A REVERSE OF JENSEN TENSORIAL INEQUALITY FOR SEQUENCES OF SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space. In this paper we show among others that, if ψ is differentiable convex on the open interval I, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators with $\mathrm{Sp}(A_i) \subseteq [m, M] \subset I$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \quad \sum_{i=1}^{n} p_i \widehat{\mathbf{A}_i} \Bigg) \\ &\leq \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i}\right)\left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right) - \quad \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \Bigg) \left(\sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right) \\ &\leq \frac{1}{2}\left[\psi'(M) - \psi'(m)\right] \sum_{i=1}^{n} w_i \left|\widehat{\mathbf{A}_i} - \sum_{j=1}^{n} w_j \widehat{\mathbf{A}_j}\right| \\ &\leq \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M - m), \end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of A_i in position $i = 1, ..., n$ and with 1 in the other positions. Let $\left(B_t^{(i)}\right)_{t \in \Omega}$, $i = 1, ..., n$ be continuous fields of positive operators in $B(H)$ with $\mathrm{Sp}\left(B_t^{(i)}\right) \subseteq [k, K] \subset (0, \infty)$ for $t \in \Omega$, $i = 1, ..., n$, and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then

$$\sum_{i=1}^{n} w_i \left(\widehat{\int_{\Omega} \mathbf{B}_{t_i}^{(i)} d\mu(t_i)}\right) - \bigotimes_{i=1}^{n} \left(\int_{\Omega} \left[B_{t_i}^{(i)}\right]^{w_i} d\mu(t_i)\right) \leq \frac{1}{4}(K - k)(\ln K - \ln k)$$

and

$$\sum_{i=1}^{n} w_i \left(\widehat{\int_{\Omega} \mathbf{B}_{t_i}^{(i)} d\mu(t_i)}\right) \leq \exp\left[\frac{1}{4kK}(K - k)^2\right] \bigotimes_{i=1}^{n} \left(\int_{\Omega} \left[B_{t_i}^{(i)}\right]^{w_i} d\mu(t_i)\right).$$

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.

RGMIA Res. Rep. Coll. **25** (2022), Art. 94, 19 pp. Received 21/09/22

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k), \tag{1.2}$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [9, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [12] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} [(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.5}$$

for $A, B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [8], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.6}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [9, p. 173]

$$f (A \circ B) \geq (\leq) f (A) \circ f (B) \text{ for all } A,\ B \geq 0. \tag{1.7}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A + B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [10] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if ψ is differentiable convex on the open interval I, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subseteq [m, M] \subset I$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$, then

$$\begin{aligned}
0 &\leq \sum_{i=1}^n w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi \left(\sum_{i=1}^n p_i \widehat{\mathbf{A}}_i \right) \\
&\leq \sum_{i=1}^n w_i \left(\widehat{\mathbf{A}}_i \right) \left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)} \right) - \left(\sum_{i=1}^n w_i \widehat{\mathbf{A}}_i \right) \left(\sum_{i=1}^n w_i \widehat{\boldsymbol{\psi}'(\mathbf{A}_i)} \right) \\
&\leq \frac{1}{2} \left[\psi'(M) - \psi'(m) \right] \sum_{i=1}^n w_i \left| \widehat{\mathbf{A}}_i - \sum_{j=1}^n w_j \widehat{\mathbf{A}}_j \right| \\
&\leq \frac{1}{4} \left[\psi'(M) - \psi'(m) \right] (M - m),
\end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of A_i in position $i = 1, ..., n$ and with 1 in the other positions. Let $\left(B_t^{(i)}\right)_{t \in \Omega}$, $i = 1, ..., n$ be continuous fields of positive operators in $B(H)$ with $\operatorname{Sp}\left(B_t^{(i)}\right) \subseteq [k, K] \subset (0, \infty)$ for $t \in \Omega$, $i = 1, ..., n$, and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$, then

$$\sum_{i=1}^n w_i \left(\widehat{\int_\Omega \mathbf{B}_{t_i}^{(i)} d\mu (t_i)} \right) - \bigotimes_{i=1}^n \left(\int_\Omega \left[B_{t_i}^{(i)} \right]^{w_i} d\mu (t_i) \right) \leq \frac{1}{4} (K - k) (\ln K - \ln k)$$

and

$$\sum_{i=1}^n w_i \left(\widehat{\int_\Omega \mathbf{B}_{t_i}^{(i)} d\mu (t_i)} \right) \leq \exp \left[\frac{1}{4kK} (K - k)^2 \right] \bigotimes_{i=1}^n \left(\int_\Omega \left[B_{t_i}^{(i)} \right]^{w_i} d\mu (t_i) \right).$$

2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A,\ B,\ C,\ D\in B(H)$.

If we take $C=A$ and $D=B$, then we get

$$A^2\otimes B^2=(A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n=(A\otimes B)^n \text{ for natural } n\geq 0. \tag{2.2}$$

In particular

$$A^n\otimes 1=(A\otimes 1)^n \text{ and } 1\otimes B^n=(1\otimes B)^n \tag{2.3}$$

for all $n\geq 0$.

We also observe that, by (2.1), the operators $A\otimes 1$ and $1\otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B)=(1\otimes B)(A\otimes 1)=A\otimes B. \tag{2.4}$$

Moreover, for two natural numbers $m,\ n$ we have

$$(A\otimes 1)^m(1\otimes B)^n=(1\otimes B)^n(A\otimes 1)^m=A^m\otimes B^n. \tag{2.5}$$

By induction over m, we derive

$$(A_1\otimes A_2\otimes...\otimes A_m)^n=A_1^n\otimes A_2^n\otimes...\otimes A_m^n \text{ for natural } n\geq 0 \tag{2.6}$$

and

$$\begin{aligned} &A_1\otimes A_2\otimes...\otimes A_m \\ &=(A_1\otimes 1\otimes...\otimes 1)(1\otimes A_2\otimes...\otimes 1)...(1\otimes 1\otimes...\otimes A_m) \end{aligned} \tag{2.7}$$

and the m operators $(A_1\otimes 1\otimes...\otimes 1),\ (1\otimes A_2\otimes...\otimes 1),\ ...$ and $(1\otimes 1\otimes...\otimes A_m)$ are commutative between them.

We define for $A_i,\ B_i\in B(H),\ i\in\{1,...,n\},\ \bigotimes_{i=1}^n B_i:=B_1\otimes...\otimes B_n$,

$$\hat{\mathbf{A}}_i:=1\otimes...\otimes A_i\otimes...\otimes 1,\ i=2,...,n-1,$$

and

$$\hat{\mathbf{A}}_1:=A_1\otimes 1\otimes...\otimes 1 \text{ while } \hat{\mathbf{A}}_n:=1\otimes...\otimes 1\otimes A_n.$$

Basically $\hat{\mathbf{A}}_i$ is defined as a tensorial product of A_i in position $i=1,...,n$ and with 1 in the other positions.

We need the following identity for the tensorial product, see also [6]:

Lemma 1. *Assume $A_i,\ i\in\{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i)\subset I_i$, $i\in\{1,...,m\}$ and with the spectral resolutions*

$$A_i=\int_{I_i}t_i dE(t_i),\ i\in\{1,...,m\}.$$

Let $f_i,\ i\in\{1,...,m\}$ be continuous on I_i and φ continuous on an interval K that contains the product of the intervals $f(I_1)...f(I_m)$, then

$$\varphi\left(\bigotimes_{i=1}^m f_i(A_i)\right)=\int_{I_1}...\int_{I_m}\varphi\left(\prod_{i=1}^m f_i(t_i)\right)dE(t_1)\otimes...\otimes dE(t_m). \tag{2.8}$$

We also have

$$\varphi\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right)=\boldsymbol{\varphi}\,\widehat{\left(\mathbf{f}_i\left(\mathbf{A}_i\right)\right)} \tag{2.9}$$

for all $i=1,...,m.$

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t)=t^n$ with n any natural number.

Then, by (1.1) and (2.6) we obtain

$$\begin{aligned}&\int_{I_1}...\int_{I_m}\left[f_1(t_1)...f_m(t_m)\right]^n dE(t_1)\otimes...\otimes dE(t_m)\\&=\int_{I_1}...\int_{I_m}\left[f_1(t_1)\right]^n...\left[f_m(t_m)\right]^n dE(t_1)\otimes...\otimes dE(t_m)\\&=\left[f(A_1)\right]^n\otimes...\otimes\left[f_m(A_m)\right]^n=\left[f_1(A_1)\otimes...\otimes f_m(A_m)\right]^n,\end{aligned}$$

which shows that the identity (2.6) is valid for the power function.

This proves the identity (2.8)

By taking $f_j\equiv 1$ for $j=1,...,m$ and $j\neq i$ in (2.8) we get

$$\begin{aligned}\varphi(1\otimes...\otimes f_i(A_i)\otimes...\otimes 1)&=\int_{I_1}...\int_{I_m}\varphi(f_i(t_i))\,dE(t_1)\otimes...\otimes dE(t_m)\\&=1\otimes...\otimes\varphi(f_i(A_i))\otimes...\otimes 1,\end{aligned}$$

which proves (2.9). □

Corollary 1. *Assume* A_i, $i\in\{1,...,m\}$ *are selfadjoint operators with* $\operatorname{Sp}(A_i)\subset I_i$ *and* f_i, $i\in\{1,...,m\}$ *are continuous and positive on* I_i, *then*

$$\ln\left(\bigotimes_{i=1}^{m}f_i(A_i)\right)=\sum_{i=1}^{n}\ln\widehat{\left(\mathbf{f}_i\left(\mathbf{A}_i\right)\right)}. \tag{2.10}$$

Also

$$\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right)=\ln\widehat{\left(\mathbf{f}_i\left(\mathbf{A}_i\right)\right)} \tag{2.11}$$

for all $i=1,...,m.$

Proof. Assume that

$$A_i=\int_I t_i\,dE(t_i)$$

are the spectral resolutions for A_i, $i=1,...,m.$

We have for $\varphi(u) = \ln u$, $u > 0$, in (2.8) that

$$\begin{aligned}
&\ln(f_1(A) \otimes ... \otimes f_m(A_m)) \\
&= \int_{I_1} ... \int_{I_m} \ln(f_1(t_1) ... f_m(t_m)) \, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} [\ln f_1(t_1) + ... + \ln f_m(t_m)] \, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} \ln f_1(t_1) \, dE(t_1) \otimes ... \otimes dE(t_m) + ... \\
&+ \int_{I_1} ... \int_{I_m} \ln f_m(t_m) \, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= (\ln f_1(A_1)) \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes (\ln f_m(A_m))
\end{aligned}$$

and the identity (2.10) is proved. □

Corollary 2. *Assume A_i, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$ and f_i, $i \in \{1, ..., m\}$ are continuous on I_i, then for $r > 0$*

$$\left| \bigotimes_{i=1}^{m} f_i(A_i) \right|^r = \bigotimes_{i=1}^{m} |f_i(A_i)|^r \tag{2.12}$$

and

$$\left| \widehat{\mathbf{f}_i(\mathbf{A}_i)} \right|^r = |\widehat{\mathbf{f}_i(\mathbf{A}_i)}|^r \tag{2.13}$$

for all $i = 1, ..., m$.

Proof. From (2.8) we have for the function $\varphi(t) = |t|^r$ that

$$\begin{aligned}
&|f_1(A_1) \otimes ... \otimes f_m(A_m)|^r \\
&= \int_{I_1} ... \int_{I_m} |f_1(t_1) ... f_m(t_m)|^r \, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} |f_1(t_1)|^r ... |f_m(t_m)|^r \, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= |f_1(A_1)|^r \otimes ... \otimes |f_m(A_m)|^r,
\end{aligned}$$

which proves (2.12).

The identity (2.13) follows in a similar way. □

Corollary 3. *Assume A_i, $i \in \{1, ..., m\}$ are positive operators and $q_i \geq 0$, $i \in \{1, ..., m\}$, then*

$$\varphi\left(\bigotimes_{i=1}^{m} A_i^{q_i} \right) = \int_{I_1} ... \int_{I_m} \varphi\left(\prod_{i=1}^{m} t_i^{q_i} \right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.14}$$

We also have the additive result [6]:

Lemma 2. *Assume A_i, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$, $i \in \{1, ..., m\}$ and with the spectral resolutions*

$$A_i = \int_{I_i} t_i dE(t_i), \; i \in \{1, ..., m\}.$$

Let g_i, $i \in \{1, ..., m\}$ be continuous on I_i and ψ continuous on an interval K that contains the sum of the intervals $g(I_1) + ... + g(I_m)$, then

$$\psi\left(\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right) = \int_{I_1} ... \int_{I_m} \psi\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.15}$$

Proof. Let f_i, continuous, positive and such that $g_i(t_i) = \ln f_i(t_i)$, $t_i \in I_i$, $i \in \{1, ..., m\}$. Then

$$\sum_{i=1}^{m} g_i(t_i) = \sum_{i=1}^{m} \ln f_i(t_i) = \ln\left(\prod_{i=1}^{m} f_i(t_i)\right).$$

By (2.8) we get for $\varphi = \psi \circ \ln$ that

$$\begin{aligned}
&\int_{I_1} ... \int_{I_m} \psi\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} \psi\left(\ln\left(\prod_{i=1}^{m} f_i(t_i)\right)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} (\psi \circ \ln)\left(\prod_{i=1}^{m} f_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= (\psi \circ \ln)\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \psi\left(\ln\left(\bigotimes_{i=1}^{m} f_i(A_i)\right)\right).
\end{aligned}$$

By (2.10) we also have

$$\ln\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \sum_{i=1}^{n} \ln\widehat{(\mathbf{f}_i(\mathbf{A}_i))} = \sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}$$

and the identity (2.15) is obtained. □

Corollary 4. *Assume that A_i, $i \in \{1, ..., m\}$ and g_i, $i \in \{1, ..., m\}$ are as in Lemma 2 and $r > 0$, then*

$$\left|\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right|^r = \int_{I_1} ... \int_{I_m} \left|\sum_{i=1}^{m} g_i(t_i)\right|^r dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.16}$$

Also, if we take $\psi = \exp$, then we get

$$\begin{aligned}
\exp\left(\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right) &= \int_{I_1} ... \int_{I_m} \exp\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \bigotimes_{i=1}^{m} \exp[g_i(A_i)].
\end{aligned} \tag{2.17}$$

The case of convex combination is as follows:

Corollary 5. *Assume A_i, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I$, $i \in \{1, ..., m\}$ and with the spectral resolutions*

$$A_i = \int_I t_i dE(t_i), \ i \in \{1, ..., m\}.$$

If $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$, and ψ continuous on I, then

$$\psi\left(\sum_{i=1}^{m} p_i \widehat{\mathbf{A}_i}\right) = \int_I ... \int_I \psi\left(\sum_{i=1}^{n} p_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.18}$$

Follows by (2.4) for $g_i(t_i) = p_i t_i$, $i \in \{1, ..., m\}$.

3. Main Results

Our first main result is as follows:

Theorem 1. *Assume ψ is differentiable convex on the open interval I, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subseteq [m, M] \subset I$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi\left(\sum_{i=1}^{n} p_i \widehat{\mathbf{A}_i}\right) \\ &\leq \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i}\right)\left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right) - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right) \\ &\leq \frac{1}{2}\left[\psi'(M) - \psi'(m)\right] \sum_{i=1}^{n} w_i \left|\widehat{\mathbf{A}_i} - \sum_{j=1}^{n} w_j \widehat{\mathbf{A}_j}\right| \\ &\leq \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M - m). \end{aligned} \tag{3.1}$$

Proof. We use the following reverse of Jensen's inequality, see for instance [5, p. 198]

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \psi(x_i) - \psi\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq \sum_{i=1}^{n} w_i \psi'(x_i) x_i - \sum_{i=1}^{n} w_i \psi'(x_i) \sum_{i=1}^{n} w_i x_i \\ &\leq \frac{1}{2}\left[\psi'(M) - \psi'(m)\right] \sum_{i=1}^{n} w_i \left|x_i - \sum_{j=1}^{n} w_j x_j\right| \\ &\leq \frac{1}{2}\left[\psi'(M) - \psi'(m)\right] \left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i\right)^2\right]^{1/2} \\ &\leq \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M - m), \end{aligned} \tag{3.2}$$

where ψ is differentiable convex on the open interval I, $[m, M] \subset I$, $x_i \in [m, M]$, and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$.

Assume that we have the spectral resolutions

$$A_i = \int_m^M t_i dE(t_i), \ i \in \{1, ..., m\}.$$

By (3.2) we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \psi\left(t_i\right) - \psi\left(\sum_{i=1}^{n} w_i t_i\right) \\
&\leq \sum_{i=1}^{n} w_i \psi'\left(t_i\right) t_i - \sum_{i=1}^{n} w_i \psi'\left(t_i\right) \sum_{i=1}^{n} w_i t_i \\
&\leq \frac{1}{2}\left[\psi'(M) - \psi'(m)\right] \sum_{i=1}^{n} w_i \left| t_i - \sum_{j=1}^{n} w_j t_j \right| \\
&\leq \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M-m),
\end{aligned} \tag{3.3}
$$

for all $t_i \in [m, M]$, $i \in \{1, ..., m\}$.

If we take the integral $\int_m^M ... \int_m^M$ over $dE(t_1) \otimes ... \otimes dE(t_n)$ in (3.3), then we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \int_m^M ... \int_m^M \psi\left(t_i\right) dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) \\
&- \int_m^M ... \int_m^M \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) \\
&\leq \sum_{i=1}^{n} w_i \int_m^M ... \int_m^M \psi'\left(t_i\right) t_i dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) \\
&- \int_m^M ... \int_m^M \left(\sum_{i=1}^{n} w_i \psi'\left(t_i\right)\right)\left(\sum_{i=1}^{n} w_i t_i\right) dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) \\
&\leq \frac{1}{2}\left[\psi'(M) - \psi'(m)\right] \\
&\times \sum_{i=1}^{n} w_i \int_m^M ... \int_m^M \left| t_i - \sum_{j=1}^{n} w_j t_j \right| dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) \\
&\leq \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M-m) \int_m^M ... \int_m^M dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right).
\end{aligned} \tag{3.4}
$$

By (1.2) we have

$$
\int_m^M ... \int_m^M \psi\left(t_i\right) dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) = \widehat{\boldsymbol{\psi}\left(A_i\right)},\ i \in \{1, ..., m\},
$$

by (2.18) we have

$$
\int_I ... \int_I \psi\left(\sum_{i=1}^{n} p_i t_i\right) dE\left(t_1\right) \otimes ... \otimes dE\left(t_n\right) = \psi\left(\sum_{i=1}^{m} p_i \widehat{\mathbf{A}}_i\right)
$$

and by (2.16)

$$\begin{aligned}&\int_m^M ... \int_m^M \left|t_i - \sum_{j=1}^n w_j t_j\right| dE(t_1) \otimes ... \otimes dE(t_n)\\&= \int_m^M ... \int_m^M \left|\sum_{j=1}^n w_j (t_i - t_j)\right| dE(t_1) \otimes ... \otimes dE(t_n)\\&= \sum_{i=1}^n w_i \left|\widehat{\mathbf{A}}_i - \sum_{j=1}^n w_j \widehat{\mathbf{A}}_j\right|.\end{aligned}$$

Also,

$$\int_m^M ... \int_m^M \psi'(t_i) t_i dE(t_1) \otimes ... \otimes dE(t_n) = \boldsymbol{\psi}'\widehat{(\mathbf{A}_i)} \mathbf{A}_i, \; i \in \{1, ..., m\}$$

and

$$\int_m^M ... \int_m^M dE(t_1) \otimes ... \otimes dE(t_n) = 1 \otimes ... \otimes 1 = 1.$$

Further,

$$\begin{aligned}&\int_m^M ... \int_m^M \left(\sum_{i=1}^n w_i \psi'(t_i)\right)\left(\sum_{i=1}^n w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n)\\&= \int_m^M ... \int_m^M \left(\sum_{i=1}^n \sum_{j=1}^n w_i w_j \psi'(t_i) t_j\right) dE(t_1) \otimes ... \otimes dE(t_n)\\&= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \int_m^M ... \int_m^M \psi'(t_i) t_j dE(t_1) \otimes ... \otimes dE(t_n).\end{aligned}$$

Since, by the properties of tensorial product

$$\begin{aligned}&\int_m^M ... \int_m^M \psi'(t_i) t_j dE(t_1) \otimes ... \otimes dE(t_n)\\&= 1 \otimes ... \otimes \psi'(A_i) \otimes ... \otimes A_j \otimes ... \otimes 1\\&= \left(1 \otimes ... \otimes \psi'(A_i) \otimes ... \otimes 1\right)(1 \otimes ... \otimes A_j \otimes ... \otimes 1)\\&= (1 \otimes ... \otimes A_j \otimes ... \otimes 1)\left(1 \otimes ... \otimes \psi'(A_i) \otimes ... \otimes 1\right)\\&= \left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right)\left(\widehat{\mathbf{A}}_j\right) = \left(\widehat{\mathbf{A}}_j\right)\left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right)\end{aligned}$$

for $i, j \in \{1, ..., m\}$, with the notation convention from the previous section.

Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\int_m^M\dots\int_m^M\psi'(t_i)\,t_j dE(t_1)\otimes\dots\otimes dE(t_n)$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right)\left(\widehat{\mathbf{A}_j}\right)=\left(\sum_{i=1}^{n}w_i\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right)\left(\sum_{j=1}^{n}w_j\widehat{\mathbf{A}_j}\right)$$
$$=\left(\sum_{j=1}^{n}w_j\widehat{\mathbf{A}_j}\right)\left(\sum_{i=1}^{n}w_i\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right).$$

Also

$$\int_m^M\dots\int_m^M\psi'(t_i)\,t_i dE(t_1)\otimes\dots\otimes dE(t_n)$$
$$=\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)\,\mathbf{A}_i}=\left(\widehat{\mathbf{A}_i}\right)\left(\widehat{\boldsymbol{\psi}'(\mathbf{A}_i)}\right),\ i\in\{1,\dots,m\}.$$

Then by (3.4) we deduce (3.1). □

Corollary 6. *Assume that* $0<m\le A_i\le M$ *and* $w_i\ge 0,\ i\in\{1,\dots,n\}$ *with* $\sum_{i=1}^{n}w_i=1$, *then*

$$0\le\ln\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i}\right)-\ln\left(\bigotimes_{i=1}^{n}A_i^{w_i}\right)\tag{3.5}$$
$$\le\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i^{-1}}\right)-1$$
$$\le\frac{M-m}{2mM}\sum_{i=1}^{n}w_i\left|\widehat{\mathbf{A}_i}-\sum_{j=1}^{n}w_j\widehat{\mathbf{A}_j}\right|\le\frac{1}{4mM}(M-m)^2.$$

The proof follows by (3.1) by taking $\psi(t)=\ln t,\ t>0$ and using (2.10).

Corollary 7. *Assume that* $m\le A_i\le M$ *and* $w_i\ge 0,\ i\in\{1,\dots,n\}$ *with* $\sum_{i=1}^{n}w_i=1$, *then*

$$0\le\sum_{i=1}^{n}w_i\widehat{\exp(\mathbf{A}_i)}-\bigotimes_{i=1}^{n}[\exp(A_i)]^{w_i}\tag{3.6}$$
$$\le\sum_{i=1}^{n}w_i\left(\widehat{\mathbf{A}_i}\right)\left(\widehat{\exp(\mathbf{A}_i)}\right)-\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n}w_i\widehat{\exp(\mathbf{A}_i)}\right)$$
$$\le\frac{1}{2}(\exp M-\exp m)\sum_{i=1}^{n}w_i\left|\widehat{\mathbf{A}_i}-\sum_{j=1}^{n}w_j\widehat{\mathbf{A}_j}\right|$$
$$\le\frac{1}{4}(\exp M-\exp m)(M-m).$$

The proof follows by (3.1) for $\psi(t)=\exp t,\ t\in\mathbb{R}$ and by (2.17).

Now, by taking $A_i = \ln B_i$, where $0 < k \le B_i \le K$ for $i \in \{1, ..., n\}$, then by (3.6) we get

$$
\begin{aligned}
0 &\le \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i} - \bigotimes_{i=1}^{n} B_i^{w_i} \\
&\le \sum_{i=1}^{n} w_i \left(\widehat{\ln \mathbf{B}_i}\right)\left(\widehat{\mathbf{B}_i}\right) - \left(\sum_{i=1}^{n} w_i \widehat{\ln \mathbf{B}_i}\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i}\right) \\
&\le \frac{1}{2}(K-k)\sum_{i=1}^{n} w_i \left|\widehat{\ln \mathbf{B}_i} - \sum_{j=1}^{n} w_j \widehat{\ln \mathbf{B}_j}\right| \\
&\le \frac{1}{4}(K-k)(\ln K - \ln k).
\end{aligned}
\tag{3.7}
$$

The case of power function is as follows:

Corollary 8. *Assume that $0 < m \le A_i \le M$ and $w_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then for $r \in (-\infty, 0) \cup [0, \infty)$*

$$
\begin{aligned}
0 &\le \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^r} - \left(\sum_{i=1}^{n} p_i \widehat{\mathbf{A}_i}\right)^r \\
&\le r\left(\sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i^r}\right) - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{r-1}}\right)\right) \\
&\le \frac{1}{2} r\left(M^{r-1} - m^{r-1}\right)\sum_{i=1}^{n} w_i \left|\widehat{\mathbf{A}_i} - \sum_{j=1}^{n} w_j \widehat{\mathbf{A}_j}\right| \\
&\le \frac{1}{4} r\left(M^{r-1} - m^{r-1}\right)(M-m).
\end{aligned}
\tag{3.8}
$$

The proof follows by (3.1) by taking $\psi(t) = t^r$, $t > 0$.

Now, if we take $r = -1$ in (3.8), then we get

$$
\begin{aligned}
0 &\le \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} - \left(\sum_{i=1}^{n} p_i \widehat{\mathbf{A}_i}\right)^{-1} \\
&\le \left(\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-2}}\right) - \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i^{-1}}\right)\right) \\
&\le \frac{M^2 - m^2}{2m^2 M^2}\sum_{i=1}^{n} w_i \left|\widehat{\mathbf{A}_i} - \sum_{j=1}^{n} w_j \widehat{\mathbf{A}_j}\right| \le \frac{M^2 - m^2}{4m^2 M^2}(M-m),
\end{aligned}
\tag{3.9}
$$

provided that $0 < m \le A_i \le M$ and $w_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$.

Assume that $0 < k \le B_i \le K$ and $w_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$, then by taking $A_i = B_i^{-1}$, $m = K^{-1}$, $M = k^{-1}$ we get from (3.9) that

$$\begin{aligned} 0 &\le \sum_{i=1}^n w_i \widehat{\mathbf{B}_i} - \left(\sum_{i=1}^n p_i \widehat{\mathbf{B}_i^{-1}}\right)^{-1} \\ &\le \left(\sum_{i=1}^n w_i \widehat{\mathbf{B}_i^{-1}}\right)\left(\sum_{i=1}^n w_i \widehat{\mathbf{B}_i^{2}}\right) - \sum_{i=1}^n w_i \left(\widehat{\mathbf{B}_i}\right) \\ &\le \frac{K^2 - k^2}{2k^2 K^2} \sum_{i=1}^n w_i \left| \widehat{\mathbf{B}_i^{-1}} - \sum_{j=1}^n w_j \widehat{\mathbf{B}_j^{-1}} \right| \le \frac{K^2 - k^2}{4k^3 K^3} (K - k) . \end{aligned} \tag{3.10}$$

Further, from the inequality (3.2) for the logarithmic function $\psi(t) = -\ln t$ we get

$$0 \le \ln \left(\sum_{i=1}^n w_i x_i\right) - \sum_{i=1}^n w_i \ln (x_i) \le \frac{1}{4mM} (M - m)^2 ,$$

which is equivalent to

$$\sum_{i=1}^n w_i x_i \le \exp\left[\frac{1}{4mM} (M - m)^2\right] \prod_{i=1}^n x_i^{p_i}. \tag{3.11}$$

By utilizing a similar argument to the one in the proof of Theorem 1, we get

$$\sum_{i=1}^n w_i \widehat{\mathbf{B}_i} \le \exp\left[\frac{1}{4kK} (K - k)^2\right] \bigotimes_{i=1}^n B_i^{w_i} \tag{3.12}$$

provided that $0 < k \le B_i \le K$ and $w_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$.

4. Integral Inequalities

For $m = 2$, $w_1 = 1 - \nu$, $w_2 = \nu$, $A_1 = A$, $A_2 = B$ we get from (3.1) that

$$\begin{aligned} 0 &\le (1 - \nu) \psi (A) \otimes 1 + \nu 1 \otimes \psi (B) - \psi ((1 - \nu) A \otimes 1 + \nu 1 \otimes B) \\ &\le (1 - \nu) \left[A\psi' (A)\right] \otimes 1 + \nu 1 \otimes \left[B\psi' (B)\right] \\ &- ((1 - \nu) A \otimes 1 + \nu 1 \otimes B) \left((1 - \nu) \psi' (A) \otimes 1 + \nu 1 \otimes \psi' (B)\right) \\ &= \nu (1 - \nu) (A \otimes 1 - 1 \otimes B) \left[\psi' (A) \otimes 1 - 1 \otimes \psi' (B)\right] \\ &= \nu (1 - \nu) \\ &\times \left(\left[A\psi' (A)\right] \otimes 1 + 1 \otimes \left[B\psi' (B)\right] - A \otimes \psi' (B) - \psi' (A) \otimes B\right) \\ &\le \nu (1 - \nu) \left[\psi' (M) - \psi' (m)\right] |A \otimes 1 - 1 \otimes B| \\ &\le \frac{1}{4} \left[\psi' (M) - \psi' (m)\right] (M - m) , \end{aligned} \tag{4.1}$$

provided that ψ is differentiable convex on the open interval I, A, B are selfadjoint operators with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m, M] \subset I$.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) =$

$\int_{\Omega}\varphi(A_t)\,d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_{\Omega}1d\mu(t)=1$.

Proposition 1. *Assume ψ is differentiable convex on the open interval I. Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be continuous fields of positive operators in $B(H)$ with $\operatorname{Sp}(A_t)$, $\operatorname{Sp}(B_t)\subseteq[m,M]\subset I$ for $t\in\Omega$, then for $\nu\in[0,1]$,*

$$
\begin{aligned}
&(1-\nu)\left(\int_{\Omega}\psi(A_t)\,d\mu(t)\right)\otimes 1+\nu 1\otimes\left(\int_{\Omega}\psi(B_t)\,d\mu(t)\right)\\
&-\int_{\Omega}\int_{\Omega}\psi((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)\,d\mu(t)\,d\mu(s)\\
&\leq\nu(1-\nu)\\
&\times\left[\left(\int_{\Omega}A_t\psi'(A_t)\,d\mu(t)\right)\otimes 1+1\otimes\left(\int_{\Omega}B_t\psi'(B_t)\,d\mu(t)\right)\right.\\
&-\left(\int_{\Omega}A_t d\mu(t)\right)\otimes\left(\int_{\Omega}\psi'(B_t)\,d\mu(t)\right)\\
&\left.-\left(\int_{\Omega}\psi'(A_t)\,d\mu(t)\right)\otimes\left(\int_{\Omega}B_t d\mu(t)\right)\right]\\
&\leq\frac{1}{4}\left[\psi'(M)-\psi'(m)\right](M-m).
\end{aligned}\tag{4.2}
$$

In particular,

$$
\begin{aligned}
&(1-\nu)\left(\int_{\Omega}\psi(A_t)\,d\mu(t)\right)\otimes 1+\nu 1\otimes\left(\int_{\Omega}\psi(A_t)\,d\mu(s)\right)\\
&-\int_{\Omega}\int_{\Omega}\psi((1-\nu)A_t\otimes 1+\nu 1\otimes A_s)\,d\mu(t)\,d\mu(s)\\
&\leq\nu(1-\nu)\\
&\times\left[\left(\int_{\Omega}A_t\psi'(A_t)\,d\mu(t)\right)\otimes 1+1\otimes\left(\int_{\Omega}A_t\psi'(A_t)\,d\mu(t)\right)\right.\\
&-\left(\int_{\Omega}A_t d\mu(t)\right)\otimes\left(\int_{\Omega}\psi'(A_t)\,d\mu(t)\right)\\
&\left.-\left(\int_{\Omega}\psi'(A_t)\,d\mu(t)\right)\otimes\left(\int_{\Omega}A_t d\mu(t)\right)\right]\\
&\leq\frac{1}{4}\left[\psi'(M)-\psi'(m)\right](M-m).
\end{aligned}\tag{4.3}
$$

Proof. From (4.1) we get

$$
\begin{aligned}
0&\leq(1-\nu)\psi(A)\otimes 1+\nu 1\otimes\psi(B)-\psi((1-\nu)A\otimes 1+\nu 1\otimes B)\\
&\leq\nu(1-\nu)(A\otimes 1-1\otimes B)\left[\psi'(A)\otimes 1-1\otimes\psi'(B)\right]\\
&=\nu(1-\nu)\\
&\times\left[\left[A\psi'(A)\right]\otimes 1+1\otimes\left[B\psi'(B)\right]-A\otimes\psi'(B)-\psi'(A)\otimes B\right]\\
&\leq\frac{1}{4}\left[\psi'(M)-\psi'(m)\right](M-m),
\end{aligned}
$$

which gives that

$$
\begin{aligned}
(4.4) \qquad & (1-\nu)\,\psi(A_t)\otimes 1+\nu 1\otimes\psi(B_s)-\psi((1-\nu)\,A_t\otimes 1+\nu 1\otimes B_s) \\
& \leq \nu(1-\nu) \\
& \times\left[\left[A_t\psi'(A_t)\right]\otimes 1+1\otimes\left[B_s\psi'(B_s)\right]-A_t\otimes\psi'(B_s)-\psi'(A_t)\otimes B_s\right] \\
& \leq \frac{1}{4}\left[\psi'(M)-\psi'(m)\right](M-m),
\end{aligned}
$$

for all t, $s\in\Omega$ and $\nu\in[0,1]$.

Fix $s\in\Omega$. If we take the $\int_\Omega$ over $d\mu(t)$ in (4.4), then by the properties of tensorial product and Bochner's integral we get

$$
\begin{aligned}
& (1-\nu)\left(\int_\Omega\psi(A_t)\,d\mu(t)\right)\otimes 1+\nu 1\otimes\psi(B_s) \\
& -\int_\Omega\psi((1-\nu)\,A_t\otimes 1+\nu 1\otimes B_s)\,d\mu(t) \\
& \leq \nu(1-\nu) \\
& \times\left[\left(\int_\Omega A_t\psi'(A_t)\,d\mu(t)\right)\otimes 1+1\otimes\left[B_s\psi'(B_s)\right]\right. \\
& \left.-\left(\int_\Omega A_t d\mu(t)\right)\otimes\psi'(B_s)-\left(\int_\Omega\psi'(A_t)\,d\mu(t)\right)\otimes B_s\right] \\
& \leq \frac{1}{4}\left[\psi'(M)-\psi'(m)\right](M-m),
\end{aligned}
$$

for all $s\in\Omega$.

If we take the $\int_\Omega$ over $d\mu(s)$, then we get (4.2). □

We have the representation for X, $Y\in B(H)$,

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\,\mathcal{U}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

Corollary 9. *With the assumptions of Proposition 1 we have*

$$
\begin{aligned}
(4.5) \qquad & \int_\Omega((1-\nu)\,\psi(A_t)+\nu\psi(B_t))\,d\mu(t)\circ 1 \\
& -\int_\Omega\int_\Omega\mathcal{U}^*\psi((1-\nu)\,A_t\otimes 1+\nu 1\otimes B_s)\,\mathcal{U}d\mu(t)\,d\mu(s) \\
& \leq \nu(1-\nu) \\
& \times\left[\left(\int_\Omega A_t\psi'(A_t)\,d\mu(t)+\int_\Omega B_t\psi'(B_t)\,d\mu(t)\right)\circ 1\right. \\
& -\left(\int_\Omega A_t d\mu(t)\right)\circ\left(\int_\Omega\psi'(B_t)\,d\mu(t)\right) \\
& \left.-\left(\int_\Omega\psi'(A_t)\,d\mu(t)\right)\circ\left(\int_\Omega B_t d\mu(t)\right)\right] \\
& \leq \frac{1}{4}\left[\psi'(M)-\psi'(m)\right](M-m).
\end{aligned}
$$

In particular,

$$\int_{\Omega}\psi\left(A_{t}\right)d\mu\left(t\right)\circ 1-\int_{\Omega}\int_{\Omega}\mathcal{U}^{*}\psi\left(\left(1-\nu\right)A_{t}\otimes 1+\nu 1\otimes A_{s}\right)\mathcal{U}d\mu\left(t\right)d\mu\left(s\right) \tag{4.6}$$
$$\leq 2\nu\left(1-\nu\right)$$
$$\times\left[\left(\int_{\Omega}A_{t}\psi'\left(A_{t}\right)d\mu\left(t\right)\right)\circ 1-\left(\int_{\Omega}A_{t}d\mu\left(t\right)\right)\circ\left(\int_{\Omega}\psi'\left(A_{t}\right)d\mu\left(t\right)\right)\right]$$
$$\leq\frac{1}{4}\left[\psi'\left(M\right)-\psi'\left(m\right)\right]\left(M-m\right)$$

for $\nu\in[0,1]$.

Proof. If we take $\mathcal{U}^{*}$ to the left in the inequality (4.2) and $\mathcal{U}$ to the right and use the properties of the integral and the continuity of $\mathcal{U}^{*}$ and $\mathcal{U}$, we get

$$(1-\nu)\mathcal{U}^{*}\left[\left(\int_{\Omega}\psi\left(A_{t}\right)d\mu\left(t\right)\right)\otimes 1\right]\mathcal{U} \tag{4.7}$$
$$+\nu\mathcal{U}^{*}\left[1\otimes\left(\int_{\Omega}\psi\left(B_{t}\right)d\mu\left(t\right)\right)\right]\mathcal{U}$$
$$-\int_{\Omega}\int_{\Omega}\mathcal{U}^{*}\psi\left(\left(1-\nu\right)A_{t}\otimes 1+\nu 1\otimes B_{s}\right)\mathcal{U}d\mu\left(t\right)d\mu\left(s\right)$$
$$\leq\nu\left(1-\nu\right)$$
$$\times\left[\mathcal{U}^{*}\left[\left(\int_{\Omega}A_{t}\psi'\left(A_{t}\right)d\mu\left(t\right)\right)\otimes 1\right]\mathcal{U}\right.$$
$$+\mathcal{U}^{*}\left[1\otimes\left(\int_{\Omega}B_{t}\psi'\left(B_{t}\right)d\mu\left(t\right)\right)\right]\mathcal{U}$$
$$-\mathcal{U}^{*}\left[\left(\int_{\Omega}A_{t}d\mu\left(t\right)\right)\otimes\left(\int_{\Omega}\psi'\left(B_{t}\right)d\mu\left(t\right)\right)\right]\mathcal{U}$$
$$\left.-\mathcal{U}^{*}\left[\left(\int_{\Omega}\psi'\left(A_{t}\right)d\mu\left(t\right)\right)\otimes\left(\int_{\Omega}B_{t}d\mu\left(t\right)\right)\right]\mathcal{U}\right]$$
$$\leq\frac{1}{4}\left[\psi'\left(M\right)-\psi'\left(m\right)\right]\left(M-m\right),$$

which is equivalent to (4.5). □

Proposition 2. *Let* $\left(B_{t}^{(i)}\right)_{t\in\Omega}$, $i=1,...,n$ *be continuous fields of positive operators in* $B\left(H\right)$ *with* $\operatorname{Sp}\left(B_{t}^{(i)}\right)\subseteq[k,K]\subset(0,\infty)$ *for* $t\in\Omega$, $i=1,...,n$, *and* $w_{i}\geq 0$, $i\in\{1,...,n\}$ *with* $\sum_{i=1}^{n}w_{i}=1$, *then,*

$$\sum_{i=1}^{n}w_{i}\widehat{\left(\int_{\Omega}\mathbf{B}_{t_{i}}^{(i)}d\mu\left(t_{i}\right)\right)}-\bigotimes_{i=1}^{n}\left(\int_{\Omega}\left[B_{t_{i}}^{(i)}\right]^{w_{i}}d\mu\left(t_{i}\right)\right) \tag{4.8}$$
$$\leq\frac{1}{4}\left(K-k\right)\left(\ln K-\ln k\right)$$

and

$$\sum_{i=1}^{n}w_{i}\widehat{\left(\int_{\Omega}\mathbf{B}_{t_{i}}^{(i)}d\mu\left(t_{i}\right)\right)}\leq\exp\left[\frac{1}{4kK}\left(K-k\right)^{2}\right]\bigotimes_{i=1}^{n}\left(\int_{\Omega}\left[B_{t_{i}}^{(i)}\right]^{w_{i}}d\mu\left(t_{i}\right)\right). \tag{4.9}$$

Proof. From the inequality (3.7) we have

$$\begin{aligned}(4.10)\qquad & w_1\mathbf{B}_{t_1}^{(1)}\otimes 1\otimes ...\otimes 1+...+w_n 1\otimes ...\otimes 1\otimes \mathbf{B}_{t_n}^{(n)} \\ & -\left[\mathbf{B}_{t_1}^{(1)}\right]^{w_1}\otimes ...\otimes \left[\mathbf{B}_{t_n}^{(n)}\right]^{w_n} \\ & \leq \frac{1}{4}(K-k)(\ln K-\ln k)\end{aligned}$$

for all $t_i\in\Omega$, $i=1,...,n$.

