frac{1}{2b}\sum_{i=1}^{m} q_i (x_i - \bar{x}_q)^2 \leq \bar{x}_q - \prod_{i=1}^{m} x_i^{q_i} \leq \frac{1}{2a}\sum_{i=1}^{m} q_i (x_i - \bar{x})^2, \tag{3.13}$$

where $x_i \in [a,b] \subset (0,\infty)$, $q_i \geq 0$, $i \in \{1,...,m\}$ with $\sum_{i=1}^m q_i = 1$ and $\bar{x}_q := \sum_{i=1}^m q_i x_i$.

Observe that

$$\begin{aligned}\sum_{i=1}^m q_i (x_i - \bar{x}_q)^2 &= \sum_{i=1}^m q_i x_i^2 - 2\bar{x}_q \sum_{i=1}^m q_i x_i + (\bar{x}_q)^2 \\ &= \sum_{i=1}^m q_i x_i^2 - \left(\sum_{i=1}^m q_i x_i\right)^2 .\end{aligned}$$

Now, if $t_i \in [k,K] \subset (0,\infty)$, $i \in \{1,...,m\}$ then by (3.13) we get

$$\begin{aligned}&0 \leq \frac{1}{2K}\left[\sum_{i=1}^m q_i t_i^2 - \left(\sum_{i=1}^m q_i t_i\right)^2\right] \\ &\leq \sum_{i=1}^m q_i t_i - \prod_{i=1}^m t_i^{q_i} \leq \frac{1}{2k}\left[\sum_{i=1}^m q_i t_i^2 - \left(\sum_{i=1}^m q_i t_i\right)^2\right],\end{aligned} \tag{3.14}$$

Assume $A_i$, $i \in \{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset [k,K] \subset (0,\infty)$, $i \in \{1,...,m\}$ and with the spectral resolutions

$$A_i = \int_k^K t_i dE(t_i), \ i \in \{1,...,m\}.$$

If we take the integral $\int_k^K ... \int_k^K$ over $dE(t_1) \otimes ... \otimes dE(t_m)$ in (3.10), then we get

$$\begin{aligned}&0 \leq \frac{1}{2M}\int_k^K ... \int_k^K \left[\sum_{i=1}^m q_i t_i^2 - \left(\sum_{i=1}^m q_i t_i\right)^2\right] dE(t_1) \otimes ... \otimes dE(t_m) \\ &\leq \sum_{i=1}^m q_i \int_k^K ... \int_k^K t_i dE(t_1) \otimes ... \otimes dE(t_m) \\ &- \int_k^K ... \int_k^K \prod_{i=1}^m t_i^{q_i} dE(t_1) \otimes ... \otimes dE(t_m) \\ &\leq \frac{1}{2k}\int_k^K ... \int_k^K \left[\sum_{i=1}^m q_i t_i^2 - \left(\sum_{i=1}^m q_i t_i\right)^2\right] dE(t_1) \otimes ... \otimes dE(t_m).\end{aligned} \tag{3.15}$$

Observe that

$$\begin{aligned}
&\int_k^K \cdots \int_k^K \left[\sum_{i=1}^m q_i t_i^2 - \left(\sum_{i=1}^m q_i t_i\right)^2\right] dE(t_1) \otimes \ldots \otimes dE(t_m) \\
&= \sum_{i=1}^m q_i \int_k^K \cdots \int_k^K t_i^2 dE(t_1) \otimes \ldots \otimes dE(t_m) \\
&- \int_k^K \cdots \int_k^K \left(\sum_{i=1}^m q_i t_i\right)^2 dE(t_1) \otimes \ldots \otimes dE(t_m) \\
&= \sum_{i=1}^m q_i \widehat{\mathbf{A}}_i^2 - \left(\sum_{i=1}^m q_i \widehat{\mathbf{A}}_i\right)^2
\end{aligned}$$

and by (3.15) we obtain (3.12). □

**Corollary 8.** *Assume that* $m \leq A, B \leq M$ *for some constants* $m, M$ *and* $\nu \in [0,1]$, *then*

$$\begin{aligned}
0 &\leq \frac{1}{2M}\left(A^2 \otimes 1 + 1 \otimes B^2 - 2A \otimes B\right) \\
&\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\
&\leq \frac{1}{2m}\left(A^2 \otimes 1 + 1 \otimes B^2 - 2A \otimes B\right)
\end{aligned} \tag{3.16}$$

*and*

$$\begin{aligned}
0 &\leq \frac{1}{M}\left(\frac{A^2+B^2}{2} \circ 1 - A \circ B\right) \\
&\leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \\
&\leq \frac{1}{m}\left(\frac{A^2+B^2}{2} \circ 1 - A \circ B\right).
\end{aligned} \tag{3.17}$$

## 4. Integral Inequalities

Let $\Omega$ be a locally compact Hausdorff space endowed with a Radon measure $\mu$. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on $\Omega$, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t) d\mu(t)$ for every bounded linear functional $\varphi$ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

**Proposition 1.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be continuous fields of positive operators in $B(H)$, then for $\nu\in[0,1]$,*

$$
\begin{aligned}
(4.1)\qquad & 0\le \min\{\nu,1-\nu\}\left[\left(\int_\Omega A_t d\mu(t)\right)\otimes 1+1\otimes\left(\int_\Omega B_t d\mu(t)\right)\right.\\
&\left.-2\left(\int_\Omega A_t^{1/2}d\mu(t)\right)\otimes\left(\int_\Omega B_t^{1/2}d\mu(t)\right)\right]\\
&\le (1-\nu)\left(\int_\Omega A_t d\mu(t)\right)\otimes 1+\nu 1\otimes\int_\Omega B_t d\mu(t)\\
&-\left(\int_\Omega A_t^{1-\nu}d\mu(t)\right)\otimes\left(\int_\Omega B_t^{\nu}d\mu(t)\right)\\
&\le \max\{\nu,1-\nu\}\left[\left(\int_\Omega A_t d\mu(t)\right)\otimes 1+1\otimes\left(\int_\Omega B_t d\mu(t)\right)\right.\\
&\left.-2\left(\int_\Omega A_t^{1/2}d\mu(t)\right)\otimes\left(\int_\Omega B_t^{1/2}d\mu(t)\right)\right].
\end{aligned}
$$

*Proof.* From (3.5) we have

$$
\begin{aligned}
(4.2)\qquad & 0\le \min\{\nu,1-\nu\}\left(A_t\otimes 1+1\otimes B_s-2A_t^{1/2}\otimes B_s^{1/2}\right)\\
&\le (1-\nu)A_t\otimes 1+\nu 1\otimes B_s-A_t^{1-\nu}\otimes B_s^{\nu}\\
&\le \max\{\nu,1-\nu\}\left(A_t\otimes 1+1\otimes B_s-2A_t^{1/2}\otimes B_s^{1/2}\right)
\end{aligned}
$$

for all $t$, $s\in\Omega$.

Fix $s\in\Omega$. If we take the $\int_\Omega$ over $d\mu(t)$ in (4.2), then we get

$$
\begin{aligned}
(4.3)\qquad & 0\le \min\{\nu,1-\nu\}\int_\Omega\left(A_t\otimes 1+1\otimes B_s-2A_t^{1/2}\otimes B_s^{1/2}\right)d\mu(t)\\
&\le \int_\Omega\left[(1-\nu)A_t\otimes 1+\nu 1\otimes B_s-A_t^{1-\nu}\otimes B_s^{\nu}\right]d\mu(t)\\
&\le \max\{\nu,1-\nu\}\int_\Omega\left(A_t\otimes 1+1\otimes B_s-2A_t^{1/2}\otimes B_s^{1/2}\right)d\mu(t)
\end{aligned}
$$

for all $s\in\Omega$.

Since, by the properties of tensorial product and Bochner's integral

$$
\begin{aligned}
&\int_\Omega\left(A_t\otimes 1+1\otimes B_s-2A_t^{1/2}\otimes B_s^{1/2}\right)d\mu(t)\\
&=\left(\int_\Omega A_t d\mu(t)\right)\otimes 1+1\otimes B_s-2\left(\int_\Omega A_t^{1/2}d\mu(t)\right)\otimes B_s^{1/2}d\mu(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega\left[(1-\nu)A_t\otimes 1+\nu 1\otimes B_s-A_t^{1-\nu}\otimes B_s^{\nu}\right]d\mu(t)\\
&=(1-\nu)\left(\int_\Omega A_t d\mu(t)\right)\otimes 1+\nu 1\otimes B_s\\
&-\left(\int_\Omega A_t^{1-\nu}d\mu(t)\right)\otimes B_s^{\nu}d\mu(t).
\end{aligned}
$$

By (4.3) we get

$$
\begin{aligned}
(4.4)\qquad & 0 \leq \min\{\nu, 1-\nu\} \\
& \times \left[\left(\int_\Omega A_t d\mu(t)\right) \otimes 1 + 1 \otimes B_s - 2\left(\int_\Omega A_t^{1/2} d\mu(t)\right) \otimes B_s^{1/2} d\mu(t)\right] \\
& \leq (1-\nu)\left(\int_\Omega A_t d\mu(t)\right) \otimes 1 + \nu 1 \otimes B_s \\
& - \left(\int_\Omega A_t^{1-\nu} d\mu(t)\right) \otimes B_s^{\nu} d\mu(t) \\
& \leq \max\{\nu, 1-\nu\} \\
& \times \left[\left(\int_\Omega A_t d\mu(t)\right) \otimes 1 + 1 \otimes B_s - 2\left(\int_\Omega A_t^{1/2} d\mu(t)\right) \otimes B_s^{1/2} d\mu(t)\right]
\end{aligned}
$$

for all $s \in \Omega$.

Further, by taking the integral $\int_\Omega$ over $d\mu(s)$, and conducting a similar argument, we derive the desired result (4.1). □

**Corollary 9.** *With the assumptions of Proposition 1, we have the following inequalities for the Hadamard product*

$$
\begin{aligned}
(4.5)\qquad & 0 \leq 2\min\{\nu, 1-\nu\}\left[\left(\int_\Omega \frac{A_t + B_t}{2} d\mu(t)\right) \circ 1\right. \\
& \left. - \left(\int_\Omega A_t^{1/2} d\mu(t)\right) \circ \left(\int_\Omega B_t^{1/2} d\mu(t)\right)\right] \\
& \leq \left(\int_\Omega [(1-\nu) A_t + \nu B_t]\, d\mu(t)\right) \circ 1 \\
& - \left(\int_\Omega A_t^{1-\nu} d\mu(t)\right) \circ \left(\int_\Omega B_t^{\nu} d\mu(t)\right) \\
& \leq 2\max\{\nu, 1-\nu\}\left[\left(\int_\Omega \frac{A_t + B_t}{2} d\mu(t)\right) \circ 1\right. \\
& \left. - \left(\int_\Omega A_t^{1/2} d\mu(t)\right) \circ \left(\int_\Omega B_t^{1/2} d\mu(t)\right)\right].
\end{aligned}
$$

We observe that, if we take $B_t = A_t$, for $t \in \Omega$ in (4.5), then we get the simpler inequalities

$$
\begin{aligned}
(4.6)\qquad & 0 \leq 2\min\{\nu, 1-\nu\} \\
& \times \left[\left(\int_\Omega A_t d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t^{1/2} d\mu(t)\right) \circ \left(\int_\Omega A_t^{1/2} d\mu(t)\right)\right] \\
& \leq \left(\int_\Omega A_t d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t^{1-\nu} d\mu(t)\right) \circ \left(\int_\Omega A_t^{\nu} d\mu(t)\right) \\
& \leq 2\max\{\nu, 1-\nu\} \\
& \times \left[\left(\int_\Omega A_t d\mu(t)\right) \circ 1 - \left(\int_\Omega A_t^{1/2} d\mu(t)\right) \circ \left(\int_\Omega A_t^{1/2} d\mu(t)\right)\right].
\end{aligned}
$$

**Proposition 2.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be continuous fields of positive operators in $B(H)$ with $\operatorname{Sp}(A_t), \operatorname{Sp}(B_t) \subseteq [m,M] \subset (0,\infty)$ for each $t\in\Omega$, then for $\nu\in[0,1]$,*

$$
\begin{aligned}
(4.7)\qquad 0 &\le \frac{1}{2M}\left[\left(\int_\Omega A_t^2 d\mu(t)\right)\otimes 1 + 1\otimes\left(\int_\Omega B_t^2 d\mu(t)\right)^2\right.\\
&\quad \left.-2\left(\int_\Omega A_t d\mu(t)\right)\otimes\left(\int_\Omega B_t d\mu(t)\right)\right]\\
&\le (1-\nu)\left(\int_\Omega A_t d\mu(t)\right)\otimes 1 + \nu 1\otimes\int_\Omega B_t d\mu(t)\\
&\quad -\left(\int_\Omega A_t^{1-\nu} d\mu(t)\right)\otimes\left(\int_\Omega B_t^{\nu} d\mu(t)\right)\\
&\le \frac{1}{2m}\left[\left(\int_\Omega A_t^2 d\mu(t)\right)\otimes 1 + 1\otimes\left(\int_\Omega B_t^2 d\mu(t)\right)^2\right.\\
&\quad \left.-2\left(\int_\Omega A_t d\mu(t)\right)\otimes\left(\int_\Omega B_t d\mu(t)\right)\right].
\end{aligned}
$$

*We also have the Hadamard product inequalities*

$$
\begin{aligned}
(4.8)\qquad 0 &\le \frac{1}{2M}\left[\left(\int_\Omega \frac{A_t^2+B_t^2}{2} d\mu(t)\right)\circ 1 - \left(\int_\Omega A_t d\mu(t)\right)\circ\left(\int_\Omega B_t d\mu(t)\right)\right]\\
&\le \left(\int_\Omega [(1-\nu)A_t+\nu B_t]\, d\mu(t)\right)\circ 1\\
&\quad -\left(\int_\Omega A_t^{1-\nu} d\mu(t)\right)\circ\left(\int_\Omega B_t^{\nu} d\mu(t)\right)\\
&\le \frac{1}{2m}\left[\left(\int_\Omega \frac{A_t^2+B_t^2}{2} d\mu(t)\right)\circ 1 - \left(\int_\Omega A_t d\mu(t)\right)\circ\left(\int_\Omega B_t d\mu(t)\right)\right].
\end{aligned}
$$

In particular, we also have

$$
\begin{aligned}
(4.9)\qquad 0 &\le \frac{1}{2M}\left[\left(\int_\Omega A_t^2 d\mu(t)\right)\circ 1 - \left(\int_\Omega A_t d\mu(t)\right)\circ\left(\int_\Omega A_t d\mu(t)\right)\right]\\
&\le \left(\int_\Omega A_t d\mu(t)\right)\circ 1 - \left(\int_\Omega A_t^{1-\nu} d\mu(t)\right)\circ\left(\int_\Omega A_t^{\nu} d\mu(t)\right)\\
&\le \frac{1}{2m}\left[\left(\int_\Omega A_t^2 d\mu(t)\right)\circ 1 - \left(\int_\Omega A_t d\mu(t)\right)\circ\left(\int_\Omega A_t d\mu(t)\right)\right].
\end{aligned}
$$

## References


[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] D. I. Cartwright, M. J. Field, A refinement of the arithmetic mean-geometric mean inequality, *Proc. Amer. Math. Soc.*, **71** (1978), 36-38.

[5] P. H. Diananda, On some inequalities of H. Kober, *Proc. Camb. Phil. Soc.* **59** (1963), , 341-346.

[6] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[7] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[8] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[11] H. Kober, On the arithmetic and geometric means and on Hölder's inequality. *Proc. American Math. Soc.* **9** (1958), 452-459.

[12] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

*E-mail address*: sever.dragomir@vu.edu.au

*URL*: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa