REFINEMENTS AND REVERSES OF JENSEN TENSORIAL INEQUALITY FOR TWICE DIFFERENTIABLE FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space. In this paper we show among others that, if $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a twice differentiable convex function on $\mathring{I}$, $\phi : I \to \mathbb{R}$ is twice differentiable and $m\psi''(t) \le \phi''(t) \le M\psi''(t)$, $t \in \mathring{I}$, for some constants $m < M$, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators with $\mathrm{Sp}(A_i) \subset I$, $i \in \{1, ..., n\}$ and $w_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} & m\left[\sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\right] \\ & \le \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi \quad \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right) \le M\left[\sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi \quad \sum_{i=1}^{n} w_i \widehat{\mathbf{A}}_i\right)\right], \end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of A_i in position $i = 1, ..., n$ and with 1 in the other positions. Let $(A_t)_{t\in\Omega}$ be a continuous field of operators in $B(H)$ with $\mathrm{Sp}(A_t) \subseteq [m, M]$ for $t \in \Omega$, then for $\nu \in [0, 1]$ we have the inequalities for the Hadamard product

$$\begin{aligned} & 0 \le (1-\nu)\nu \exp(m) \\ & \times \left[\int_\Omega A_t^2 d\mu(t) \circ 1 - \int_\Omega A_t d\mu(t) \circ \int_\Omega A_t d\mu(t)\right] \\ & \le \int_\Omega \exp(A_t) d\mu(t) \circ 1 - \int_\Omega (\exp(1-\nu) A_t) d\mu(t) \circ \int_\Omega \exp(\nu A_t) d\mu(t) \\ & \le (1-\nu)\nu \exp(M) \\ & \times \left[\int_\Omega A_t^2 d\mu(t) \circ 1 - \int_\Omega A_t d\mu(t) \circ \int_\Omega A_t d\mu(t)\right]. \end{aligned}$$

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.

RGMIA Res. Rep. Coll. **25** (2022), Art. 97, 21 pp. Received 21/09/22

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k), \tag{1.2}$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [13, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) \, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [16] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} \left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.5}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [12], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.6}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [13, p. 173]

$$f(A\circ B)\geq(\leq)f(A)\circ f(B)\ \text{ for all } A,\ B\geq 0. \tag{1.7}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\geq 0$$

and Fiedler inequality

$$A\circ A^{-1}\geq 1 \text{ for } A>0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}\ \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^{2}\circ B^{2}\right)^{1/2}\ \text{ for } A,\ B\geq 0.$$

It has been shown in [14] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if $\psi:I\subseteq\mathbb{R}\to\mathbb{R}$ is a twice differentiable convex function on $\mathring{I}$, $\phi:I\to\mathbb{R}$ is twice differentiable and $m\psi''(t)\leq\phi''(t)\leq M\psi''(t)$, $t\in\mathring{I}$, for some constants $m<M$, A_i, $i\in\{1,...,n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i)\subset I$, $i\in\{1,...,n\}$ and $w_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}w_i=1$, then

$$\begin{aligned}m\left[\sum_{i=1}^{n}w_i\widehat{\boldsymbol{\psi}(\mathbf{A}_i)}-\psi\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i}\right)\right]&\leq\sum_{i=1}^{n}w_i\widehat{\boldsymbol{\phi}(\mathbf{A}_i)}-\phi\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i}\right)\\&\leq M\left[\sum_{i=1}^{n}w_i\widehat{\boldsymbol{\psi}(\mathbf{A}_i)}-\psi\left(\sum_{i=1}^{n}w_i\widehat{\mathbf{A}_i}\right)\right],\end{aligned}$$

where $\hat{\mathbf{A}}_i$ is defined as a tensorial product of A_i in position $i=1,...,n$ and with 1 in the other positions. Let $(A_t)_{t\in\Omega}$ be a continuous field of operators in $B(H)$ with $\operatorname{Sp}(A_t)\subseteq[m,M]$ for $t\in\Omega$, then for $\nu\in[0,1]$ we have the inequalities for the Hadamard product

$$\begin{aligned}0&\leq(1-\nu)\nu\exp(m)\\&\times\left[\int_{\Omega}A_t^2d\mu(t)\circ 1-\int_{\Omega}A_td\mu(t)\circ\int_{\Omega}A_td\mu(t)\right]\\&\leq\int_{\Omega}\exp(A_t)\,d\mu(t)\circ 1-\int_{\Omega}(\exp(1-\nu)A_t)\,d\mu(t)\circ\int_{\Omega}\exp(\nu A_t)\,d\mu(t)\\&\leq(1-\nu)\nu\exp(M)\\&\times\left[\int_{\Omega}A_t^2d\mu(t)\circ 1-\int_{\Omega}A_td\mu(t)\circ\int_{\Omega}A_td\mu(t)\right].\end{aligned}$$

2. Some Facts Related Jessen's Inequality

Let L be a linear class of real-valued functions $g:E\to\mathbb{R}$ having the properties

(L1) $f,g\in L$ imply $(\alpha f+\beta g)\in L$ for all $\alpha,\beta\in\mathbb{R}$;

(L2) $\mathbf{1}\in L$, i.e., if $f_0(t)=1$, $t\in E$ then $f_0\in L$.

An *isotonic linear functional* $\mathcal{A}:L\to\mathbb{R}$ is a functional satisfying

(A1) $\mathcal{A}(\alpha f+\beta g)=\alpha\mathcal{A}(f)+\beta\mathcal{A}(g)$ for all $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$.
(A2) If $f\in L$ and $f\geq 0$, then $\mathcal{A}(f)\geq 0$.
The mapping $\mathcal{A}$ is said to be *normalised if*
(A3) $\mathcal{A}(\mathbf{1})=1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality.

We recall Jessen's inequality ([4], see also [10]).

Theorem 1. *Let $\phi : I\subseteq\mathbb{R}\to\mathbb{R}$ (I is an interval), be a convex function and $f : E\to I$ such that $\phi\circ f,\ f\in L$. If $\mathcal{A}: L\to\mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\phi(\mathcal{A}(f))\leq\mathcal{A}(\phi\circ f). \tag{2.1}$$

In we obtained among others, the following refinement and reverse of Jessen's inequality:

Theorem 2. *Let $\psi : I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E\to I$ such that $\psi\circ f,\ f\in L$ and $\mathcal{A}: L\to\mathbb{R}$ be an isotonic linear and normalised functional. If $\phi : I\to\mathbb{R}$ is twice differentiable and $m\psi''(t)\leq\phi''(t)\leq M\psi''(t)$, $t\in\mathring{I}$, then*

$$\begin{aligned} m[\mathcal{A}(\psi\circ f)-\psi(\mathcal{A}(f))] &\leq\mathcal{A}(\phi\circ f)-\phi(\mathcal{A}(f)) \\ &\leq M[\mathcal{A}(\psi\circ f)-\psi(\mathcal{A}(f))], \end{aligned} \tag{2.2}$$

provided $\phi\circ f,\ f\in L$.

Let $p\in(-\infty,0)\cup(1,\infty)$ and define $g_p : I\subset(0,\infty)\to\mathbb{R}$, $g_p(t)=\phi''(t)t^{2-p}$. Assume that $-\infty<\gamma\leq g_p(t)\leq\Gamma<\infty$, $t\in I$. This implies that $\gamma t^{p-2}\leq\phi''(t)\leq\Gamma t^{p-2}$, so if we take $m=\frac{\gamma}{p(p-1)}$, $M=\frac{\Gamma}{p(p-1)}$ and $\psi(t)=\ell^p(t)$, where $\ell(t)=t$, then we have $m\psi''(t)\leq\phi''(t)\leq M\psi''(t)$, $t\in\mathring{I}$.

By Theorem 2 we can state the following:

Corollary 1. *If $\phi : I\subset(0,\infty)\to\mathbb{R}$ is twice differentiable and $\gamma t^{p-2}\leq\phi''(t)\leq\Gamma t^{p-2}$, $t\in\mathring{I}$, $p\in(-\infty,0)\cup(1,\infty)$, then*

$$\begin{aligned} \frac{\gamma}{p(p-1)}[\mathcal{A}(f^p)-\mathcal{A}^p(f)] &\leq\mathcal{A}(\phi\circ f)-\phi(\mathcal{A}(f)) \\ &\leq\frac{\Gamma}{p(p-1)}[\mathcal{A}(f^p)-\mathcal{A}^p(f)], \end{aligned} \tag{2.3}$$

provided $\phi\circ f,\ f,\ f^p\in L$.

In particular, if $\gamma\leq\phi''(t)\leq\Gamma$, $t\in\mathring{I}$, then

$$\frac{\gamma}{2}\left[\mathcal{A}\left(f^2\right)-\mathcal{A}^2(f)\right]\leq\mathcal{A}(\phi\circ f)-\phi(\mathcal{A}(f))\leq\frac{\Gamma}{2}\left[\mathcal{A}\left(f^2\right)-\mathcal{A}^2(f)\right], \tag{2.4}$$

provided $\phi\circ f,\ f,\ f^2\in L$.

Also, if $\gamma t^{-3}\leq\phi''(t)\leq\Gamma t^{-3}$, $t\in\mathring{I}$, then

$$\frac{\gamma}{2}\left[\mathcal{A}\left(f^{-1}\right)-\mathcal{A}^{-1}(f)\right]\leq\mathcal{A}(\phi\circ f)-\phi(\mathcal{A}(f))\leq\frac{\Gamma}{2}\left[\mathcal{A}\left(f^{-1}\right)-\mathcal{A}^{-1}(f)\right], \tag{2.5}$$

provided $\phi\circ f,\ f,\ f^{-1}\in L$.

Define $l(t) = t^2\phi''(t)$, $t \in \mathring{I} \subset (0,\infty)$ and assume that $-\infty < s \le l(t) \le S < \infty$ for $t \in \mathring{I}$. This implies that $st^{-2} \le \phi''(t) \le St^{-2}$ for $t \in \mathring{I}$, so if we take $m = s$, $M = \mathcal{A}$ and $\psi(t) = -\ln t$, then we have $m\psi''(t) \le \phi''(t) \le M\psi''(t)$, $t \in \mathring{I}$.

By Theorem 2 we can state the following:

Corollary 2. *If $\phi : I \subset (0,\infty) \to \mathbb{R}$ is twice differentiable and $st^{-2} \le \phi''(t) \le St^{-2}$ for $t \in \mathring{I}$, then*

$$s[\ln(\mathcal{A}(f)) - \mathcal{A}(\ln f)] \le \mathcal{A}(\phi \circ f) - \phi(\mathcal{A}(f)) \le S[\ln(\mathcal{A}(f)) - \mathcal{A}(\ln f)], \tag{2.6}$$

provided $\phi \circ f,\ f,\ \ln f \in L$.

Consider $k(t) = t\phi''(t)$, $t \in \mathring{I}$ and assume that $-\infty < \delta \le k(t) \le \Delta < \infty$ for $t \in \mathring{I}$. This implies that $\delta t^{-1} \le \phi''(t) \le t^{-1}\Delta$ for $t \in \mathring{I}$, so if we take $m = \delta$, $M = \Delta$ and $\psi(t) = t\ln t$, then we have $m\psi''(t) \le \phi''(t) \le M\psi''(t)$, $t \in \mathring{I}$.

By utilising Theorem 2 we can also state the following:

Corollary 3. *If $\phi : I \subset (0,\infty) \to \mathbb{R}$ is twice differentiable and $\delta t^{-1} \le \phi''(t) \le t^{-1}\Delta$ for $t \in \mathring{I}$, then*

$$\begin{aligned} \delta[\mathcal{A}(f\ln f) - \mathcal{A}(f)\ln(\mathcal{A}(f))] &\le \mathcal{A}(\phi \circ f) - \phi(\mathcal{A}(f)) \\ &\le \Delta[\mathcal{A}(f\ln f) - \mathcal{A}(f)\ln(\mathcal{A}(f))], \end{aligned} \tag{2.7}$$

provided $\phi \circ f,\ f,\ f\ln f \in L$.

A counterpart of Jessen's result was proved by Beesack and Pečarić in [4] for compact intervals $I = [\alpha,\beta]$.

Theorem 3. *Let $\phi : [\alpha,\beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to [\alpha,\beta]$ such that $\phi \circ f,\ f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$A(\phi \circ f) \le \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta). \tag{2.8}$$

In we obtained the following refinement and reverse of Beesack-Pečarić inequality (2.8):

Theorem 4. *Let $\psi : [\alpha,\beta] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (α,β), $f : E \to [\alpha,\beta]$ such that $\psi \circ f,\ f \in L$ and $\mathcal{A} : L \to \mathbb{R}$ be an isotonic linear and normalised functional. If $\phi : [\alpha,\beta] \to \mathbb{R}$ is twice differentiable and $m\psi''(t) \le \phi''(t) \le M\psi''(t)$, $t \in (\alpha,\beta)$, then*

$$\begin{aligned} & m\left[\frac{\beta - A(f)}{\beta - \alpha}\psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\psi(\beta) - A(\psi \circ f)\right] \\ &\le \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f). \\ &\le M\left[\frac{\beta - A(f)}{\beta - \alpha}\psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\psi(\beta) - A(\psi \circ f)\right]. \end{aligned} \tag{2.9}$$

We consider the p-logarithmic mean of two positive numbers given by

$$L_p(a,b) := \begin{cases} a & \text{if } \quad b = a, \\ \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } \quad a \ne b \end{cases}$$

and $p \in \mathbb{R} \diagdown \{-1,0\}$.

Corollary 4. *If* $-\infty < \gamma t^{p-2} \leq \phi''(t) \leq t^{p-2}\Gamma < \infty$*,* $t \in (\alpha, \beta) \subset (0, \infty)$*, then*

$$\begin{aligned} &\frac{\gamma}{p(p-1)} \left[pL_{p-1}^{p-1}(\alpha,\beta) A(f) - \alpha\beta(p-1) L_{p-2}^{p-2}(\alpha,\beta) - A(f^p) \right] \\ &\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \\ &\leq \frac{\Gamma}{p(p-1)} \left[pL_{p-1}^{p-1}(\alpha,\beta) A(f) - \alpha\beta(p-1) L_{p-2}^{p-2}(\alpha,\beta) - A(f^p) \right]. \end{aligned} \tag{2.10}$$

If $-\infty < k \leq \phi''(t) \leq K < \infty$, $t \in (\alpha, \beta) \subset (0, \infty)$, then by taking in (2.10) $p = 2$, we get

$$\begin{aligned} \frac{k}{2} A[(f-\alpha)(\beta - f)] &= \frac{k}{2}\left[(\alpha+\beta) A(f) - \alpha\beta - A(f^2)\right] \\ &\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \\ &\leq \frac{K}{2}\left[(\alpha+\beta) A(f) - \alpha\beta - A(f^2)\right] \\ &= \frac{K}{2} A[(f-\alpha)(\beta - f)]. \end{aligned} \tag{2.11}$$

Let $I(\cdot,\cdot)$ denotes the identric mean, i.e., we recall it

$$I(u,v) := \begin{cases} u \text{ if } v = u, \\ \\ \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}}, \quad v \neq u. \end{cases}$$

Corollary 5. *If* $-\infty < st^{-2} \leq \phi''(t) \leq t^{-2}S < \infty$ *for* $t \in (\alpha, \beta) \subset (0, \infty)$*, then*

$$\begin{aligned} &s\left[A(\ln f) + \ln\left[I\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)\right] + 1 - \frac{A(f)}{L(\alpha,\beta)}\right] \\ &\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \\ &\leq S\left[A(\ln f) + \ln\left[I\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)\right] + 1 - \frac{A(f)}{L(\alpha,\beta)}\right]. \end{aligned} \tag{2.12}$$

Assume that $G(\alpha,\beta) = \sqrt{ab}$ is the geometric mean and $L(\alpha,\beta)$ is the logarithmic mean, i.e., we recall it

$$L(\alpha,\beta) := \begin{cases} \alpha & \text{if } \beta = \alpha, \\ \\ \frac{\beta - \alpha}{\ln\beta - \ln\alpha} & \text{if } \beta \neq \alpha. \end{cases}$$

Corollary 6. *If* $-\infty < \delta t^{-1} \leq \phi''(t) \leq t^{-1}\Delta < \infty$ *for* $t \in (\alpha, \beta)$*, then*

$$\begin{aligned} &\delta\left[A(f)\ln I(\alpha,\beta) - \frac{G^2(\alpha,\beta)}{L(\alpha,\beta)} + A(f) - A(f\ln f)\right] \\ &\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \\ &\leq \Delta\left[A(f)\ln I(\alpha,\beta) - \frac{G^2(\alpha,\beta)}{L(\alpha,\beta)} + A(f) - A(f\ln f)\right]. \end{aligned} \tag{2.13}$$

3. Some Properties for Functions of Tensorial Products

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{3.1}$$

that holds for any $A, B, C, D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2 .$$

By induction and using (3.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{3.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{3.3}$$

for all $n \geq 0$.

We also observe that, by (3.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{3.4}$$

Moreover, for two natural numbers m, n we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{3.5}$$

By induction over m, we derive

$$(A_1 \otimes A_2 \otimes ... \otimes A_m)^n = A_1^n \otimes A_2^n \otimes ... \otimes A_m^n \text{ for natural } n \geq 0 \tag{3.6}$$

and

$$\begin{aligned} & A_1 \otimes A_2 \otimes ... \otimes A_m \\ & = (A_1 \otimes 1 \otimes ... \otimes 1)(1 \otimes A_2 \otimes ... \otimes 1) ... (1 \otimes 1 \otimes ... \otimes A_m) \end{aligned} \tag{3.7}$$

and the m operators $(A_1 \otimes 1 \otimes ... \otimes 1), (1 \otimes A_2 \otimes ... \otimes 1), ...$ and $(1 \otimes 1 \otimes ... \otimes A_m)$ are commutative between them.

We define for $A_i, B_i \in B(H), i \in \{1, ..., n\}, \bigotimes_{i=1}^{n} B_i := B_1 \otimes ... \otimes B_n$,

$$\hat{\mathbf{A}}_i := 1 \otimes ... \otimes A_i \otimes ... \otimes 1, \; i = 2, ..., n-1,$$

and

$$\hat{\mathbf{A}}_1 := A_1 \otimes 1 \otimes ... \otimes 1 \text{ while } \hat{\mathbf{A}}_n := 1 \otimes ... \otimes 1 \otimes A_n.$$

Basically $\hat{\mathbf{A}}_i$ is defined as a tensorial product of A_i in position $i = 1, ..., n$ and with 1 in the other positions.

We need the following identity for the tensorial product, see also [9]:

Lemma 1. *Assume A_i, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$, $i \in \{1, ..., m\}$ and with the spectral resolutions*

$$A_i = \int_{I_i} t_i dE(t_i), \; i \in \{1, ..., m\}.$$

Let f_i, $i \in \{1, ..., m\}$ be continuous on I_i and φ continuous on an interval K that contains the product of the intervals $f(I_1) ... f(I_m)$, then

$$\varphi\left(\bigotimes_{i=1}^{m} f_i(A_i)\right) = \int_{I_1} ... \int_{I_m} \varphi\left(\prod_{i=1}^{m} f_i(t_i)\right) dE(t_1) \otimes ... \otimes dE(t_m). \tag{3.8}$$

We also have

$$\varphi\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right)=\boldsymbol{\varphi}\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right) \tag{3.9}$$

for all $i=1,...,m$.

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t)=t^n$ with n any natural number.

Then, by (1.1) and (3.6) we obtain

$$\begin{aligned}&\int_{I_1}...\int_{I_m}\left[f_1\left(t_1\right)...f_m\left(t_m\right)\right]^n dE\left(t_1\right)\otimes...\otimes dE\left(t_m\right)\\&=\int_{I_1}...\int_{I_m}\left[f_1\left(t_1\right)\right]^n...\left[f_m\left(t_m\right)\right]^n dE\left(t_1\right)\otimes...\otimes dE\left(t_m\right)\\&=\left[f\left(A_1\right)\right]^n\otimes...\otimes\left[f_m\left(A_m\right)\right]^n=\left[f_1\left(A_1\right)\otimes...\otimes f_m\left(A_m\right)\right]^n,\end{aligned}$$

which shows that the identity (3.6) is valid for the power function.

This proves the identity (3.8)

By taking $f_j\equiv 1$ for $j=1,...,m$ and $j\neq i$ in (3.8) we get

$$\begin{aligned}\varphi\left(1\otimes...\otimes f_i\left(A_i\right)\otimes...\otimes 1\right)&=\int_{I_1}...\int_{I_m}\varphi\left(f_i\left(t_i\right)\right)dE\left(t_1\right)\otimes...\otimes dE\left(t_m\right)\\&=1\otimes...\otimes\varphi\left(f_i\left(A_i\right)\right)\otimes...\otimes 1,\end{aligned}$$

which proves (3.9). □

Corollary 7. *Assume A_i, $i\in\{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i)\subset I_i$ and f_i, $i\in\{1,...,m\}$ are continuous and positive on I_i, then*

$$\ln\left(\bigotimes_{i=1}^{m}f_i\left(A_i\right)\right)=\sum_{i=1}^{n}\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right). \tag{3.10}$$

Also

$$\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right)=\ln\left(\widehat{\mathbf{f}_i\left(\mathbf{A}_i\right)}\right) \tag{3.11}$$

for all $i=1,...,m$.

Proof. Assume that

$$A_i=\int_I t_i dE\left(t_i\right)$$

are the spectral resolutions for A_i, $i=1,...,m$.

We have for $\varphi(u) = \ln u$, $u > 0$, in (3.8) that

$$\begin{aligned}
&\ln (f_1 (A) \otimes ... \otimes f_m (A_m)) \\
&= \int_{I_1} ... \int_{I_m} \ln (f_1 (t_1) ... f_m (t_m)) \, dE (t_1) \otimes ... \otimes dE (t_m) \\
&= \int_{I_1} ... \int_{I_m} [\ln f_1 (t_1) + ... + \ln f_m (t_m)] \, dE (t_1) \otimes ... \otimes dE (t_m) \\
&= \int_{I_1} ... \int_{I_m} \ln f_1 (t_1) \, dE (t_1) \otimes ... \otimes dE (t_m) + ... \\
&+ \int_{I_1} ... \int_{I_m} \ln f_m (t_m) \, dE (t_1) \otimes ... \otimes dE (t_m) \\
&= (\ln f_1 (A_1)) \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes (\ln f_m (A_m))
\end{aligned}$$

and the identity (3.10) is proved. □

Corollary 8. *Assume A_i, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$ and f_i, $i \in \{1, ..., m\}$ are continuous on I_i, then for $r > 0$*

$$\left| \bigotimes_{i=1}^{m} f_i (A_i) \right|^r = \bigotimes_{i=1}^{m} |f_i (A_i)|^r \tag{3.12}$$

and

$$\left| \widehat{\mathbf{f}_i (\mathbf{A}_i)} \right|^r = \widehat{|\mathbf{f}_i (\mathbf{A}_i)|^r} \tag{3.13}$$

for all $i = 1, ..., m$.

Proof. From (3.8) we have for the function $\varphi(t) = |t|^r$ that

$$\begin{aligned}
&|f_1 (A_1) \otimes ... \otimes f_m (A_m)|^r \\
&= \int_{I_1} ... \int_{I_m} |f_1 (t_1) ... f_m (t_m)|^r \, dE (t_1) \otimes ... \otimes dE (t_m) \\
&= \int_{I_1} ... \int_{I_m} |f_1 (t_1)|^r ... |f_m (t_m)|^r \, dE (t_1) \otimes ... \otimes dE (t_m) \\
&= |f_1 (A_1)|^r \otimes ... \otimes |f_m (A_m)|^r ,
\end{aligned}$$

which proves (3.12).

The identity (3.13) follows in a similar way. □

Corollary 9. *Assume A_i, $i \in \{1, ..., m\}$ are positive operators and $q_i \geq 0$, $i \in \{1, ..., m\}$, then*

$$\varphi \left(\bigotimes_{i=1}^{m} A_i^{q_i} \right) = \int_{I_1} ... \int_{I_m} \varphi \left(\prod_{i=1}^{m} t_i^{q_i} \right) dE (t_1) \otimes ... \otimes dE (t_m) . \tag{3.14}$$

We also have the additive result [9]:

Lemma 2. *Assume A_i, $i \in \{1, ..., m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I_i$, $i \in \{1, ..., m\}$ and with the spectral resolutions*

$$A_i = \int_{I_i} t_i dE (t_i) , \; i \in \{1, ..., m\} .$$

Let g_i, $i \in \{1,...,m\}$ be continuous on I_i and ψ continuous on an interval K that contains the sum of the intervals $g(I_1)+...+g(I_m)$, then

$$\psi\left(\sum_{i=1}^{m}\widehat{\mathbf{g}_i(\mathbf{A}_i)}\right)=\int_{I_1}...\int_{I_m}\psi\left(\sum_{i=1}^{m}g_i(t_i)\right)dE(t_1)\otimes...\otimes dE(t_m). \tag{3.15}$$

Proof. Let f_i, continuous, positive and such that $g_i(t_i)=\ln f_i(t_i)$, $t_i\in I_i$, $i\in\{1,...,m\}$. Then

$$\sum_{i=1}^{m}g_i(t_i)=\sum_{i=1}^{m}\ln f_i(t_i)=\ln\left(\prod_{i=1}^{m}f_i(t_i)\right).$$

By (3.8) we get for $\varphi=\psi\circ\ln$ that

$$\begin{aligned}
&\int_{I_1}...\int_{I_m}\psi\left(\sum_{i=1}^{m}g_i(t_i)\right)dE(t_1)\otimes...\otimes dE(t_m)\\
&=\int_{I_1}...\int_{I_m}\psi\left(\ln\left(\prod_{i=1}^{m}f_i(t_i)\right)\right)dE(t_1)\otimes...\otimes dE(t_m)\\
&=\int_{I_1}...\int_{I_m}(\psi\circ\ln)\left(\prod_{i=1}^{m}f_i(t_i)\right)dE(t_1)\otimes...\otimes dE(t_m)\\
&=(\psi\circ\ln)\left(\bigotimes_{i=1}^{m}f_i(A_i)\right)=\psi\left(\ln\left(\bigotimes_{i=1}^{m}f_i(A_i)\right)\right).
\end{aligned}$$

By (3.10) we also have

$$\ln\left(\bigotimes_{i=1}^{m}f_i(A_i)\right)=\sum_{i=1}^{n}\widehat{\ln(\mathbf{f}_i(\mathbf{A}_i))}=\sum_{i=1}^{m}\widehat{\mathbf{g}_i(\mathbf{A}_i)}$$

and the identity (3.15) is obtained. □

Corollary 10. *Assume that A_i, $i\in\{1,...,m\}$ and g_i, $i\in\{1,...,m\}$ are as in Lemma 2 and $r>0$, then*

$$\left|\sum_{i=1}^{m}\widehat{\mathbf{g}_i(\mathbf{A}_i)}\right|^r=\int_{I_1}...\int_{I_m}\left|\sum_{i=1}^{m}g_i(t_i)\right|^r dE(t_1)\otimes...\otimes dE(t_m). \tag{3.16}$$

Also, if we take $\psi=\exp$, then we get

$$\begin{aligned}
\exp\left(\sum_{i=1}^{m}\widehat{\mathbf{g}_i(\mathbf{A}_i)}\right)&=\int_{I_1}...\int_{I_m}\exp\left(\sum_{i=1}^{m}g_i(t_i)\right)dE(t_1)\otimes...\otimes dE(t_m)\\
&=\bigotimes_{i=1}^{m}\exp\left[g_i(A_i)\right].
\end{aligned} \tag{3.17}$$

The case of convex combination is as follows:

Corollary 11. *Assume A_i, $i\in\{1,...,m\}$ are selfadjoint operators with $\operatorname{Sp}(A_i)\subset I$, $i\in\{1,...,m\}$ and with the spectral resolutions*

$$A_i=\int_I t_i dE(t_i),\ i\in\{1,...,m\}.$$

If $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$, and ψ continuous on I, then

$$\psi\left(\sum_{i=1}^{m} p_i \widehat{\mathbf{A}_i}\right) = \int_I \dots \int_I \psi\left(\sum_{i=1}^{n} p_i t_i\right) dE(t_1) \otimes \dots \otimes dE(t_m). \tag{3.18}$$

Follows by (3.4) for $g_i(t_i) = p_i t_i$, $i \in \{1, \dots, m\}$.

4. Main Results

Our first main result is as follows:

Theorem 5. *Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$. If $\phi : I \to \mathbb{R}$ is twice differentiable and $m\psi''(t) \leq \phi''(t) \leq M\psi''(t)$, $t \in \mathring{I}$, for some constants $m < M$, A_i, $i \in \{1, \dots, n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I$, $i \in \{1, \dots, n\}$ and $w_i \geq 0$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^{n} w_i = 1$, then*

$$\begin{aligned} m\left[\sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\right] &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \\ &\leq M\left[\sum_{i=1}^{n} w_i \widehat{\boldsymbol{\psi}(\mathbf{A}_i)} - \psi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\right]. \end{aligned} \tag{4.1}$$

Proof. If we write (3.2) for the weighted dicrete sum, then we get

$$\begin{aligned} m\left[\sum_{i=1}^{n} w_i \psi(t_i) - \psi\left(\sum_{i=1}^{n} w_i t_i\right)\right] &\leq \sum_{i=1}^{n} w_i \phi(t_i) - \phi\left(\sum_{i=1}^{n} w_i t_i\right) \\ &\leq M\left[\sum_{i=1}^{n} w_i \psi(t_i) - \psi\left(\sum_{i=1}^{n} w_i t_i\right)\right], \end{aligned} \tag{4.2}$$

for all $t_i \in I$, $i \in \{1, \dots, n\}$.

Assume that we have the spectral resolutions

$$A_i = \int_I t_i dE(t_i), \ i \in \{1, \dots, n\}.$$

Now, if we take the integral $\int_I \dots \int_I$ over $dE(t_1) \otimes \dots \otimes dE(t_n)$ in (4.2), then we get

$$\begin{aligned} m\Bigg[\sum_{i=1}^{n} w_i \int_I \dots \int_I \psi(t_i) \, dE(t_1) \otimes \dots \otimes dE(t_n) \\ - \int_I \dots \int_I \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes \dots \otimes dE(t_n)\Bigg] \end{aligned} \tag{4.3}$$

$$\leq \sum_{i=1}^{n} w_i \int_I ... \int_I \phi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n)$$
$$- \int_I ... \int_I \phi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n)$$
$$\leq M \left[\sum_{i=1}^{n} w_i \int_I ... \int_I \psi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n)\right.$$
$$\left. - \int_I ... \int_I \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n)\right].$$

Since, by (1.1)

$$\int_I ... \int_I \psi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n)$$
$$= 1 \otimes ... \otimes \int_I \psi(t_i)\, dE(t_i) \otimes ... \otimes 1 = \widehat{\boldsymbol{\psi}(\mathbf{A}_i)},$$

by (3.18)

$$\int_I ... \int_I \psi\left(\sum_{i=1}^{n} w_i t_i\right) dE(t_1) \otimes ... \otimes dE(t_n) = \psi\left(\sum_{i=1}^{m} w_i \widehat{\mathbf{A}_i}\right)$$

and the similar equalities for the function ϕ, then by (4.3) we get (4.1). □

Corollary 12. *If $\phi : I \subset (0, \infty) \to \mathbb{R}$ is twice differentiable and $\gamma t^{p-2} \leq \phi''(t) \leq \Gamma t^{p-2}$, $t \in \mathring{I}$, $p \in (-\infty, 0) \cup (1, \infty)$, then*

$$\frac{\gamma}{p(p-1)} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^p} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^p\right] \tag{4.4}$$
$$\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)$$
$$\leq \frac{\Gamma}{p(p-1)} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^p} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^p\right].$$

If $\gamma \leq \phi''(t) \leq \Gamma$, $t \in \mathring{I}$, then from (4.4) for $p = 2$ we get

$$\frac{\gamma}{2} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^2\right] \leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \tag{4.5}$$
$$\leq \frac{\Gamma}{2} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^2\right].$$

If $\gamma t^{-3} \leq \phi''(t) \leq \Gamma t^{-3}$, $t \in \mathring{I}$, then for $p = -1$ in (4.4) we get

$$\frac{\gamma}{2}\left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} - \left(\sum_{i=1}^{m} w_i \widehat{\mathbf{A}_i}\right)^{-1}\right] \leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \leq \frac{\Gamma}{2}\left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)^{-1}\right]. \tag{4.6}$$

Corollary 13. *If $\phi : I \subset (0, \infty) \to \mathbb{R}$ is twice differentiable and $st^{-2} \leq \phi''(t) \leq St^{-2}$ for $t \in \mathring{I}$, then*

$$s\left[\ln\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) - \ln\left(\bigotimes_{i=1}^{n} A_i^{w_i}\right)\right] \leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \leq S\left[\ln\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) - \ln\left(\bigotimes_{i=1}^{n} A_i^{w_i}\right)\right]. \tag{4.7}$$

Corollary 14. *If $\phi : I \subset (0, \infty) \to \mathbb{R}$ is twice differentiable and $\delta t^{-1} \leq \phi''(t) \leq t^{-1}\Delta$ for $t \in \mathring{I}$, then*

$$\begin{aligned} &m\left[\sum_{i=1}^{n} w_i \mathbf{A}_i \widehat{\ln(\mathbf{A}_i)} - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \ln\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\right] \\ &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \\ &\leq M\left[\sum_{i=1}^{n} w_i \mathbf{A}_i \widehat{\ln(\mathbf{A}_i)} - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \ln\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right)\right]. \end{aligned} \tag{4.8}$$

Corollary 15. *If $\phi : I \to \mathbb{R}$ is twice differentiable and $\theta \exp t \leq \phi''(t) \leq \Theta \exp t$, $t \in \mathring{I}$, for some constants $\theta < \Theta$, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then*

$$\begin{aligned} &\theta\left[\sum_{i=1}^{n} w_i \widehat{\exp(\mathbf{A}_i)} - \bigotimes_{i=1}^{n} \exp(w_i A_i)\right] \\ &\leq \sum_{i=1}^{n} w_i \widehat{\boldsymbol{\phi}(\mathbf{A}_i)} - \phi\left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}\right) \\ &\leq \Theta\left[\sum_{i=1}^{n} w_i \widehat{\exp(\mathbf{A}_i)} - \bigotimes_{i=1}^{n} \exp(w_i A_i)\right]. \end{aligned} \tag{4.9}$$

We also have:

Theorem 6. *Let $\psi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on (α, β). If $\phi : [\alpha, \beta] \to \mathbb{R}$ is twice differentiable and $m\psi''(t) \leq \phi''(t) \leq M\psi''(t)$, $t \in (\alpha, \beta)$, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators with $\operatorname{Sp}(A_i) \subset [\alpha, \beta]$, $i \in$*

$\{1,...,n\}$ *and* $w_i \geq 0,\ i \in \{1,...,n\}$ *with* $\sum_{i=1}^n w_i = 1,$ *then*

$$m\left[\psi(\alpha)\frac{\beta - \sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha} + \psi(\beta)\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i} - \alpha}{\beta-\alpha} - \sum_{i=1}^n w_i\widehat{\boldsymbol{\psi}(\mathbf{A}_i)}\right] \tag{4.10}$$
$$\leq \phi(\alpha)\frac{\beta - \sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha} + \phi(\beta)\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i} - \alpha}{\beta-\alpha} - \sum_{i=1}^n w_i\widehat{\boldsymbol{\phi}(\mathbf{A}_i)}$$
$$\leq M\left[\psi(\alpha)\frac{\beta - \sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha} + \psi(\beta)\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i} - \alpha}{\beta-\alpha} - \sum_{i=1}^n w_i\widehat{\boldsymbol{\psi}(\mathbf{A}_i)}\right].$$

Proof. If we write (2.9) for the weighted dicrete sum, then we get

$$m\left[\frac{\beta - \sum_{i=1}^n w_i t_i}{\beta-\alpha}\psi(\alpha) + \frac{\sum_{i=1}^n w_i t_i - \alpha}{\beta-\alpha}\psi(\beta) - \sum_{i=1}^n w_i\psi(t_i)\right] \tag{4.11}$$
$$\leq \frac{\beta - \sum_{i=1}^n w_i t_i}{\beta-\alpha}\phi(\alpha) + \frac{\sum_{i=1}^n w_i t_i - \alpha}{\beta-\alpha}\phi(\beta) - \sum_{i=1}^n w_i\phi(t_i)$$
$$\leq M\left[\frac{\beta - \sum_{i=1}^n w_i t_i}{\beta-\alpha}\psi(\alpha) + \frac{\sum_{i=1}^n w_i t_i - \alpha}{\beta-\alpha}\psi(\beta) - \sum_{i=1}^n w_i\psi(t_i)\right]$$

for all $t_i \in I,\ i \in \{1,...,n\}$.

Assume that we have the spectral resolutions

$$A_i = \int_I t_i dE(t_i),\ i \in \{1,...,n\}.$$

Now, if we take the integral $\int_I ... \int_I$ over $dE(t_1) \otimes ... \otimes dE(t_n)$ in (4.11), then we get

$$m\left[\psi(\alpha)\frac{\beta - \int_I ... \int_I \sum_{i=1}^n w_i t_i dE(t_1) \otimes ... \otimes dE(t_n)}{\beta-\alpha}\right.$$
$$+\psi(\beta)\frac{\sum_{i=1}^n \int_I ... \int_I \sum_{i=1}^n w_i t_i dE(t_1) \otimes ... \otimes dE(t_n) - \alpha}{\beta-\alpha}$$
$$\left.-\int_I ... \int_I \sum_{i=1}^n w_i\psi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n)\right]$$

$$\leq \phi(\alpha)\frac{\beta - \int_I ... \int_I \sum_{i=1}^n w_i t_i dE(t_1) \otimes ... \otimes dE(t_n)}{\beta-\alpha}$$
$$+\phi(\beta)\frac{\int_I ... \int_I \sum_{i=1}^n w_i t_i dE(t_1) \otimes ... \otimes dE(t_n) - \alpha}{\beta-\alpha}$$
$$-\int_I ... \int_I \sum_{i=1}^n w_i\phi(t_i)\, dE(t_1) \otimes ... \otimes dE(t_n)$$

$$\leq M\left[\frac{\beta-\int_I\cdots\int_I\sum_{i=1}^n w_i t_i dE(t_1)\otimes\ldots\otimes dE(t_n)}{\beta-\alpha}\psi(\alpha)\right.$$
$$\frac{\sum_{i=1}^n\int_I\cdots\int_I\sum_{i=1}^n w_i t_i dE(t_1)\otimes\ldots\otimes dE(t_n)-\alpha}{\beta-\alpha}\psi(\beta)$$
$$\left.-\int_I\cdots\int_I\sum_{i=1}^n w_i\psi(t_i)\,dE(t_1)\otimes\ldots\otimes dE(t_n)\right],$$

which, as in the proof of Theorem 5 gives (4.10). □

Corollary 16. *If* $-\infty<k\leq\phi''(t)\leq K<\infty$, $t\in(\alpha,\beta)\subset(0,\infty)$, *then*

$$\frac{k}{2}\left[\alpha^2\frac{\beta-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha}+\beta^2\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}-\alpha}{\beta-\alpha}-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i^2}\right] \tag{4.12}$$
$$\leq\phi(\alpha)\frac{\beta-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha}+\phi(\beta)\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}-\alpha}{\beta-\alpha}-\sum_{i=1}^n w_i\widehat{\boldsymbol{\phi}(\mathbf{A}_i)}$$
$$\leq\frac{K}{2}\left[\alpha^2\frac{\beta-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha}+\beta^2\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}-\alpha}{\beta-\alpha}-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i^2}\right].$$

By employing the inequalities from Corollaries 4, 5 and 6 for the weighted sum, one can derive other similar inequalities. The details are omitted.

We observe that the inequality (4.12) is equivalent to

$$\frac{k}{2}\sum_{i=1}^n w_i\left(\widehat{\mathbf{A}_i}-\alpha\right)\left(\beta-\widehat{\mathbf{A}_i}\right) \tag{4.13}$$
$$\leq\phi(\alpha)\frac{\beta-\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}}{\beta-\alpha}+\phi(\beta)\frac{\sum_{i=1}^n w_i\widehat{\mathbf{A}_i}-\alpha}{\beta-\alpha}-\sum_{i=1}^n w_i\widehat{\boldsymbol{\phi}(\mathbf{A}_i)}$$
$$\leq\frac{K}{2}\sum_{i=1}^n w_i\left(\widehat{\mathbf{A}_i}-\alpha\right)\left(\beta-\widehat{\mathbf{A}_i}\right).$$

5. Some Examples

Consider the function $\phi(t)=t^p$, $p\in(-\infty,0)\cup[1,\infty)$. We observe that $\phi''(t)=p(p-1)t^{p-2}$, $t>0$. For $0<m<M$, put

$$B_{m,M,p}:=p(p-1)\times\begin{cases}M^{p-2}\text{ if }p\geq 2\\ \\ m^{p-2}\text{ if }(-\infty,0)\cup[1,2)\end{cases}$$

and

$$b_{m,M,p}:=p(p-1)\times\begin{cases}m^{p-2}\text{ if }p\geq 2\\ \\ M^{p-2}\text{ if }(-\infty,0)\cup[1,2).\end{cases}$$

If $\operatorname{Sp}(A_i) \subset [m, M] \subset (0, \infty)$, $i \in \{1, ..., n\}$ and $w_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then by (4.5)

$$\frac{1}{2} b_{m,M,p} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^2 \right] \leq \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^p} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^p \leq \frac{1}{2} B_{m,M,p} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^2 \right], \tag{5.1}$$

while from (4.13) we get

$$\frac{1}{2} b_{m,M,p} \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i} - m \right) \left(M - \widehat{\mathbf{A}_i} \right) \leq \frac{M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}}{M - m} m^p + \frac{\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m}{M - m} M^p - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^p} \leq \frac{1}{2} B_{m,M,p} \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i} - m \right) \left(M - \widehat{\mathbf{A}_i} \right). \tag{5.2}$$

For $p = -1$, we get

$$M^{-3} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^2 \right] \leq \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^{-1} \leq m^{-3} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^2 \right] \tag{5.3}$$

and

$$M^{-3} \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i} - m \right) \left(M - \widehat{\mathbf{A}_i} \right) \leq m^{-1} \frac{M - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i}}{M - m} + M^{-1} \frac{\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} - m}{M - m} - \sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^{-1}} \leq m^{-3} \sum_{i=1}^{n} w_i \left(\widehat{\mathbf{A}_i} - m \right) \left(M - \widehat{\mathbf{A}_i} \right). \tag{5.4}$$

Consider the function $\phi(t) = -\ln t$, $t > 0$. Then $\phi''(t) = \frac{1}{t^2} \in \left[\frac{1}{M^2}, \frac{1}{m^2} \right]$ for $t \in [m, M]$. Then by (4.5) we get

$$\frac{1}{2M^2} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^2 \right] \leq \ln \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right) - \sum_{i=1}^{n} w_i \widehat{\ln(\mathbf{A}_i)} \leq \frac{1}{2m^2} \left[\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i \widehat{\mathbf{A}_i} \right)^2 \right], \tag{5.5}$$

while from (4.12)

$$\begin{aligned}(5.6)\qquad & \frac{1}{2M^2}\sum_{i=1}^{n} w_i\left(\widehat{\mathbf{A}}_i - m\right)\left(M-\widehat{\mathbf{A}}_i\right)\\ & \leq \sum_{i=1}^{n} w_i \widehat{\ln(\mathbf{A}_i)} - \ln M \frac{\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i - m}{M-m} - \ln m \frac{M-\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i}{M-m}\\ & \leq \frac{1}{2m^2}\sum_{i=1}^{n} w_i\left(\widehat{\mathbf{A}}_i - m\right)\left(M-\widehat{\mathbf{A}}_i\right).\end{aligned}$$

Consider the function $\phi(t) = \exp t$, $t \in \mathbb{R}$. Then $\phi''(t) = \exp t \in [\exp m, \exp M]$ for $t \in [m, M]$. From (4.5) we then get

$$\begin{aligned}(5.7)\qquad & \frac{1}{2}\exp(m)\left[\sum_{i=1}^{n} w_i\widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i\right)^2\right]\\ & \leq \sum_{i=1}^{n} w_i\widehat{\exp(\mathbf{A}_i)} - \exp\left(\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i\right)\\ & \leq \frac{1}{2}\exp(M)\left[\sum_{i=1}^{n} w_i\widehat{\mathbf{A}_i^2} - \left(\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i\right)^2\right],\end{aligned}$$

while from (4.13) we get

$$\begin{aligned}(5.8)\qquad & \frac{1}{2}\exp(m)\sum_{i=1}^{n} w_i\left(\widehat{\mathbf{A}}_i - m\right)\left(M-\widehat{\mathbf{A}}_i\right)\\ & \leq \exp(m)\frac{M-\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i}{M-m} + \exp(M)\frac{\sum_{i=1}^{n} w_i\widehat{\mathbf{A}}_i - m}{M-m} - \sum_{i=1}^{n} w_i\widehat{\exp(\mathbf{A}_i)}\\ & \leq \frac{1}{2}\exp(M)\sum_{i=1}^{n} w_i\left(\widehat{\mathbf{A}}_i - m\right)\left(M-\widehat{\mathbf{A}}_i\right).\end{aligned}$$

6. Integral Inequalities

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

If $\gamma \leq \phi''(t) \leq \Gamma$, $t \in \mathring{I}$, then from (4.5) for $n = 2$ we get for A, B with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$ and $\nu \in [0,1]$ that

$$\begin{aligned}(6.1)\qquad & \frac{\gamma}{2}\left[(1-\nu)A^2\otimes 1 + \nu 1\otimes B^2 - ((1-\nu)A\otimes 1 + \nu 1\otimes B)^2\right]\\ & \leq (1-\nu)\phi(A)\otimes 1 + \nu 1\otimes\phi(B) - \phi((1-\nu)A\otimes 1 + \nu 1\otimes B)\\ & \leq \frac{\Gamma}{2}\left[(1-\nu)A^2\otimes 1 + \nu 1\otimes B^2 - ((1-\nu)A\otimes 1 + \nu 1\otimes B)^2\right].\end{aligned}$$

Observe that, since $A\otimes 1$ commutes with $1\otimes B$, then

$$\begin{aligned}
&(1-\nu)A^{2}\otimes 1+\nu 1\otimes B^{2}-((1-\nu)A\otimes 1+\nu 1\otimes B)^{2}\\
&=(1-\nu)A^{2}\otimes 1+\nu 1\otimes B^{2}\\
&-(1-\nu)^{2}(A\otimes 1)^{2}-2(1-\nu)\nu(A\otimes 1)(1\otimes B)+\nu^{2}(1\otimes B)^{2}\\
&=(1-\nu)\nu(A\otimes 1)^{2}+(1-\nu)\nu(1\otimes B)^{2}\\
&-2(1-\nu)\nu(A\otimes 1)(1\otimes B)\\
&=(1-\nu)\nu(A\otimes 1-1\otimes B)^{2}
\end{aligned}$$

and we get by (6.1) that

$$\begin{aligned}
&\frac{\gamma}{2}(1-\nu)\nu(A\otimes 1-1\otimes B)^{2}\\
&\leq(1-\nu)\phi(A)\otimes 1+\nu 1\otimes\phi(B)-\phi((1-\nu)A\otimes 1+\nu 1\otimes B)\\
&\leq\frac{\Gamma}{2}(1-\nu)\nu(A\otimes 1-1\otimes B)^{2}.
\end{aligned}\tag{6.2}$$

We have:

Proposition 1. *Assume that ϕ is twice differentiable and $\gamma\leq\phi''(t)\leq\Gamma$, $t\in\mathring{I}$. Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be continuous fields of operators in $B(H)$ with $\operatorname{Sp}(A_t)$, $\operatorname{Sp}(B_t)\subset I$, $t\in\Omega$, then for $\nu\in[0,1]$,*

$$\begin{aligned}
&\frac{\gamma}{2}(1-\nu)\nu\left[\left(\int_{\Omega}A_t^{2}d\mu(t)\right)\otimes 1+1\otimes\left(\int_{\Omega}B_s^{2}d\mu(s)\right)\right.\\
&\left.-2\int_{\Omega}A_t d\mu(t)\otimes\int_{\Omega}B_s d\mu(s)\right]\\
&\leq(1-\nu)\int_{\Omega}\phi(A_t)d\mu(t)\otimes 1+\nu 1\otimes\int_{\Omega}\phi(B_s)\\
&-\int_{\Omega}\int_{\Omega}\phi((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)d\mu(t)d\mu(s)\\
&\leq\frac{\Gamma}{2}(1-\nu)\nu\left[\int_{\Omega}A_t^{2}d\mu(t)\otimes 1+1\otimes\int_{\Omega}B_s^{2}d\mu(s)\right.\\
&\left.-2\int_{\Omega}A_t d\mu(t)\otimes\int_{\Omega}B_s d\mu(s)\right].
\end{aligned}\tag{6.3}$$

Proof. From (6.2) we get

$$\begin{aligned}
&\frac{\gamma}{2}(1-\nu)\nu\left(A_t^{2}\otimes 1+1\otimes B_s^{2}-2A_t\otimes B_s\right)\\
&\leq(1-\nu)\phi(A_t)\otimes 1+\nu 1\otimes\phi(B_s)-\phi((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)\\
&\leq\frac{\Gamma}{2}(1-\nu)\nu\left(A_t^{2}\otimes 1+1\otimes B_s^{2}-2A_t\otimes B_s\right)
\end{aligned}\tag{6.4}$$

for all $t,s\in\Omega$.

Fix $s \in \Omega$. If we take the $\int_\Omega$ over $d\mu(t)$ in (6.4), then we get

$$\begin{aligned}(6.5)\qquad &\frac{\gamma}{2}(1-\nu)\nu\left(\left(\int_\Omega A_t^2 d\mu(t)\right)\otimes 1 + 1\otimes B_s^2 - 2\int_\Omega A_t d\mu(t)\otimes B_s\right)\\ &\le (1-\nu)\int_\Omega \phi(A_t)\,d\mu(t)\otimes 1 + \nu 1\otimes\phi(B_s)\\ &-\int_\Omega \phi((1-\nu)A_t\otimes 1 + \nu 1\otimes B_s)\,d\mu(t)\\ &\le \frac{\Gamma}{2}(1-\nu)\nu\left(\int_\Omega A_t^2 d\mu(t)\otimes 1 + 1\otimes B_s^2 - 2\int_\Omega A_t d\mu(t)\otimes B_s\right)\end{aligned}$$

for all $s \in \Omega$.

Further, if we take the $\int_\Omega$ over $d\mu(s)$ in (6.5), then we get the desired result (6.3). □

We have the representation

$$X \circ Y = \mathcal{U}^*(X \otimes Y)\,\mathcal{U}$$

where $\mathcal{U}: H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

Corollary 17. *With the assumptions of Proposition 1 we have the inequalities*

$$\begin{aligned}(6.6)\qquad &\gamma(1-\nu)\nu\\ &\times\left[\int_\Omega\left(\frac{A_t^2+B_t^2}{2}\right)d\mu(t)\circ 1 - \int_\Omega A_t d\mu(t)\circ\int_\Omega B_s d\mu(s)\right]\\ &\le \int_\Omega [(1-\nu)\phi(A_t)+\nu\phi(B_s)]\,d\mu(t)\circ 1\\ &-\int_\Omega\int_\Omega \mathcal{U}^*\phi((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)\,\mathcal{U}d\mu(t)\,d\mu(s)\\ &\le \Gamma(1-\nu)\nu\\ &\times\left[\int_\Omega\left(\frac{A_t^2+B_t^2}{2}\right)d\mu(t)\circ 1 - \int_\Omega A_t d\mu(t)\circ\int_\Omega B_s d\mu(s)\right].\end{aligned}$$

The proof follows by (6.3) by taking $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right and performing the required calculation.

Remark 1. *If we consider the exponential function $\phi(u) = \exp u$, then we get*

$$\begin{aligned}&\exp((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)\\ &=\exp[(1-\nu)A_t\otimes 1]\exp[\nu 1\otimes B_s]\\ &=[(\exp(1-\nu)A_t)\otimes 1](1\otimes\exp(\nu B_s)) = (\exp(1-\nu)A_t)\otimes\exp(\nu B_s).\end{aligned}$$

Therefore

$$\begin{aligned}&\mathcal{U}^*\exp((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)\,\mathcal{U}\\ &=\mathcal{U}^*[(\exp(1-\nu)A_t)\otimes\exp(\nu B_s)]\,\mathcal{U} = (\exp(1-\nu)A_t)\circ\exp(\nu B_s)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\int_\Omega \mathcal{U}^*\exp((1-\nu)A_t\otimes 1+\nu 1\otimes B_s)\,\mathcal{U}d\mu(t)\,d\mu(s)\\ &=\int_\Omega(\exp(1-\nu)A_t)\,d\mu(t)\circ\int_\Omega\exp(\nu B_s)\,d\mu(s).\end{aligned}$$

In this case the inequality (6.6) becomes

$$\begin{aligned}(1-\nu)\nu\exp(m) & \\ \times\left[\int_\Omega\left(\frac{A_t^2+B_t^2}{2}\right)d\mu(t)\circ 1-\int_\Omega A_t d\mu(t)\circ\int_\Omega B_t d\mu(t)\right] & \\ \leq\int_\Omega\left[(1-\nu)\exp(A_t)+\nu\exp(B_t)\right]d\mu(t)\circ 1 & \\ -\int_\Omega\left(\exp(1-\nu)A_t\right)d\mu(t)\circ\int_\Omega\exp(\nu B_t)\,d\mu(t) & \\ \leq(1-\nu)\nu\exp(M) & \\ \times\left[\int_\Omega\left(\frac{A_t^2+B_t^2}{2}\right)d\mu(t)\circ 1-\int_\Omega A_t d\mu(t)\circ\int_\Omega B_t d\mu(s)\right], & \end{aligned}\tag{6.7}$$

provided that $\operatorname{Sp}(A_t),\ \operatorname{Sp}(B_t)\subset[m,M],\ t\in\Omega.$

Finally, if we take $B_t=A_t,\ t\in\Omega$, *then we get the simpler inequality*

$$\begin{aligned}(1-\nu)\nu\exp(m) & \\ \times\left[\int_\Omega A_t^2 d\mu(t)\circ 1-\int_\Omega A_t d\mu(t)\circ\int_\Omega A_t d\mu(t)\right] & \\ \leq\int_\Omega\exp(A_t)\,d\mu(t)\circ 1-\int_\Omega\left(\exp(1-\nu)A_t\right)d\mu(t)\circ\int_\Omega\exp(\nu A_t)\,d\mu(t) & \\ \leq(1-\nu)\nu\exp(M) & \\ \times\left[\int_\Omega A_t^2 d\mu(t)\circ 1-\int_\Omega A_t d\mu(t)\circ\int_\Omega A_t d\mu(t)\right], & \end{aligned}\tag{6.8}$$

provided that $\operatorname{Sp}(A_t)\subset[m,M],\ t\in\Omega.$

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] P. R. Beesack and J. E. Pečarić, On Jessen's inequality for convex functions, *J. Math. Anal. Appl.,* **110** (1985), 536-552.

[5] S. S. Dragomir, On a reverse of Jessen's inequality for isotonic linear functionals, *Journal of Inequalities in Pure and Applied Mathematics*, Volume **2** (2001), Issue 3, Article 36.S.S.

[6] S. S. Dragomir, On the Lupaş-Beesack-Pečarić inequality for isotonic linear functionals, *Nonlinear Funct. Anal. & Appl.,* **7**(2)(2002), 285-298.

[7] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74** (3) (2006), 471-476.

[8] S. S. Dragomir, Reverses of Jensen's integral inequality and applications: a survey of recent results, T. M. Rassias (ed.), *Applications of Nonlinear Analysis, Springer Optimization and Its Applications 134*, pp. 193-263. `https://doi.org/10.1007/978-3-319-89815-5_8`

[9] S. S. Dragomir, Refinements and reverses of tensorial arithmetic mean-geometric mean inequalities for selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art.

[10] S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić, On Jessen's and related inequalities for isotonic sublinear functionals, *Acta. Sci. Math.* (Szeged), **61** (1995), 373-382.

[11] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[12] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[13] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[14] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[15] H. Kober, On the arithmetic and geometric means and on Hölder's inequality. *Proc. American Math. Soc.* **9** (1958), 452-459.

[16] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa