# TWO NEW REVERSES OF JENSEN TENSORIAL INEQUALITY FOR SEQUENCES OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let $H$ be a Hilbert space. In this paper we show among others that, if $\psi$ is differentiable convex on the open interval $I$, $A_i$, $i \in \{1, ..., n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subseteq [m, M] \subset I$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \quad \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}) \\ &\leq \frac{\psi'_-(M) - \psi'_+(m)}{M - m} \quad M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}) \left( \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m \right) \\ &\leq \frac{1}{4} \left[ \psi'(M) - \psi'(m) \right] (M - m), \end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of $A_i$ in position $i = 1, ..., n$ and with 1 in the other positions. Let $(A_t)_{t \in \Omega}$ and $(B_t)_{t \in \Omega}$be continuous fields of positive operators in $B(H)$ with $\operatorname{Sp}(A_t), \operatorname{Sp}(B_t) \subseteq [m, M] \subset I$ for $t \in \Omega$, then for $\nu \in [0, 1]$ we have the inequalities for the Hadamard product

$$\begin{aligned} &\int_{\Omega} [(1 - \nu) \exp A_t + \nu \exp B_t] \, d\mu(t) \circ 1 \\ &- \left( \int_{\Omega} \exp((1 - \nu) A_t) \, d\mu(t) \right) \circ \left( \int_{\Omega} \exp(\nu B_t) \, d\mu(t) \right) \\ &\leq \frac{\exp M - \exp m}{M - m} \left( M - \int_{\Omega} [(1 - \nu)(A_t) + \nu B_t] \, d\mu(t) \circ 1 \right) \\ &\times \left( \int_{\Omega} [(1 - \nu)(A_t) + \nu B_t] \, d\mu(t) \circ 1 - m \right) \\ &\leq \frac{1}{4} (\exp M - \exp m)(M - m). \end{aligned}$$

## 1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a $k$-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of $A_i$ is contained in $I_i$ for $i = 1, ..., k$. We say that such a $k$-tuple is in the domain of $f$. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

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is the spectral resolution of $A_i$ for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k), \tag{1.2}$$

whenever $f$ can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of $k$ functions each depending on only one variable.

It is know that, *if $f$ is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if $f$ is continuous on $[0, \infty)$, then [9, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of $A$ and $B$, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function $f$ on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B)\,\#\,(B \otimes A).$$

In 2007, S. Wada [12] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B) \otimes (A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A \otimes B + B \otimes A)\end{aligned} \tag{1.5}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of $A$ and $B$ in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle Ae_j, e_j \rangle \langle Be_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space $H$.

It is known that, see [8], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.6}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If $f$ *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [9, p. 173]

$$f (A \circ B) \geq (\leq) f (A) \circ f (B) \text{ for all } A,\ B \geq 0. \tag{1.7}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left( \frac{A+B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left( A^2 \circ 1 \right)^{1/2} \left( B^2 \circ 1 \right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left( A^2 \circ B^2 \right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [10] that $\left( A^2 \circ 1 \right)^{1/2} \left( B^2 \circ 1 \right)^{1/2}$ and $\left( A^2 \circ B^2 \right)^{1/2}$ are incomparable for 2-square positive definite matrices $A$ and $B$.

Motivated by the above results, in this paper we show among others that, if $\psi$ is differentiable convex on the open interval $I$, $A_i$, $i \in \{1, ..., n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subseteq [m, M] \subset I$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n w_i = 1$, thenthen

$$\begin{aligned}
0 &\leq \sum_{i=1}^n w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi \left( \sum_{i=1}^n w_i \widehat{\mathbf{A}}_i \right) \\
&\leq \frac{\psi'_- (M) - \psi'_+ (m)}{M - m} \left( M - \sum_{i=1}^n w_i \widehat{\mathbf{A}}_i \right) \left( \sum_{i=1}^n w_i \widehat{\mathbf{A}}_i - m \right) \\
&\leq \frac{1}{4} \left[ \psi' (M) - \psi' (m) \right] (M - m),
\end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of $A_i$ in position $i = 1, ..., n$ and with 1 in the other positions. Let $(A_t)_{t \in \Omega}$ and $(B_t)_{t \in \Omega}$be continuous fields of positive operators in $B(H)$ with $\operatorname{Sp}(A_t), \operatorname{Sp}(B_t) \subseteq [m, M] \subset I$ for $t \in \Omega$, then for $\nu \in [0, 1]$ we have the inequalities for the Hadamard product

$$\begin{aligned}
&\int_\Omega [(1-\nu) \exp A_t + \nu \exp B_t] \, d\mu (t) \circ 1 \\
&- \left( \int_\Omega \exp ((1-\nu) A_t) \, d\mu (t) \right) \circ \left( \int_\Omega \exp (\nu B_t) \, d\mu (t) \right) \\
&\leq \frac{\exp M - \exp m}{M - m} \left( M - \int_\Omega [(1-\nu)(A_t) + \nu B_t] \, d\mu (t) \circ 1 \right) \\
&\times \left( \int_\Omega [(1-\nu)(A_t) + \nu B_t] \, d\mu (t) \circ 1 - m \right) \\
&\leq \frac{1}{4} (\exp M - \exp m) (M - m).
\end{aligned}$$

## 2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A,\ B,\ C,\ D\in B(H)$.

If we take $C=A$ and $D=B$, then we get

$$A^2\otimes B^2=(A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n=(A\otimes B)^n \text{ for natural } n\geq 0. \tag{2.2}$$

In particular

$$A^n\otimes 1=(A\otimes 1)^n \text{ and } 1\otimes B^n=(1\otimes B)^n \tag{2.3}$$

for all $n\geq 0$.

We also observe that, by (2.1), the operators $A\otimes 1$ and $1\otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B)=(1\otimes B)(A\otimes 1)=A\otimes B. \tag{2.4}$$

Moreover, for two natural numbers $m,\ n$ we have

$$(A\otimes 1)^m(1\otimes B)^n=(1\otimes B)^n(A\otimes 1)^m=A^m\otimes B^n. \tag{2.5}$$

By induction over $m$, we derive

$$(A_1\otimes A_2\otimes ...\otimes A_m)^n=A_1^n\otimes A_2^n\otimes ...\otimes A_m^n \text{ for natural } n\geq 0 \tag{2.6}$$

and

$$\begin{aligned} &A_1\otimes A_2\otimes ...\otimes A_m \\ &=(A_1\otimes 1\otimes ...\otimes 1)(1\otimes A_2\otimes ...\otimes 1)...(1\otimes 1\otimes ...\otimes A_m) \end{aligned} \tag{2.7}$$

and the $m$ operators $(A_1\otimes 1\otimes ...\otimes 1),\ (1\otimes A_2\otimes ...\otimes 1),\ ...$ and $(1\otimes 1\otimes ...\otimes A_m)$ are commutative between them.

We define for $A_i,\ B_i\in B(H),\ i\in\{1,...,n\},\ \bigotimes_{i=1}^n B_i:=B_1\otimes ...\otimes B_n$,

$$\hat{\mathbf{A}}_i:=1\otimes ...\otimes A_i\otimes ...\otimes 1,\ i=2,...,n-1,$$

and

$$\hat{\mathbf{A}}_1:=A_1\otimes 1\otimes ...\otimes 1 \text{ while } \hat{\mathbf{A}}_n:=1\otimes ...\otimes 1\otimes A_n.$$

Basically $\hat{\mathbf{A}}_i$ is defined as a tensorial product of $A_i$ in position $i=1,...,n$ and with 1 in the other positions.

We need the following identity for the tensorial product, see also [6]:

**Lemma 1.** *Assume $A_i,\ i\in\{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i)\subset I_i$, $i\in\{1,...,m\}$ and with the spectral resolutions*

$$A_i=\int_{I_i}t_i dE(t_i),\ i\in\{1,...,m\}.$$

*Let $f_i,\ i\in\{1,...,m\}$ be continuous on $I_i$ and $\varphi$ continuous on an interval $K$ that contains the product of the intervals $f(I_1)...f(I_m)$, then*

$$\varphi\left(\bigotimes_{i=1}^m f_i(A_i)\right)=\int_{I_1}...\int_{I_m}\varphi\left(\prod_{i=1}^m f_i(t_i)\right)dE(t_1)\otimes ...\otimes dE(t_m). \tag{2.8}$$

*We also have*

$$\varphi\left(\widehat{\mathbf{f}_i(\mathbf{A}_i)}\right)=\boldsymbol{\varphi}\,(\widehat{\mathbf{f}_i(\mathbf{A}_i)}) \tag{2.9}$$

*for all* $i=1,...,m$.

*Proof.* By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t)=t^n$ with $n$ any natural number.

Then, by (1.1) and (2.6) we obtain

$$\begin{aligned}&\int_{I_1}...\int_{I_m}[f_1(t_1)...f_m(t_m)]^n\,dE(t_1)\otimes...\otimes dE(t_m)\\&=\int_{I_1}...\int_{I_m}[f_1(t_1)]^n...[f_m(t_m)]^n\,dE(t_1)\otimes...\otimes dE(t_m)\\&=[f(A_1)]^n\otimes...\otimes[f_m(A_m)]^n=[f_1(A_1)\otimes...\otimes f_m(A_m)]^n,\end{aligned}$$

which shows that the identity (2.6) is valid for the power function.

This proves the identity (2.8)

By taking $f_j\equiv 1$ for $j=1,...,m$ and $j\neq i$ in (2.8) we get

$$\begin{aligned}\varphi(1\otimes...\otimes f_i(A_i)\otimes...\otimes 1)&=\int_{I_1}...\int_{I_m}\varphi(f_i(t_i))\,dE(t_1)\otimes...\otimes dE(t_m)\\&=1\otimes...\otimes\varphi(f_i(A_i))\otimes...\otimes 1,\end{aligned}$$

which proves (2.9). □

**Corollary 1.** *Assume* $A_i$, $i\in\{1,...,m\}$ *are selfadjoint operators with* $\mathrm{Sp}(A_i)\subset I_i$ *and* $f_i$, $i\in\{1,...,m\}$ *are continuous and positive on* $I_i$, *then*

$$\ln\left(\bigotimes_{i=1}^{m}f_i(A_i)\right)=\sum_{i=1}^{n}\ln(\widehat{\mathbf{f}_i(\mathbf{A}_i)}). \tag{2.10}$$

*Also*

$$\ln\left(\widehat{\mathbf{f}_i(\mathbf{A}_i)}\right)=\ln(\widehat{\mathbf{f}_i(\mathbf{A}_i)}) \tag{2.11}$$

*for all* $i=1,...,m$.

*Proof.* Assume that

$$A_i=\int_I t_i dE(t_i)$$

are the spectral resolutions for $A_i$, $i=1,...,m$.

We have for $\varphi(u)=\ln u$, $u>0$, in (2.8) that

$$\begin{aligned}&\ln(f_1(A)\otimes...\otimes f_m(A_m))\\&=\int_{I_1}...\int_{I_m}\ln(f_1(t_1)...f_m(t_m))\,dE(t_1)\otimes...\otimes dE(t_m)\end{aligned}$$

$$
\begin{aligned}
&= \int_{I_1} ... \int_{I_m} [\ln f_1(t_1) + ... + \ln f_m(t_m)]\, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} \ln f_1(t_1)\, dE(t_1) \otimes ... \otimes dE(t_m) + ... \\
&+ \int_{I_1} ... \int_{I_m} \ln f_m(t_m)\, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= (\ln f_1(A_1)) \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes (\ln f_m(A_m))
\end{aligned}
$$

and the identity (2.10) is proved. □

**Corollary 2.** *Assume $A_i$, $i \in \{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$ and $f_i$, $i \in \{1,...,m\}$ are continuous on $I_i$, then for $r > 0$*

$$
\left| \bigotimes_{i=1}^{m} f_i(A_i) \right|^r = \bigotimes_{i=1}^{m} |f_i(A_i)|^r \tag{2.12}
$$

*and*

$$
\left| \widehat{\mathbf{f}_i(\mathbf{A}_i)} \right|^r = |\widehat{\mathbf{f}_i(\mathbf{A}_i)}|^r \tag{2.13}
$$

*for all $i = 1, ..., m$.*

*Proof.* From (2.8) we have for the function $\varphi(t) = |t|^r$ that

$$
\begin{aligned}
&|f_1(A_1) \otimes ... \otimes f_m(A_m)|^r \\
&= \int_{I_1} ... \int_{I_m} |f_1(t_1) ... f_m(t_m)|^r\, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} ... \int_{I_m} |f_1(t_1)|^r ... |f_m(t_m)|^r\, dE(t_1) \otimes ... \otimes dE(t_m) \\
&= |f_1(A_1)|^r \otimes ... \otimes |f_m(A_m)|^r,
\end{aligned}
$$

which proves (2.12).

The identity (2.13) follows in a similar way. □

**Corollary 3.** *Assume $A_i$, $i \in \{1,...,m\}$ are positive operators and $q_i \geq 0$, $i \in \{1,...,m\}$, then*

$$
\varphi\left( \bigotimes_{i=1}^{m} A_i^{q_i} \right) = \int_{I_1} ... \int_{I_m} \varphi\left( \prod_{i=1}^{m} t_i^{q_i} \right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.14}
$$

We also have the additive result [6]:

**Lemma 2.** *Assume $A_i$, $i \in \{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$, $i \in \{1,...,m\}$ and with the spectral resolutions*

$$
A_i = \int_{I_i} t_i dE(t_i), \; i \in \{1,...,m\}.
$$

*Let $g_i$, $i \in \{1,...,m\}$ be continuous on $I_i$ and $\psi$ continuous on an interval $K$ that contains the sum of the intervals $g(I_1) + ... + g(I_m)$, then*

$$
\psi\left( \sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)} \right) = \int_{I_1} ... \int_{I_m} \psi\left( \sum_{i=1}^{m} g_i(t_i) \right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.15}
$$

*Proof.* Let $f_i$, continuous, positive and such that $g_i(t_i) = \ln f_i(t_i)$, $t_i \in I_i$, $i \in \{1, ..., m\}$. Then

$$\sum_{i=1}^{m} g_i(t_i) = \sum_{i=1}^{m} \ln f_i(t_i) = \ln\left(\prod_{i=1}^{m} f_i(t_i)\right).$$

By (2.8) we get for $\varphi = \psi \circ \ln$ that

$$\begin{aligned}
&\int_{I_1} \cdots \int_{I_m} \psi\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} \cdots \int_{I_m} \psi\left(\ln\left(\prod_{i=1}^{m} f_i(t_i)\right)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \int_{I_1} \cdots \int_{I_m} (\psi \circ \ln)\left(\prod_{i=1}^{m} f_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= (\psi \circ \ln)\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \psi\left(\ln\left(\bigotimes_{i=1}^{m} f_i(A_i)\right)\right).
\end{aligned}$$

By (2.10) we also have

$$\ln\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \sum_{i=1}^{n} \widehat{\ln(\mathbf{f}_i(\mathbf{A}_i))} = \sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}$$

and the identity (2.15) is obtained. □

**Corollary 4.** *Assume that* $A_i$, $i \in \{1, ..., m\}$ *and* $g_i$, $i \in \{1, ..., m\}$ *are as in Lemma 2 and* $r > 0$, *then*

$$\left|\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right|^r = \int_{I_1} \cdots \int_{I_m} \left|\sum_{i=1}^{m} g_i(t_i)\right|^r dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.16}$$

*Also, if we take* $\psi = \exp$, *then we get*

$$\begin{aligned}
\exp\left(\sum_{i=1}^{m} \widehat{\mathbf{g}_i(\mathbf{A}_i)}\right) &= \int_{I_1} \cdots \int_{I_m} \exp\left(\sum_{i=1}^{m} g_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m) \\
&= \bigotimes_{i=1}^{m} \exp[g_i(A_i)].
\end{aligned} \tag{2.17}$$

The case of convex combination is as follows:

**Corollary 5.** *Assume* $A_i$, $i \in \{1, ..., m\}$ *are selfadjoint operators with* $\operatorname{Sp}(A_i) \subset I$, $i \in \{1, ..., m\}$ *and with the spectral resolutions*

$$A_i = \int_I t_i dE(t_i), \ i \in \{1, ..., m\}.$$

*If* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$, *and* $\psi$ *continuous on* $I$, *then*

$$\psi\left(\sum_{i=1}^{m} p_i \widehat{\mathbf{A}_i}\right) = \int_I \cdots \int_I \psi\left(\sum_{i=1}^{n} p_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{2.18}$$

Follows by (2.4) for $g_i(t_i) = p_i t_i$, $i \in \{1, ..., m\}$.

## 3. Main Results

Our first main result is as follows:

**Theorem 1.** *Assume $\psi$ is differentiable convex on the open interval $I$, $A_i$, $i \in \{1,...,n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subseteq [m,M] \subset I$, $i \in \{1,...,n\}$ and $w_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n w_i = 1$, then*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right) \\ &\leq \sup_{t\in(m,M)} \Psi_\psi(t;m,M)\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - m\right) \\ &\leq \frac{\psi'_-(M) - \psi'_+(m)}{M-m}\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - m\right) \\ &\leq \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M-m), \end{aligned} \tag{3.1}$$

*where $\Psi_\psi(\cdot;m,M) : (m,M) \to \mathbb{R}$ is defined by*

$$\Psi_\psi(t;m,M) = \frac{\psi(M) - \psi(t)}{M-t} - \frac{\psi(t) - \psi(m)}{t-m}.$$

*Proof.* We use the following reverse of Jensen's inequality, see for instance [5, p. 228]

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \psi(x_i) - \psi\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq \frac{\left(M - \sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i x_i - m\right)}{M-m} \sup_{t\in(m,M)} \Psi_\psi(t;m,M) \\ &\leq \left(M - \sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i x_i - m\right)\frac{\psi'_-(M) - \psi'_+(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\psi'_-(M) - \psi'_+(m)\right], \end{aligned} \tag{3.2}$$

where $\psi$ is differentiable convex on the open interval $I$, $[m,M] \subset I$, $x_i \in [m,M]$, and $w_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n w_i = 1$.

Assume that we have the spectral resolutions

$$A_i = \int_m^M t_i dE(t_i), \ i \in \{1,...,n\}.$$

By (3.2) we get

$$
\begin{aligned}
0 &\le \sum_{i=1}^{n} w_i \psi(t_i) - \psi\left(\sum_{i=1}^{n} w_i t_i\right) \\
&\le \frac{\left(M - \sum_{i=1}^{n} w_i t_i\right)\left(\sum_{i=1}^{n} w_i t_i - m\right)}{M-m} \sup_{t \in (m,M)} \Psi_{\psi}(t; m, M) \\
&\le \left(M - \sum_{i=1}^{n} w_i t_i\right)\left(\sum_{i=1}^{n} w_i t_i - m\right) \frac{\psi'_-(M) - \psi'_+(m)}{M-m} \\
&\le \frac{1}{4}(M-m)\left[\psi'_-(M) - \psi'_+(m)\right],
\end{aligned} \tag{3.3}
$$

for all $t_i \in [m, M]$, $i \in \{1, ..., n\}$.

If we take the integral $\int_m^M ... \int_m^M$ over $dE(t_1) \otimes ... \otimes dE(t_n)$ in (3.3), then we get

$$
\begin{aligned}
0 &\le \sum_{i=1}^{n} w_i \int_m^M ... \int_m^M \psi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n) \\
&- \int_m^M ... \int_m^M \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n) \\
&\le \sup_{t \in (m,M)} \Psi_{\psi}(t; m, M) \\
&\times \int_m^M ... \int_m^M \frac{\left(M - \sum_{i=1}^{n} w_i t_i\right)\left(\sum_{i=1}^{n} w_i t_i - m\right)}{M-m} dE(t_1) \otimes ... \otimes dE(t_n) \\
&\le \frac{\psi'_-(M) - \psi'_+(m)}{M-m} \\
&\times \int_m^M ... \int_m^M \left(M - \sum_{i=1}^{n} w_i t_i\right)\left(\sum_{i=1}^{n} w_i t_i - m\right) E(t_1) \otimes ... \otimes dE(t_n) \\
&\le \frac{1}{4}\left[\psi'(M) - \psi'(m)\right](M-m) \int_m^M ... \int_m^M dE(t_1) \otimes ... \otimes dE(t_n).
\end{aligned} \tag{3.4}
$$

By (1.2) we have

$$
\int_m^M ... \int_m^M \psi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n) = \widehat{\boldsymbol{\psi}(A_i)},\ i \in \{1, ..., m\},
$$

by (2.18) we have

$$
\int_I ... \int_I \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n) = \psi\left(\sum_{i=1}^{m} w_i \widehat{\mathbf{A}}_i\right),
$$

also

$$
\int_I ... \int_I dE(t_1) \otimes ... \otimes dE(t_n) = 1 \otimes ... \otimes 1 = 1
$$

and, by (2.16) for $\psi(u)=u^2$,

$$\begin{aligned}&\int_m^M\dots\int_m^M\left(M-\sum_{i=1}^n w_it_i\right)\left(\sum_{i=1}^n w_it_i-m\right)dE(t_1)\otimes\dots\otimes dE(t_n)\\&=\int_m^M\dots\int_m^M\left[(m+M)\sum_{i=1}^n w_it_i-mM-\left(\sum_{i=1}^n w_it_i\right)^2\right]\\&\times dE(t_1)\otimes\dots\otimes dE(t_n)\\&=(m+M)\sum_{i=1}^m w_i\widehat{\mathbf{A}_i}-mM-\left(\sum_{i=1}^m w_i\widehat{\mathbf{A}_i}\right)^2\\&=\left(M-\sum_{i=1}^m w_i\widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^m w_i\widehat{\mathbf{A}_i}-m\right).\end{aligned}$$

Then by (3.4) we deduce (3.1). □

We also have:

**Theorem 2.** *Assume $\psi$ is convex and monotonic nondecreasing on the open interval $I$, $A_i$, $i\in\{1,...,n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i)\subseteq[m,M]\subset I$, $i\in\{1,...,n\}$ and $w_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n w_i=1$, then*

$$\begin{aligned}0&\leq\sum_{i=1}^n w_i\widehat{\boldsymbol{\psi}(\mathbf{A}_i)}-\psi\left(\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}\right)\\&\leq\frac{\left(M-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}\right)\left(\psi\left(\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}\right)-\psi(m)\right)}{M-m}\\&+\frac{\left(\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}-m\right)\left(\psi(M)-\psi\left(\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}\right)\right)}{M-m}\\&\leq\left[\frac{1}{2}+\frac{\left|\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}-\frac{m+M}{2}\right|}{M-m}\right][\psi(M)-\psi(m)].\end{aligned}\tag{3.5}$$

*Proof.* If $x_i\in I$ and $w_i\geq 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n w_i=1$, then we also have the inequality [5, p. 219]:

$$\begin{aligned}0&\leq\sum_{i=1}^n w_i\psi(x_i)-\psi(\bar{x}_w)\\&\leq\frac{(M-\bar{x}_w)\int_m^{\bar{x}_w}\left|\psi'(t)\right|dt+(\bar{x}_w-m)\int_{\bar{x}_w}^M\left|\psi'(t)\right|dt}{M-m}\\&\leq\begin{cases}\left[\frac{1}{2}+\frac{\left|\bar{x}_w-\frac{m+M}{2}\right|}{M-m}\right]\int_m^M\left|\psi'(t)\right|dt,\\[2ex]\left[\frac{1}{2}\int_m^M\left|\psi'(t)\right|dt+\frac{1}{2}\left|\int_{\bar{x}_w}^M\left|\psi'(t)\right|dt-\int_m^{\bar{x}_w}\left|\psi'(t)\right|dt\right|\right],\end{cases}\end{aligned}\tag{3.6}$$

where $\psi$ is differentiable convex on $I$.

Moreover, if $\psi$ is monotonic nondecreasing or nonincreasing on $I$, then we have from 3.6 the simpler inequalities

$$
\begin{aligned}
(3.7)\qquad 0 &\le \sum_{i=1}^{n} w_i \psi(x_i) - \psi(\bar{x}_w) \\
&\le \frac{(M-\bar{x}_w)\left|\psi(\bar{x}_w)-\psi(m)\right| + (\bar{x}_w - m)\left|\psi(M)-\psi(\bar{x}_w)\right|}{M-m} \\
&\le \left[\frac{1}{2} + \frac{\left|\bar{x}_w - \frac{m+M}{2}\right|}{M-m}\right]\left|\psi(M)-\psi(m)\right|
\end{aligned}
$$

for $x_i \in I$ and $w_i \ge 0$ for $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} w_i = 1$.

Assume that we have the spectral resolutions

$$A_i = \int_m^M t_i dE(t_i), \; i \in \{1,...,n\}.$$

By (3.7) in the case of nondecreasing functions we get

$$
\begin{aligned}
(3.8)\qquad 0 &\le \sum_{i=1}^{n} w_i \psi(t_i) - \psi\left(\sum_{i=1}^{n} w_i t_i\right) \\
&\le \frac{1}{M-m}\left(\left(M - \sum_{i=1}^{n} w_i t_i\right)\left[\psi\left(\sum_{i=1}^{n} w_i t_i\right) - \psi(m)\right]\right. \\
&\left. + \left(\sum_{i=1}^{n} w_i t_i - m\right)\left[\psi(M) - \psi\left(\sum_{i=1}^{n} w_i t_i\right)\right]\right) \\
&\le \left[\frac{1}{2} + \frac{\left|\sum_{i=1}^{n} w_i t_i - \frac{m+M}{2}\right|}{M-m}\right]\left[\psi(M)-\psi(m)\right],
\end{aligned}
$$

for all $t_i \in [m,M]$, $i \in \{1,...,n\}$.

If we take the integral $\int_m^M ... \int_m^M$ over $dE(t_1) \otimes ... \otimes dE(t_n)$ in (3.8), then we get

$$
\begin{aligned}
(3.9)\qquad 0 &\le \sum_{i=1}^{n} w_i \int_m^M ... \int_m^M \psi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n) \\
&- \int_m^M ... \int_m^M \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{M-m} \\
&\times \left(\int_m^M ... \int_m^M \left(M - \sum_{i=1}^{n} w_i t_i\right)\left[\psi\left(\sum_{i=1}^{n} w_i t_i\right) - \psi(m)\right]\right. \\
&\times dE(t_1) \otimes ... \otimes dE(t_n) \\
&+ \int_m^M ... \int_m^M \left(\sum_{i=1}^{n} w_i t_i - m\right)\left[\psi(M) - \psi\left(\sum_{i=1}^{n} w_i t_i\right)\right] \\
&\left.\times dE(t_1) \otimes ... \otimes dE(t_n)\right)
\end{aligned}
$$

$$\leq \left[\frac{1}{2}+\frac{\int_m^M \dots \int_m^M \left|\sum_{i=1}^n w_i t_i - \frac{m+M}{2}\right| dE\left(t_1\right)\otimes \dots \otimes dE\left(t_n\right)}{M-m}\right]$$
$$\times\left[\psi\left(M\right)-\psi\left(m\right)\right]$$

Now, observe that (2.18)

$$\begin{aligned}
&\int_m^M \dots \int_m^M \left(M-\sum_{i=1}^n w_i t_i\right)\left[\psi\left(\sum_{i=1}^n w_i t_i\right)-\psi\left(m\right)\right] \\
&\times dE\left(t_1\right)\otimes \dots \otimes dE\left(t_n\right) \\
&=\int_m^M \dots \int_m^M \left[M\psi\left(\sum_{i=1}^n w_i t_i\right)-M\psi\left(m\right)\right. \\
&\left.-\sum_{i=1}^n w_i t_i \psi\left(\sum_{i=1}^n w_i t_i\right)+\psi\left(m\right)\sum_{i=1}^n w_i t_i\right] dE\left(t_1\right)\otimes \dots \otimes dE\left(t_n\right) \\
&=M\psi\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)-M\psi\left(m\right) \\
&-\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\psi\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)+\psi\left(m\right)\sum_{i=1}^n w_i \widehat{\mathbf{A}_i} \\
&=\left(M-\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)\left[\psi\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)-\psi\left(m\right)\right].
\end{aligned}$$

From (2.18) for $r=1$ we also have

$$\int_m^M \dots \int_m^M \left|\sum_{i=1}^n w_i t_i - \frac{m+M}{2}\right| dE\left(t_1\right)\otimes \dots \otimes dE\left(t_n\right)=\left|\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}-\frac{m+M}{2}\right|$$

and by (3.9) we derive (3.5). □

**Remark 1.** *We observe that, if $\psi$ is convex and monotonic nonincreasing on the open interval $I$, $A_i$, $i\in\{1,...,n\}$ are selfadjoint operators with $\operatorname{Sp}\left(A_i\right)\subseteq[m,M]\subset I$, $i\in\{1,...,n\}$ and $w_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n w_i=1$, then*

$$\begin{aligned}
(3.10)\qquad 0 &\leq \sum_{i=1}^n w_i \widehat{\boldsymbol{\psi}\left(\mathbf{A}_i\right)}-\psi\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right) \\
&\leq \frac{\left(M-\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)\left(\psi\left(m\right)-\psi\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)\right)}{M-m} \\
&+\frac{\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}-m\right)\left(\psi\left(\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}\right)-\psi\left(M\right)\right)}{M-m} \\
&\leq \left[\frac{1}{2}+\frac{\left|\sum_{i=1}^n w_i \widehat{\mathbf{A}_i}-\frac{m+M}{2}\right|}{M-m}\right]\left[\psi\left(m\right)-\psi\left(M\right)\right].
\end{aligned}$$

**Remark 2.** *If $\psi$ is convex and change monotonicity on $[m,M]$, then by (3.6) we can also get the inequality*

$$0 \leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right) \leq \left[\frac{1}{2} + \frac{\left|\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - \frac{m+M}{2}\right|}{M-m}\right] \int_m^M \left|\psi'(t)\right| dt, \tag{3.11}$$

*where $A_i$, $i \in \{1,...,n\}$ are selfadjoint operators with $\mathrm{Sp}(A_i) \subseteq [m,M]$, $i \in \{1,...,n\}$ and $w_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} w_i = 1$.*

## 4. Some Examples

Consider the convex function $\psi(t) = -\ln t$, $t > 0$ and assume that $A_i$, $i \in \{1,...,n\}$ are selfadjoint operators with $\mathrm{Sp}(A_i) \subseteq [m,M] \subset (0,\infty)$, $i \in \{1,...,n\}$ and $w_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} w_i = 1$. By (3.1) we get

$$0 \leq \ln\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right) - \ln\left(\bigotimes_{i=1}^{n} A_i^{w_i}\right) \leq \frac{1}{mM}\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - m\right) \leq \frac{(M-m)^2}{4mM}, \tag{4.1}$$

while from (3.10)

$$\begin{aligned} 0 &\leq \ln\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right) - \ln\left(\bigotimes_{i=1}^{n} A_i^{w_i}\right) \\ &\leq \frac{\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\left(\ln\left(\bigotimes_{i=1}^{n} A_i^{w_i}\right) - \ln m\right)}{M-m} \\ &\quad + \frac{\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - m\right)\left(\ln M - \ln\left(\bigotimes_{i=1}^{n} A_i^{w_i}\right)\right)}{M-m} \\ &\leq \left[\frac{1}{2} + \frac{\left|\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - \frac{m+M}{2}\right|}{M-m}\right] (\ln M - \ln m). \end{aligned} \tag{4.2}$$

Consider the convex function $\psi(t) = \exp t$, $t \in \mathbb{R}$ and assume that $A_i$, $i \in \{1,...,n\}$ are selfadjoint operators with $\mathrm{Sp}(A_i) \subseteq [m,M] \subset (0,\infty)$, $i \in \{1,...,n\}$ and $w_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} w_i = 1$. By (3.1) we get

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \widehat{\exp(\mathbf{A}_i)} - \exp\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right) \\ &\leq \frac{\exp M - \exp m}{M-m}\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i - m\right) \\ &\leq \frac{1}{4}(\exp M - \exp m)(M-m) \end{aligned} \tag{4.3}$$

and by (3.5) we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\exp(\mathbf{A}_i)} - \exp\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \\
&\leq \frac{\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\exp\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) - \exp m\right)}{M-m} \\
&+ \frac{\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m\right)\left(\exp M - \exp\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\right)}{M-m} \\
&\leq \left[\frac{1}{2} + \frac{\left|\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - \frac{m+M}{2}\right|}{M-m}\right](\exp M - \exp m)\,.
\end{aligned} \tag{4.4}
$$

Now, by taking $A_i = \ln B_i$, where $0 < k \leq B_i \leq K$ for $i \in \{1,...,n\}$, then by (4.3) we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i} - \bigotimes_{i=1}^{n} B_i^{w_i} \\
&\leq \frac{K-k}{\ln K - \ln k}\left(\ln K - \ln\left(\bigotimes_{i=1}^{n} B_i^{w_i}\right)\right)\left(\ln\left(\bigotimes_{i=1}^{n} B_i^{w_i}\right) - \ln k\right) \\
&\leq \frac{1}{4}(K-k)(\ln K - \ln k)\,,
\end{aligned} \tag{4.5}
$$

while from (4.4)

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i} - \bigotimes_{i=1}^{n} B_i^{w_i} \\
&\leq \frac{\left(\ln K - \ln\left(\bigotimes_{i=1}^{n} B_i^{w_i}\right)\right)\left(\bigotimes_{i=1}^{n} B_i^{w_i} - k\right)}{\ln K - \ln k} \\
&+ \frac{\left(\ln\left(\bigotimes_{i=1}^{n} B_i^{w_i}\right) - \ln k\right)\left(K - \bigotimes_{i=1}^{n} B_i^{w_i}\right)}{\ln K - \ln k} \\
&\leq \left[\frac{1}{2} + \frac{\left|\ln\left(\bigotimes_{i=1}^{n} B_i^{w_i}\right) - \frac{\ln k + \ln K}{2}\right|}{\ln K - \ln k}\right](K-k)\,.
\end{aligned} \tag{4.6}
$$

Assume that $0 < m \leq A_i \leq M$ and $w_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} w_i = 1$, then for $r \in (-\infty, 0) \cup [0, \infty)$ we get from (3.1) that

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^r} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^r \\
&\leq r\frac{M^{r-1} - m^{r-1}}{M-m}\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m\right) \\
&\leq \frac{1}{4} r\left(M^{r-1} - m^{r-1}\right)(M-m)\,,
\end{aligned} \tag{4.7}
$$

while from (3.5)

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^r} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^r \\
&\leq \frac{\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^r - m^r\right)}{M-m} \\
&+ \frac{\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m\right)\left(M^r - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^r\right)}{M-m} \\
&\leq \left[\frac{1}{2} + \frac{\left|\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - \frac{m+M}{2}\right|}{M-m}\right](M^r - m^r)
\end{aligned}
\tag{4.8}
$$

for $r \geq 1$ and a similar inequality for $r < 0$.

Now, if we take $r = -1$ in (4.7), then we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^{-1} \\
&\leq \frac{M+m}{M^2 m^2}\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m\right) \\
&\leq \frac{1}{4}\left(\frac{M+m}{m^2 M^2}\right)(M-m)^2,
\end{aligned}
\tag{4.9}
$$

Now, if we take $r = -1$ in (4.8), then we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^{-1} \\
&\leq \frac{\left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\left(m^{-1} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^{-1}\right)}{M-m} \\
&+ \frac{\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m\right)\left(\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^{-1} - M^{-1}\right)}{M-m} \\
&\leq \left[\frac{1}{2} + \frac{\left|\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - \frac{m+M}{2}\right|}{M-m}\right]\frac{M-m}{mM}.
\end{aligned}
\tag{4.10}
$$

Assume that $0 < k \leq B_i \leq K$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then by taking $A_i = B_i^{-1}$, $m = K^{-1}$, $M = k^{-1}$ we get from (4.9) that

$$0 \leq \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i} - \left( \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i^{-1}} \right)^{-1} \tag{4.11}$$
$$\leq kK(k+K) \left( k^{-1} - \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i^{-1}} \right) \left( \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_i^{-1}} - K^{-1} \right)$$
$$\leq \frac{(K-k)^2}{4kK}(k+K),$$

## 5. Some Integral Inequalities

Let $\Omega$ be a locally compact Hausdorff space endowed with a Radon measure $\mu$. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on $\Omega$, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t) d\mu(t)$ for every bounded linear functional $\varphi$ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

**Theorem 3.** *Let* $\left(B_t^{(i)}\right)_{t\in\Omega}$, $i = 1, ..., n$ *be continuous fields of positive operators in* $B(H)$ *with* $\operatorname{Sp}\left(B_t^{(i)}\right) \subseteq [k, K] \subset (0, \infty)$ *for* $t \in \Omega$, $i = 1, ..., n$, *and* $w_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} w_i = 1$, *then,*

$$\sum_{i=1}^{n} w_i \int_\Omega \widehat{\exp\left(\mathbf{B}_{t_i}^{(i)}\right)} d\mu(t_i) - \bigotimes_{i=1}^{n} \left( \int_\Omega \exp\left(w_i B_{t_i}^{(i)}\right) d\mu(t_i) \right) \tag{5.1}$$
$$\leq \frac{\exp M - \exp m}{M-m} \left( M - \int_\Omega ... \int_\Omega \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_{t_i}^{(i)}} d\mu(t_1) ... d\mu(t_n) \right)$$
$$\times \left( \int_\Omega ... \int_\Omega \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_{t_i}^{(i)}} - m \right) d\mu(t_1) ... d\mu(t_n)$$
$$\leq \frac{1}{4}(\exp M - \exp m)(M-m).$$

*Proof.* From (4.3) we have, by the comutativity of $\mathbf{A}_i$ with $\mathbf{A}_j$ for all $i, j \in \{1, ..., n\}$

$$0 \leq \sum_{i=1}^{n} w_i \widehat{\exp(\mathbf{A}_i)} - \exp\left(w_1 \widehat{\mathbf{A}_1}\right) ... \exp\left(w_n \widehat{\mathbf{A}_n}\right) \tag{5.2}$$
$$\leq \frac{\exp M - \exp m}{M-m} \left( M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right) \left( \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m \right)$$
$$\leq \frac{1}{4}(\exp M - \exp m)(M-m).$$

Also, since

$$\exp\left(w_i \widehat{\mathbf{A}_i}\right) = \exp(1 \otimes ... \otimes (w_i A_i) \otimes 1 ... \otimes 1)$$
$$= 1 \otimes ... \otimes \exp(w_i A_i) \otimes 1 ... \otimes 1,$$

then

$$\begin{aligned}
&\exp\left(w_1\widehat{\mathbf{A}_1}\right)...\exp\left(w_n\widehat{\mathbf{A}_n}\right)\\
&=\left(\exp\left(w_1A_1\right)\otimes 1...\otimes 1\right)...\left(1...\otimes 1\otimes\exp\left(w_nA_n\right)\right)\\
&=\exp\left(w_1A_1\right)\otimes...\otimes\exp\left(w_nA_n\right)=\bigotimes_{i=1}^{n}\exp\left(w_iA_i\right)
\end{aligned}$$

Now, if we take $A_i=B_{t_i}^{(i)}$, $t_i\in\Omega$ in (5.2), then we get

$$\begin{aligned}
&w_1\exp\left(B_{t_1}^{(1)}\otimes 1...\otimes 1\right)+...+w_n\exp\left(1...\otimes 1\otimes B_{t_n}^{(n)}\right)\\
&-\exp\left(w_1B_{t_1}^{(1)}\right)\otimes...\otimes\exp\left(w_nB_{t_n}^{(n)}\right)\\
&\leq\frac{\exp M-\exp m}{M-m}\left(M-\sum_{i=1}^{n}w_i\widehat{\mathbf{B}_{t_i}^{(i)}}\right)\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{B}_{t_i}^{(i)}}-m\right)\\
&\leq\frac{1}{4}\left(\exp M-\exp m\right)\left(M-m\right).
\end{aligned}\tag{5.3}$$

If we take in (5.3) the integral $\int_\Omega...\int_\Omega$ over $d\mu\left(t_1\right)...d\mu\left(t_n\right)$ and use the properties of tensorial products and integrals, then we get

$$\begin{aligned}
&w_1\int_\Omega...\int_\Omega\exp\left(B_{t_1}^{(1)}\otimes 1...\otimes 1\right)d\mu\left(t_1\right)...d\mu\left(t_n\right)\\
&+...+w_n\int_\Omega...\int_\Omega\exp\left(1...\otimes 1\otimes B_{t_n}^{(n)}\right)d\mu\left(t_1\right)...d\mu\left(t_n\right)\\
&-\int_\Omega...\int_\Omega\exp\left(w_1B_{t_1}^{(1)}\right)\otimes...\otimes\exp\left(w_nB_{t_n}^{(n)}\right)d\mu\left(t_1\right)...d\mu\left(t_n\right)\\
&\leq\frac{\exp M-\exp m}{M-m}\\
&\times\int_\Omega...\int_\Omega\left(M-\sum_{i=1}^{n}w_i\widehat{\mathbf{B}_{t_i}^{(i)}}\right)\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{B}_{t_i}^{(i)}}-m\right)d\mu\left(t_1\right)...d\mu\left(t_n\right)\\
&\leq\frac{1}{4}\left(\exp M-\exp m\right)\left(M-m\right).
\end{aligned}\tag{5.4}$$

Observe that

$$\begin{aligned}
&w_1\int_\Omega...\int_\Omega\exp\left(B_{t_1}^{(1)}\otimes 1...\otimes 1\right)d\mu\left(t_1\right)...d\mu\left(t_n\right)\\
&+...+w_n\int_\Omega...\int_\Omega\exp\left(1...\otimes 1\otimes B_{t_n}^{(n)}\right)d\mu\left(t_1\right)...d\mu\left(t_n\right)\\
&=w_1\int_\Omega\exp\left(B_{t_1}^{(1)}\otimes 1...\otimes 1\right)d\mu\left(t_1\right)\\
&+...+w_n\int_\Omega\exp\left(1...\otimes 1\otimes B_{t_n}^{(n)}\right)d\mu\left(t_n\right)\\
&=\sum_{i=1}^{n}w_i\int_\Omega\widehat{\exp\left(\mathbf{B}_{t_i}^{(i)}\right)}d\mu\left(t_i\right),
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega ... \int_\Omega \exp\left(w_1 B_{t_1}^{(1)}\right) \otimes ... \otimes \exp\left(w_n B_{t_n}^{(n)}\right) d\mu(t_1) ... d\mu(t_n) \\
&= \left(\int_\Omega \exp\left(w_1 B_{t_1}^{(1)}\right) d\mu(t_1)\right) \otimes ... \otimes \left(\int_\Omega \exp\left(w_n B_{t_n}^{(n)}\right) d\mu(t_n)\right) \\
&= \bigotimes_{i=1}^{n} \left(\int_\Omega \exp\left(w_i B_{t_i}^{(i)}\right) d\mu(t_i)\right).
\end{aligned}$$

The function $g(t) = (M - t)(t - m)$ is operator concave on $[m, M]$, then

$$\begin{aligned}
&\int_\Omega ... \int_\Omega \left(M - \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_{t_i}^{(i)}}\right)\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{B}_{t_i}^{(i)}} - m\right) d\mu(t_1) ... d\mu(t_n) \\
&\leq \left(M - \int_\Omega ... \int_\Omega \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_{t_i}^{(i)}} d\mu(t_1) ... d\mu(t_n)\right) \\
&\times \left(\int_\Omega ... \int_\Omega \sum_{i=1}^{n} w_i \widehat{\mathbf{B}_{t_i}^{(i)}} - m\right) d\mu(t_1) ... d\mu(t_n) \\
&\leq \frac{1}{4}(M - m)^2
\end{aligned}$$

and by (5.4) we derive (5.1). □

The case of two fields of operators is as follows:

**Corollary 6.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be continuous fields of positive operators in $B(H)$ with $\operatorname{Sp}(A_t), \operatorname{Sp}(B_t) \subseteq [m, M] \subset I$ for $t \in \Omega$, then for $\nu \in [0, 1]$,*

$$\begin{aligned}
&(1 - \nu)\int_\Omega (\exp A_t \otimes 1) d\mu(t) + \nu \int_\Omega (1 \otimes \exp B_t) d\mu(t) \\
&- \left(\int_\Omega \exp((1 - \nu) A_t) d\mu(t)\right) \otimes \left(\int_\Omega \exp(\nu B_t) d\mu(t)\right) \\
&\leq \frac{\exp M - \exp m}{M - m}\left(M - \int_\Omega [(1 - \nu)(A_t \otimes 1) + \nu 1 \otimes B_t] d\mu(t)\right) \\
&\times \left(\int_\Omega [(1 - \nu)(A_t \otimes 1) + \nu 1 \otimes B_t] d\mu(t) - m\right) \\
&\leq \frac{1}{4}(\exp M - \exp m)(M - m).
\end{aligned} \tag{5.5}$$

We also have the following inequality for the Hadamard product of operators:

**Corollary 7.** *With the assumptions of Corollary 6, we have*

$$
\begin{aligned}
&\int_\Omega \left[(1-\nu)\exp A_t + \nu \exp B_t\right] d\mu(t) \circ 1 \\
&-\left(\int_\Omega \exp\left((1-\nu)A_t\right) d\mu(t)\right) \circ \left(\int_\Omega \exp\left(\nu B_t\right) d\mu(t)\right) \\
&\le \frac{\exp M - \exp m}{M-m}\left(M - \int_\Omega \left[(1-\nu)A_t + \nu B_t\right] d\mu(t) \circ 1\right) \\
&\times \left(\int_\Omega \left[(1-\nu)A_t + \nu B_t\right] d\mu(t) \circ 1 - m\right) \\
&\le \frac{1}{4}\left(\exp M - \exp m\right)(M-m).
\end{aligned} \tag{5.6}
$$

*Proof.* We have the representation for the operators $A,\ B \in B(H),\ A \circ B = \mathcal{U}^*(A \otimes B)\mathcal{U}$, where $\mathcal{U}: H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (5.5)

$$
\begin{aligned}
&(1-\nu)\int_\Omega \mathcal{U}^*\left(\exp A_t \otimes 1\right)\mathcal{U} d\mu(t) + \nu \int_\Omega \mathcal{U}^*\left(1 \otimes \exp B_t\right)\mathcal{U} d\mu(t) \\
&-\mathcal{U}^*\left(\int_\Omega \exp\left((1-\nu)A_t\right) d\mu(t)\right) \otimes \left(\int_\Omega \exp\left(\nu B_t\right) d\mu(t)\right)\mathcal{U} \\
&\le \frac{\exp M - \exp m}{M-m}\mathcal{U}^*\left(M - \int_\Omega \left[(1-\nu)(A_t \otimes 1) + \nu 1 \otimes B_t\right] d\mu(t)\right) \\
&\times \left(\int_\Omega \left[(1-\nu)(A_t \otimes 1) + \nu 1 \otimes B_t\right] d\mu(t) - m\right)\mathcal{U} \\
&\le \frac{1}{4}\left(\exp M - \exp m\right)(M-m).
\end{aligned} \tag{5.7}
$$

Observe that

$$
\mathcal{U}^*\left(\exp A_t \otimes 1\right)\mathcal{U} = \exp A_t \circ 1,\ \mathcal{U}^*\left(1 \otimes \exp B_t\right)\mathcal{U} = 1 \circ \exp B_t
$$

and

$$
\begin{aligned}
&\mathcal{U}^*\left(\int_\Omega \exp\left((1-\nu)A_t\right) d\mu(t)\right) \otimes \left(\int_\Omega \exp\left(\nu B_t\right) d\mu(t)\right)\mathcal{U} \\
&= \left(\int_\Omega \exp\left((1-\nu)A_t\right) d\mu(t)\right) \circ \left(\int_\Omega \exp\left(\nu B_t\right) d\mu(t)\right).
\end{aligned}
$$

Also, by Davies-Choi-Jensen's inequality for the operator concave function $g(t) = (M-t)(t-m)$ on $[m, M]$ we have that

$$
\begin{aligned}
&\mathcal{U}^*\left(M - \int_\Omega \left[(1-\nu)(A_t \otimes 1) + \nu 1 \otimes B_t\right] d\mu(t)\right) \\
&\times \left(\int_\Omega \left[(1-\nu)(A_t \otimes 1) + \nu 1 \otimes B_t\right] d\mu(t) - m\right)\mathcal{U}
\end{aligned}
$$

$$\leq \left(M - \int_\Omega \mathcal{U}^* \left[(1-\nu)(A_t \otimes 1) + \nu 1 \otimes B_t\right] \mathcal{U} d\mu(t)\right)$$
$$\times \left(\int_\Omega \mathcal{U}^* \left[(1-\nu)(A_t \otimes 1) + \nu 1 \otimes B_t\right] \mathcal{U} d\mu(t) - m\right)$$
$$= \left(M - \int_\Omega \left[(1-\nu)(A_t \circ 1) + \nu 1 \circ B_t\right] d\mu(t)\right)$$
$$\times \left(\int_\Omega \left[(1-\nu)(A_t \circ 1) + \nu 1 \circ B_t\right] d\mu(t) - m\right)$$

and by (5.7) we derive the desired result (5.6). □

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