If we take in (4.10) the integral $\int_\Omega ... \int_\Omega$ over $d\mu(t_1)...d\mu(t_n)$, then we get

$$\begin{aligned}(4.11)\qquad & \int_\Omega ... \int_\Omega \left[w_1\mathbf{B}_{t_1}^{(1)}\otimes 1\otimes ...\otimes 1+...+w_n 1\otimes ...\otimes 1\otimes \mathbf{B}_{t_n}^{(n)}\right] \\ & \times d\mu(t_1)...d\mu(t_n) \\ & -\int_\Omega ... \int_\Omega \left[\mathbf{B}_{t_1}^{(1)}\right]^{w_1}\otimes ...\otimes \left[\mathbf{B}_{t_n}^{(n)}\right]^{w_n} d\mu(t_1)...d\mu(t_n) \\ & \leq \frac{1}{4}(K-k)(\ln K-\ln k).\end{aligned}$$

Using the properties of integral and tensorial product, we have

$$\begin{aligned}(4.12)\qquad & \int_\Omega ... \int_\Omega \left[w_1\mathbf{B}_{t_1}^{(1)}\otimes 1\otimes ...\otimes 1+...+w_n 1\otimes ...\otimes 1\otimes \mathbf{B}_{t_n}^{(n)}\right] \\ & \times d\mu(t_1)...d\mu(t_n) \\ & = w_1\int_\Omega ... \int_\Omega \mathbf{B}_{t_1}^{(1)}\otimes 1\otimes ...\otimes 1 d\mu(t_1)...d\mu(t_n)+... \\ & + w_n\int_\Omega ... \int_\Omega 1\otimes ...\otimes 1\otimes \mathbf{B}_{t_n}^{(n)} d\mu(t_1)...d\mu(t_n) \\ & = w_1\left(\int_\Omega \mathbf{B}_{t_1}^{(1)} d\mu(t_1)\right)\otimes 1\otimes ...\otimes 1+... \\ & + w_n 1\otimes ...\otimes 1\otimes \left(\int_\Omega \mathbf{B}_{t_n}^{(n)} d\mu(t_n)\right) \\ & = \sum_{i=1}^n w_i \widehat{\int_\Omega \mathbf{B}_{t_i}^{(i)} d\mu(t_i)}\end{aligned}$$

and

$$\begin{aligned} & \int_\Omega ... \int_\Omega \left[\mathbf{B}_{t_1}^{(1)}\right]^{w_1}\otimes ...\otimes \left[\mathbf{B}_{t_n}^{(n)}\right]^{w_n} d\mu(t_1)...d\mu(t_n) \\ & = \left(\int_\Omega \left[\mathbf{B}_{t_1}^{(1)}\right]^{w_1} d\mu(t_1)\right)\otimes ...\otimes \left(\int_\Omega \left[\mathbf{B}_{t_n}^{(n)}\right]^{w_n} d\mu(t_n)\right) \\ & = \bigotimes_{i=1}^n \int_\Omega \left[B_{t_i}^{(i)}\right]^{w_i} d\mu(t_i).\end{aligned}$$

From (4.11) we derive the desired result (4.8).

The inequality (4.9) follows in a similar way from (3.12). □

The case of two operators is as follows:

$$0\leq(1-\nu)\left(\int_{\Omega}A_{t}d\mu(t)\right)\otimes 1+\nu 1\otimes\left(\int_{\Omega}B_{t}d\mu(t)\right)-\left(\int_{\Omega}A_{t}^{1-\nu}d\mu(t)\right)\otimes\left(\int_{\Omega}B_{t}^{\nu}d\mu(t)\right)\leq\frac{1}{4}(K-k)(\ln K-\ln k) \tag{4.13}$$

and

$$\left(\int_{\Omega}A_{t}^{1-\nu}d\mu(t)\right)\otimes\left(\int_{\Omega}B_{t}^{\nu}d\mu(t)\right)\leq(1-\nu)\left(\int_{\Omega}A_{t}d\mu(t)\right)\otimes 1+\nu 1\otimes\left(\int_{\Omega}B_{t}d\mu(t)\right)\leq\exp\left[\frac{1}{4kK}(K-k)^{2}\right]\left(\int_{\Omega}A_{t}^{1-\nu}d\mu(t)\right)\otimes\left(\int_{\Omega}B_{t}^{\nu}d\mu(t)\right) \tag{4.14}$$

for $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ continuous fields of positive operators in $B(H)$ with $\mathrm{Sp}(A_t), \mathrm{Sp}(B_t)\subseteq[m,M]\subset I$ for $t\in\Omega$ and $\nu\in[0,1]$.

By taking $\mathcal{U}^*$ to the left in the inequalities (4.13), (4.14) and $\mathcal{U}$ to the right and use the properties of the integral, we also derive the following inequalities for the Hadamard product

$$0\leq\int_{\Omega}((1-\nu)A_t+\nu B_t)\,d\mu(t)\circ 1-\left(\int_{\Omega}A_{t}^{1-\nu}d\mu(t)\right)\circ\left(\int_{\Omega}B_{t}^{\nu}d\mu(t)\right)\leq\frac{1}{4}(K-k)(\ln K-\ln k) \tag{4.15}$$

and

$$\left(\int_{\Omega}A_{t}^{1-\nu}d\mu(t)\right)\circ\left(\int_{\Omega}B_{t}^{\nu}d\mu(t)\right)\leq\int_{\Omega}((1-\nu)A_t+\nu B_t)\,d\mu(t)\circ 1\leq\exp\left[\frac{1}{4kK}(K-k)^{2}\right]\left(\int_{\Omega}A_{t}^{1-\nu}d\mu(t)\right)\circ\left(\int_{\Omega}B_{t}^{\nu}d\mu(t)\right). \tag{4.16}$$

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[5] S. S. Dragomir, Reverses of Jensen's integral inequality and applications: a survey of recent results, T. M. Rassias (ed.), *Applications of Nonlinear Analysis, Springer Optimization and Its Applications 134*, pp. 193-263. https://doi.org/10.1007/978-3-319-89815-5_8

[6] S. S. Dragomir, Refinements and reverses of tensorial arithmetic mean-geometric mean inequalities for selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art.

[7] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[8] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[11] H. Kober, On the arithmetic and geometric means and on Hölder's inequality. *Proc. American Math. Soc.* **9** (1958), 452-459.

[12] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa