# Fractional Calculus between Banach spaces along with Ostrowski and Grüss type inequalities

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**Abstract**

Here we present a fractional calculus Caputo type for functions between Banach spaces. Left and right fractional derivatives are defined on a segment of a Banach space, then we expand. We apply the above to new Ostrowski and Grüss type inequalities at very abstract level.

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## 1 Introduction

The problem of estimating the difference of a value of a function from its average is a top one. The answer to it are the Ostrowski type inequalities. Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals. The problem of estimating the difference between the average of a product of functions from the product of their averages is also a very important one. The answer to it are the Grüss type inequalities. Grüss type inequalities are very useful among others in Probability for estimating expected values, etc. There exists a huge literature on Ostrowski and Grüss type inequalities to all possible directions. Mathematical community is very much interested to these inequalities due to their applications. So here we derive very general fractional Ostrowski and Grüss type inequalities on a very abstract level. Our functions are between Banach spaces, and we develop first the related abstract fractional calculus, which is based on a Banach space segment. Great sources to support our goal are the books [1], [4].


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We are motivated by the following results:

**Theorem 1** *(1938. Ostrowski [3]) Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on $(a,b)$, i.e., $\|f'\|_{\infty}^{\sup}:=\sup_{t\in(a,b)}|f'(t)|<+\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,\|f'\|_{\infty}^{\sup},\tag{1}$$

*for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.*

**Theorem 2** *(1935, Grüss [2]) Let $f,g$ be integrable functions from $[a,b]$ into $\mathbb{R}$, that satisfy the conditions*

$$m\leq f(x)\leq M,\ \ n\leq g(x)\leq N,\ \ x\in[a,b],$$

*where $m,M,n,N\in\mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right|\tag{2}$$

$$\leq\frac{1}{4}(M-m)(N-n).$$

## 2 Fractionality between a pair of Banach spaces

We make

**Remark 3** *Throughout this article let $(X,\|\cdot\|_1)$ and $(Y,\|\cdot\|_2)$ be Banach spaces. Here $X^j$ denotes the $j$-fold product space $\underbrace{X\times X\times...\times X}_{j}$ endowed with the max-norm $\|x\|_{X^j}:=\max_{1\leq\lambda\leq j}\|x_\lambda\|_1$, where $x:=(x_1,...,x_j)\in X^j$.*

*Let the space of $L_j:=L_j\left(X^j,Y\right)$ of all $j$-multilinear continuous maps $h:X^j\to Y$, $j=1,...,m$, which is a Banach space with norm*

$$\|h\|=\|h\|_{L_j}:=\sup_{\left(\|x\|_{X^j}=1\right)}\|h(x)\|_2=\sup\frac{\|h(x)\|_2}{\|x_1\|_1...\|x_j\|_1}.\tag{3}$$

*Let $M$ be a non-empty convex and compact set of $X$ and $x_0\in M$ is fixed.*

*Let $O$ be an open subset of $X:M\subset O$.*

*Let $f:O\to Y$ be a continuous function, whose Fréchet derivatives ([4], pp. 87-127) $f^{(j)}:O\to L_j=L_j\left(X^j,Y\right)$ exist and are continuous for $1\leq j\leq m$, $m\in\mathbb{N}$.*

*Call* $(x-x_0)^j := (x-x_0,...,x-x_0) \in X^j$, $x \in M$.

*We will work with* $f|_M$.

*Then, by Taylor's formula ([4], p. 124) we get*

$$f(x) = \sum_{j=0}^{m-1} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} + R_m(x,x_0), \tag{4}$$

*for all* $x \in M$,

*where the remainder* $R_m(x,x_0)$ *is the Riemann integral*

$$R_m(x,x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0+u(x-x_0))\right)(x-x_0)^m\, du, \tag{5}$$

*where we set* $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

*We obtain*

$$\left\|\left(f^{(m)}(x_0+u(x-x_0))\right)(x-x_0)^m\right\|_2 \le \left\|f^{(m)}(x_0+u(x-x_0))\right\| \|x-x_0\|_1^m. \tag{6}$$

*Above* $(f|_M)^{(j)}$, $j=0,1,...,m$, *are norm bounded by continuity, and thus they are integrable.*

*By Corollary 20.3, p. 125, of [4] we obtain*

$$\|R_m(x,x_0)\|_2 \le \sup_{\overline{x}\in L(x_0,x_1)} \left\|f^{(m)}(\overline{x})\right\|_2 \frac{\|x_1-x_0\|_1^m}{m!}, \tag{7}$$

*where*

$$L(x_0,x_1) := \{\overline{x} | \overline{x} = \theta x_1 + (1-\theta)x_0,\ 0 \le \theta \le 1\} \tag{8}$$

*is the line segment joining the points* $x_0$ *and* $x_1$ *(notice that the last* $\overline{x} = x_0 + \theta(x_1-x_0)$*).*

*Denote also* $L(x_0,x_1) = \overline{x_0x_1}$.

*Here* $(\cdot - x_0)^j$ *maps* $M$ *into* $X^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $X^j$ *into* $Y$ *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* $M$ *into* $Y$.

*Let us restrict* $f$ *on the line segment* $\overline{x_0x_1}$. *Then for*

$$\overline{x}(u) = ux_1 + (1-u)x_0 = x_0 + u(x_1-x_0),\quad 0 \le u \le 1,$$

*the abstract function*

$$f(u) = f(\overline{x}(u)) = f(x_0+u(x_1-x_0))$$

*will map* $[0,1]$ *into an abstract arc in* $Y$, *which starts at* $y_0 = f(x_0)$ *and ends at* $y_1 = f(x_1)$.

*By [4], p. 124, we have that*

$$f^{(k)}(u) = f^{(k)}(x_0+u(x_1-x_0))(x_1-x_0)^k, \tag{9}$$

*for* $k = 1,2,...,m$; $u \in [0,1]$.

We need

**Definition 4** *All as in Remark 3. We define the vector left Caputo-Fréchet fractional derivative of order $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), by*

$$D_{*0}^{\alpha}\left(f\left(x_0+u\left(x_1-x_0\right)\right)\right) := J_0^{m-\alpha}\left(\left(f^{(m)}\left(x_0+u\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m\right)$$

$$:= \frac{1}{\Gamma\left(m-\alpha\right)}\int_0^u \left(u-t\right)^{m-\alpha-1}\left(f^{(m)}\left(x_0+t\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m dt, \quad (10)$$

*all* $0 \leq u \leq 1$,

*defined via the vector left Riemann-Liouville fractional integral ([1], p. 2).*

Then, we observe that

$$J_0^{\alpha} D_{*0}^{\alpha}\left(f\left(x_0+u\left(x_1-x_0\right)\right)\right) = J_0^{\alpha} J_0^{m-\alpha}\left(f^{(m)}\left(x_0+u\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m =$$

(by [1], p. 6)

$$J_0^{m}\left(f^{(m)}\left(x_0+u\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m = \quad (11)$$

$$\frac{1}{(m-1)!}\int_0^u \left(u-t\right)^{m-1}\left(f^{(m)}\left(x_0+t\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m dt,$$

true for $0 \leq u \leq 1$.

So, we have proved that

$$\frac{1}{\Gamma\left(\alpha\right)}\int_0^u \left(u-t\right)^{\alpha-1} D_{*0}^{\alpha}\left(f\left(x_0+t\left(x_1-x_0\right)\right)\right) dt =$$

$$\frac{1}{(m-1)!}\int_0^u \left(u-t\right)^{m-1}\left(f^{(m)}\left(x_0+t\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m dt, \quad (12)$$

for all $0 \leq u \leq 1$.

Consequently (by [1], p. 12) it holds the new left fractional-Fréchet Taylor formula on $\overline{x_0 x_1}$:

**Theorem 5** *All as above. Then*

$$f\left(x_0+u\left(x_1-x_0\right)\right) = \sum_{k=0}^{m-1}\frac{f^{(k)}\left(x_0\right)\left(x_1-x_0\right)^k}{k!}u^k +$$

$$\frac{1}{\Gamma\left(\alpha\right)}\int_0^u \left(u-t\right)^{\alpha-1} D_{*0}^{\alpha} f\left(x_0+t\left(x_1-x_0\right)\right) dt, \quad (13)$$

*for all* $0 \leq u \leq 1$.

In particular we notice that (the case of $u = 1$)

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} D_{*0}^{\alpha} (f(x_0 + w(x_1 - x_0))) \, dw = \tag{14}$$

$$\frac{1}{(m-1)!} \int_0^1 (1-w)^{m-1} \left( f^{(m)} (x_0 + w(x_1 - x_0)) \right) (x_1 - x_0)^m \, dw = R_m(x_1, x_0),$$

where $R_m(x_1, x_0)$ is as in (5).

So, we have the particular vector left hand side Caputo-Fréchet fractional Taylor's formula ($u = 1$).

**Corollary 6** *All as above. Then*

$$f(x_1) = \sum_{j=0}^{m-1} \frac{f^{(j)}(x_0)(x_1 - x_0)^j}{j!} + \tag{15}$$

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} D_{*0}^{\alpha} (f(x_0 + w(x_1 - x_0))) \, dw,$$

*for all $x_0, x_1 \in M$.*

We make

**Remark 7** *We are again working on the segment $\overline{x_0 x_1}$, where $x_0, x_1 \in M$; $0 \leq u \leq 1$, with $\overline{x} = \overline{x}(u) := x_0 + u(x_1 - x_0)$, and $f(u) = f(\overline{x}(u)) = f(x_0 + u(x_1 - x_0))$, i.e. $f : [0,1] \to Y$.*

*When $u = 0$, $f(0) = f(x_0)$, and when $u = 1$, $f(1) = f(x_1)$.*

*We have (by [1], pp. 121-122) the abstract Riemann integral:*

$$\int_{x_0}^{x_1} f(\overline{x}) \, d\overline{x} = \int_0^1 f(u) \, du. \tag{16}$$

*We also have that ([1], p. 122, and p. 124)*

$$f'(u) = f'(x_0 + u(x_1 - x_0))(x_1 - x_0),$$

*and*

$$f^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \tag{17}$$

$k = 1, 2, ..., m$.

Let $\alpha > 0$, such that $\lceil \alpha \rceil = m$.

We consider the vector valued right hand side Riemann-Liouville fractional integral ([1], p. 34),

$$J_{1-}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^1 (J - u)^{\alpha-1} f(J) \, dJ, \tag{18}$$

and the vector valued right hand side Caputo fractional derivative of order $\alpha > 0$ ([1], p. 42), by

$$D_{1-}^{\alpha} f(u) := (-1)^m J_{1-}^{m-\alpha} f^{(m)}(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_u^1 (J-u)^{m-\alpha-1} f^{(m)}(J)\, dJ. \tag{19}$$

(we have that $J_{1-}^{\alpha} J_{-1}^{\beta} f = J_{-1}^{\alpha+\beta} f = J_{-1}^{\beta} J_{-1}^{\alpha} f$, when $f \in C([0,1], Y)$ or $\alpha+\beta \geq 1$, see [1], p. 39).

We need right hand side Taylor's fractional formula.

**Theorem 8** *([1], p. 44, by Theorem 2.16) Let $f \in C^m([0,1], Y)$, $u \in [0,1]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then*

$$f(u) = \sum_{k=0}^{m-1} \frac{f^{(k)}(1)}{k!} (u-1)^k + \frac{1}{\Gamma(\alpha)} \int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(J)\, dJ. \tag{20}$$

Equation (20) implies the vector right hand side corresponding fractional Taylor's formula:

**Theorem 9** *All as in Remarks 3, 7. Then*

$$f(x_0 + u(x_1 - x_0)) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1 - x_0)^k}{k!} (u-1)^k + \tag{21}$$

$$\frac{1}{\Gamma(\alpha)} \int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(J)\, dJ,$$

*all $0 \leq u \leq 1$,*

*where*

$$D_{1-}^{\alpha} f(J) = D_{1-}^{\alpha} f(x_0 + J(x_1 - x_0)) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_J^1 (t-J)^{m-\alpha-1} f^{(m)}(t)\, dt \tag{22}$$

$$= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_J^1 (t-J)^{m-\alpha-1} f^{(m)}(x_0 + t(x_1 - x_0))(x_1 - x_0)^m\, dt.$$

When $u = 0$ we obtain

**Corollary 10** *All as in Remarks 3, 7. Then*

$$f(x_0) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1 - x_0)^k}{k!} (-1)^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_0^1 J^{\alpha-1} D_{1-}^{\alpha} f(x_0 + J(x_1 - x_0))\, dJ, \tag{23}$$

*for all $x_0, x_1 \in M$.*

Denote by

$$\int_{\langle x_0 x^* x_1 \rangle} f(x)\,dx := \int_{x_0}^{x^*} f(x)\,dx + \int_{x^*}^{x_1} f(x)\,dx, \tag{24}$$

where $x_0, x^*, x_1 \in M$ and $x^*$ not necessarily on the segment $\overline{x_0 x_1}$.

# 3 Main Results

## 3.1 About Ostrowski inequalities

We give

**Theorem 11** *Let $f : X \to Y$ and $g : X \to \mathbb{R}$ be as in Remarks 3, 7. Let $x_0, x^*, x_1 \in M$, $x^*$ not necessarily on the segment $\overline{x_0 x_1}$. We assume that $f^{(k)}(x^*) = g^{(k)}(x^*) = 0$, $k = 1, ..., m-1$.*

*Denote by*

$$\theta(f,g)(x^*) := 2\int_{\langle x_0 x^* x_1 \rangle} f(x)\,g(x)\,dx - f(x^*)\int_{\langle x_0 x^* x_1 \rangle} g(x)\,dx \tag{25}$$

$$-g(x^*)\int_{\langle x_0 x^* x_1 \rangle} f(x)\,dx.$$

*Then*

$$\theta(f,g)(x^*) =$$

$$\frac{1}{\Gamma(\alpha)}\left\{\int_0^1 \left[g(x^* + u(x_1 - x^*))\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} f(x^* + t(x_1 - x^*))\,dt\right.\right.$$

$$\left. + f(x^* + u(x_1 - x^*))\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} g(x^* + t(x_1 - x^*))\,dt\right] du +$$

$$\int_0^1 \left[g(x_0 + u(x^* - x_0))\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(x_0 + J(x^* - x_0))\,dJ\right.$$

$$\left.\left. + f(x_0 + u(x^* - x_0))\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} g(x_0 + J(x^* - x_0))\,dJ\right] du\right\}. \tag{26}$$

**Proof.** Here the function $g : (X, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$ has

$$D_{*0}^{\alpha}(g(x_0 + u(x_1 - x_0))) = \tag{27}$$

$$\frac{1}{\Gamma(m-\alpha)}\int_0^u (u-t)^{m-\alpha-1}\left(g^{(m)}(x_0 + t(x_1 - x_0))\right)(x_1 - x_0)^m\,dt,$$

and

$$g(x_0 + u(x_1 - x_0)) = \sum_{k=0}^{m-1} \frac{g^{(k)}(x_0)(x_1 - x_0)^k}{k!} u^k +$$

$$\frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} g(x_0+t(x_1-x_0))\,dt, \tag{28}$$

for all $0 \le u \le 1$,

and

$$g(x_0+u(x_1-x_0)) = \sum_{k=0}^{m-1} \frac{g^{(k)}(x_1)(x_1-x_0)^k}{k!}(u-1)^k +$$

$$\frac{1}{\Gamma(\alpha)}\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} g(x_0+J(x_1-x_0))\,dJ, \tag{29}$$

$0 \le u \le 1$,

where

$$D_{1-}^{\alpha} g(x_0+J(x_1-x_0)) = \tag{30}$$

$$\frac{(-1)^m}{\Gamma(m-\alpha)}\int_J^1 (t-J)^{m-\alpha-1} g^{(m)}(x_0+t(x_1-x_0))(x_1-x_0)^m\,dt.$$

Consider $x^* \in M$, not necessarily on the line segment $\overline{x_0x_1}$.

We assumed that $f^{(k)}(x^*) = g^{(k)}(x^*) = 0$, $k = 1,...,m-1$.

We will work on the segments $[x_0, x^*]$ and $[x^*, x_1]$.

By Theorem 5 we have that (on $[x^*, x_1]$)

$$f(x^*+u(x_1-x^*)) - f(x^*) = \frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} f(x^*+t(x_1-x^*))\,dt, \tag{31}$$

all $0 \le u \le 1$,

and

$$g(x^*+u(x_1-x^*)) - g(x^*) = \frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} g(x^*+t(x_1-x^*))\,dt, \tag{32}$$

all $0 \le u \le 1$.

Also (by Theorem 9) we have that (on $[x_0, x^*]$)

$$f(x_0+u(x^*-x_0)) - f(x^*) = \frac{1}{\Gamma(\alpha)}\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(x_0+J(x^*-x_0))\,dJ, \tag{33}$$

and

$$g(x_0+u(x^*-x_0)) - g(x^*) = \frac{1}{\Gamma(\alpha)}\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} g(x_0+J(x^*-x_0))\,dJ, \tag{34}$$

all $0 \le u \le 1$.

We would have (all $0 \le u \le 1$)

$$g(x^*+u(x_1-x^*))\,f(x^*+u(x_1-x^*)) - g(x^*+u(x_1-x^*))\,f(x^*) = \tag{35}$$

$$\frac{g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)}{\Gamma\left(\alpha\right)}\int_{0}^{u}\left(u-t\right)^{\alpha-1}D_{*0}^{\alpha}f\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)dt,$$

and

$$f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)-f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)g\left(x^{*}\right)= \tag{36}$$
$$\frac{f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)}{\Gamma\left(\alpha\right)}\int_{0}^{u}\left(u-t\right)^{\alpha-1}D_{*0}^{\alpha}g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)dt.$$

Furthermore we get:

$$g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)-g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)f\left(x^{*}\right)= \tag{37}$$
$$\frac{g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)}{\Gamma\left(\alpha\right)}\int_{u}^{1}\left(J-u\right)^{\alpha-1}D_{1-}^{\alpha}f\left(x_{0}+J\left(x^{*}-x_{0}\right)\right)dJ,$$

and

$$f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)-f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)g\left(x^{*}\right)= \tag{38}$$
$$\frac{f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)}{\Gamma\left(\alpha\right)}\int_{u}^{1}\left(J-u\right)^{\alpha-1}D_{1-}^{\alpha}g\left(x_{0}+J\left(x^{*}-x_{0}\right)\right)dJ,$$

all $0\leq u\leq 1$.

Adding (35) and (36) we get

$$2f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)-g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)f\left(x^{*}\right)-$$
$$f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)g\left(x^{*}\right)=$$
$$\frac{1}{\Gamma\left(\alpha\right)}\left[g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}\left(u-t\right)^{\alpha-1}D_{*0}^{\alpha}f\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)dt\right. \tag{39}$$
$$\left.+f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}\left(u-t\right)^{\alpha-1}D_{*0}^{\alpha}g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)dt\right],$$

and by adding (37) and (38) we get:

$$2f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)-g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)f\left(x^{*}\right)$$
$$-f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)g\left(x^{*}\right)=$$
$$\frac{1}{\Gamma\left(\alpha\right)}\left[g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}\left(J-u\right)^{\alpha-1}D_{1-}^{\alpha}f\left(x_{0}+J\left(x^{*}-x_{0}\right)\right)dJ\right. \tag{40}$$
$$\left.+f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}\left(J-u\right)^{\alpha-1}D_{1-}^{\alpha}g\left(x_{0}+J\left(x^{*}-x_{0}\right)\right)dJ\right],$$

all $0\leq u\leq 1$.

Next we integrate (39) and (40) to obtain:

$$2\int_{x^{*}}^{x_{1}}f\left(x\right)g\left(x\right)dx-f\left(x^{*}\right)\int_{x^{*}}^{x_{1}}g\left(x\right)dx-g\left(x^{*}\right)\int_{x^{*}}^{x_{1}}f\left(x\right)dx= \tag{41}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^1\left[g\left(x^*+u\left(x_1-x^*\right)\right)\int_0^u(u-t)^{\alpha-1}D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)dt\right.$$
$$\left.+f\left(x^*+u\left(x_1-x^*\right)\right)\int_0^u(u-t)^{\alpha-1}D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)dt\right]du,$$

and

$$2\int_{x_0}^{x^*}f(x)\,g(x)\,dx-f(x^*)\int_{x_0}^{x^*}g(x)\,dx-g(x^*)\int_{x_0}^{x^*}f(x)\,dx= \tag{42}$$
$$\frac{1}{\Gamma(\alpha)}\int_0^1\left[g\left(x_0+u\left(x^*-x_0\right)\right)\int_u^1(J-u)^{\alpha-1}D_{1-}^{\alpha}f\left(x_0+J\left(x^*-x_0\right)\right)dJ\right.$$
$$\left.+f\left(x_0+u\left(x^*-x_0\right)\right)\int_u^1(J-u)^{\alpha-1}D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)dJ\right]du.$$

Adding (41) and (42) we derive

$$\theta(f,g)(x^*):=$$
$$2\int_{\langle x_0x^*x_1\rangle}f(x)\,g(x)\,dx-f(x^*)\int_{\langle x_0x^*x_1\rangle}g(x)\,dx-g(x^*)\int_{\langle x_0x^*x_1\rangle}f(x)\,dx$$
$$=\frac{1}{\Gamma(\alpha)}\left\{\int_0^1\left[g\left(x^*+u\left(x_1-x^*\right)\right)\int_0^u(u-t)^{\alpha-1}D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)dt\right.\right.$$
$$\left.+f\left(x^*+u\left(x_1-x^*\right)\right)\int_0^u(u-t)^{\alpha-1}D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)dt\right]du+$$
$$\int_0^1\left[g\left(x_0+u\left(x^*-x_0\right)\right)\int_u^1(J-u)^{\alpha-1}D_{1-}^{\alpha}f\left(x_0+J\left(x^*-x_0\right)\right)dJ\right.$$
$$\left.\left.+f\left(x_0+u\left(x^*-x_0\right)\right)\int_u^1(J-u)^{\alpha-1}D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)dJ\right]du\right\}, \tag{43}$$

proving the claim. ■

We present the following general fractional Ostrowski type inequalities:

**Theorem 12** *All as in Theorem 11. Then*

*i)*

$$\left\|\theta(f,g)(x^*)\right\|_2\leq \tag{44}$$
$$\frac{1}{\Gamma(\alpha+2)}\left\{\left\|g\left(x^*+u\left(x_1-x^*\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)\right\|_2\right\|_{t,\infty,[0,1]}\right.$$
$$+\left\|\left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2\right\|_{u,\infty,[0,1]}\left\|D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)\right\|_{t,\infty,[0,1]}+$$
$$\left\|g\left(x_0+u\left(x^*-x_0\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{\alpha}f\left(x_0+J\left(x^*-x_0\right)\right)\right\|_2\right\|_{J,\infty,[0,1]}$$
$$\left.+\left\|\left\|f\left(x_0+u\left(x^*-x_0\right)\right)\right\|_2\right\|_{u,\infty,[0,1]}\left\|D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)\right\|_{J,\infty,[0,1]}\right\},$$

*ii) let $\alpha \geq 1$, then*

$$\|\theta(f,g)(x^*)\|_2 \leq \tag{45}$$

$$\frac{1}{\Gamma(\alpha+1)}\Big\{\|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]}\,\|\|D^{\alpha}_{*0}f(x^*+t(x_1-x^*))\|_2\|_{t,L_1([0,1])}$$
$$+\|\|f(x^*+u(x_1-x^*))\|_2\|_{u,\infty,[0,1]}\,\|D^{\alpha}_{*0}g(x^*+t(x_1-x^*))\|_{t,L_1([0,1])}+$$
$$\|g(x_0+u(x^*-x_0))\|_{u,\infty,[0,1]}\,\left\|\left\|D^{\alpha}_{1-}f(x_0+J(x^*-x_0))\right\|_2\right\|_{J,L_1([0,1])}$$
$$+\|\|f(x_0+u(x^*-x_0))\|_2\|_{u,\infty,[0,1]}\,\left\|D^{\alpha}_{1-}g(x_0+J(x^*-x_0))\right\|_{J,L_1([0,1])}\Big\},$$

*iii) let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$, then*

$$\|\theta(f,g)(x^*)\|_2 \leq \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)\left(\alpha+\frac{1}{p}\right)} \tag{46}$$

$$\Big\{\|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]}\,\|\|D^{\alpha}_{*0}f(x^*+t(x_1-x^*))\|_2\|_{t,L_q([0,1])}$$
$$+\|\|f(x^*+u(x_1-x^*))\|_2\|_{u,\infty,[0,1]}\,\|D^{\alpha}_{*0}g(x^*+t(x_1-x^*))\|_{t,L_q([0,1])}+$$
$$\|g(x_0+u(x^*-x_0))\|_{u,\infty,[0,1]}\,\left\|\left\|D^{\alpha}_{1-}f(x_0+J(x^*-x_0))\right\|_2\right\|_{J,L_q([0,1])}$$
$$+\|\|f(x_0+u(x^*-x_0))\|_2\|_{u,\infty,[0,1]}\,\left\|D^{\alpha}_{1-}g(x_0+J(x^*-x_0))\right\|_{J,L_q([0,1])}\Big\}.$$

**Proof.** We have that

$$\|\theta(f,g)(x^*)\|_2 \overset{(26)}{\leq} \frac{1}{\Gamma(\alpha)}$$

$$\Big\{\Big\|\int_0^1\Big[g(x^*+u(x_1-x^*))\int_0^u(u-t)^{\alpha-1}D^{\alpha}_{*0}f(x^*+t(x_1-x^*))\,dt \tag{47}$$
$$+f(x^*+u(x_1-x^*))\int_0^u(u-t)^{\alpha-1}D^{\alpha}_{*0}g(x^*+t(x_1-x^*))\,dt\Big]du\Big\|_2+$$
$$\Big\|\int_0^1\Big[g(x_0+u(x^*-x_0))\int_u^1(J-u)^{\alpha-1}D^{\alpha}_{1-}f(x_0+J(x^*-x_0))\,dJ$$
$$+f(x_0+u(x^*-x_0))\int_u^1(J-u)^{\alpha-1}D^{\alpha}_{1-}g(x_0+J(x^*-x_0))\,dJ\Big]du\Big\|_2\Big\}\leq$$

$$\frac{1}{\Gamma(\alpha)}\Big\{\Big\|\int_0^1\Big[g(x^*+u(x_1-x^*))\int_0^u(u-t)^{\alpha-1}D^{\alpha}_{*0}f(x^*+t(x_1-x^*))\,dt\Big]du\Big\|_2 \tag{48}$$
$$+\Big\|\int_0^1\Big[f(x^*+u(x_1-x^*))\int_0^u(u-t)^{\alpha-1}D^{\alpha}_{*0}g(x^*+t(x_1-x^*))\,dt\Big]du\Big\|_2+$$
$$\Big\|\int_0^1\Big[g(x_0+u(x^*-x_0))\int_u^1(J-u)^{\alpha-1}D^{\alpha}_{1-}f(x_0+J(x^*-x_0))\,dJ\Big]du\Big\|_2$$

$$+\left\|\int_0^1\left[f\left(x_0+u\left(x^*-x_0\right)\right)\int_u^1\left(J-u\right)^{\alpha-1}D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)dJ\right]du\right\|_2\Bigg\}\leq$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\int_0^1\left[\left|g\left(x^*+u\left(x_1-x^*\right)\right)\right|\int_0^u\left(u-t\right)^{\alpha-1}\left\|D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)\right\|_2dt\right]du\right.$$

$$+\int_0^1\left[\left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2\int_0^u\left(u-t\right)^{\alpha-1}\left|D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)\right|dt\right]du+$$

$$\int_0^1\left[\left|g\left(x_0+u\left(x^*-x_0\right)\right)\right|\int_u^1\left(J-u\right)^{\alpha-1}\left\|D_{1-}^{\alpha}f\left(x_0+J\left(x^*-x_0\right)\right)\right\|_2dJ\right]du$$

$$\left.+\int_0^1\left[\left\|f\left(x_0+u\left(x^*-x_0\right)\right)\right\|_2\int_u^1\left(J-u\right)^{\alpha-1}\left|D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)\right|dJ\right]du\right\}\tag{49}$$

$$=:\left(\xi\right).$$

i) We observe that

$$\left(\xi\right)\leq$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\int_0^1\left|g\left(x^*+u\left(x_1-x^*\right)\right)\right|\frac{u^{\alpha}}{\alpha}\left\|\left\|D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)\right\|_2\right\|_{t,\infty,[0,1]}du\right.$$

$$+\int_0^1\left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2\frac{u^{\alpha}}{\alpha}\left\|D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)\right\|_{t,\infty,[0,1]}du+$$

$$\int_0^1\left|g\left(x_0+u\left(x^*-x_0\right)\right)\right|\frac{\left(1-u\right)^{\alpha}}{\alpha}\left\|\left\|D_{1-}^{\alpha}f\left(x_0+J\left(x^*-x_0\right)\right)\right\|_2\right\|_{J,\infty,[0,1]}du$$

$$\left.+\int_0^1\left\|f\left(x_0+u\left(x^*-x_0\right)\right)\right\|_2\frac{\left(1-u\right)^{\alpha}}{\alpha}\left\|D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)\right\|_{J,\infty,[0,1]}du\right\}\leq\tag{50}$$

$$\frac{1}{\Gamma\left(\alpha+2\right)}\left\{\left\|g\left(x^*+u\left(x_1-x^*\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)\right\|_2\right\|_{t,\infty,[0,1]}\right.$$

$$+\left\|\left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2\right\|_{u,\infty,[0,1]}\left\|D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)\right\|_{t,\infty,[0,1]}+\tag{51}$$

$$\left\|g\left(x_0+u\left(x^*-x_0\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{\alpha}f\left(x_0+J\left(x^*-x_0\right)\right)\right\|_2\right\|_{J,\infty,[0,1]}$$

$$\left.+\left\|\left\|f\left(x_0+u\left(x^*-x_0\right)\right)\right\|_2\right\|_{u,\infty,[0,1]}\left\|D_{1-}^{\alpha}g\left(x_0+J\left(x^*-x_0\right)\right)\right\|_{J,\infty,[0,1]}\right\},$$

proving (i).

ii) Let $\alpha\geq1$, then

$$\left(\xi\right)\leq$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\int_0^1\left|g\left(x^*+u\left(x_1-x^*\right)\right)\right|u^{\alpha-1}\left\|\left\|D_{*0}^{\alpha}f\left(x^*+t\left(x_1-x^*\right)\right)\right\|_2\right\|_{t,L_1([0,1])}du\right.$$

$$+\int_0^1\left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2u^{\alpha-1}\left\|D_{*0}^{\alpha}g\left(x^*+t\left(x_1-x^*\right)\right)\right\|_{t,L_1([0,1])}du+\tag{52}$$

$$\int_0^1 |g(x_0+u(x^*-x_0))|(1-u)^{\alpha-1}\left\|\left\|D_{1-}^{\alpha}f(x_0+J(x^*-x_0))\right\|_2\right\|_{J,L_1([0,1])}du$$

$$\left.+\int_0^1 \|f(x_0+u(x^*-x_0))\|_2(1-u)^{\alpha-1}\left\|D_{1-}^{\alpha}g(x_0+J(x^*-x_0))\right\|_{J,L_1([0,1])}du\right\}\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]}\|\|D_{*0}^{\alpha}f(x^*+t(x_1-x^*))\|_2\|_{t,L_1([0,1])}\right.$$

$$+\|\|f(x^*+u(x_1-x^*))\|_2\|_{u,\infty,[0,1]}\|D_{*0}^{\alpha}g(x^*+t(x_1-x^*))\|_{t,L_1([0,1])}+ \tag{53}$$

$$\|g(x_0+u(x^*-x_0))\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{\alpha}f(x_0+J(x^*-x_0))\right\|_2\right\|_{J,L_1([0,1])}$$

$$\left.+\|\|f(x_0+u(x^*-x_0))\|_2\|_{u,\infty,[0,1]}\left\|D_{1-}^{\alpha}g(x_0+J(x^*-x_0))\right\|_{J,L_1([0,1])}\right\},$$

proving (ii).

iii) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. Then (by applying Hölder's inequality) we obtain:

$$(\xi)\leq$$

$$\frac{1}{\Gamma(\alpha)}\left\{\int_0^1 |g(x^*+u(x_1-x^*))|\frac{u^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\|\|D_{*0}^{\alpha}f(x^*+t(x_1-x^*))\|_2\|_{t,L_q([0,1])}du\right.$$

$$+\int_0^1 \|f(x^*+u(x_1-x^*))\|_2\frac{u^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\|D_{*0}^{\alpha}g(x^*+t(x_1-x^*))\|_{t,L_q([0,1])}du+ \tag{54}$$

$$\int_0^1 |g(x_0+u(x^*-x_0))|\frac{(1-u)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|\left\|D_{1-}^{\alpha}f(x_0+J(x^*-x_0))\right\|_2\right\|_{J,L_q([0,1])}du$$

$$\left.+\int_0^1 \|f(x_0+u(x^*-x_0))\|_2\frac{(1-u)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{1-}^{\alpha}g(x_0+J(x^*-x_0))\right\|_{J,L_q([0,1])}du\right\}$$

$$\leq\frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)\left(\alpha+\frac{1}{p}\right)}$$

$$\left\{\|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]}\|\|D_{*0}^{\alpha}f(x^*+t(x_1-x^*))\|_2\|_{t,L_q([0,1])}\right.$$

$$+\|\|f(x^*+u(x_1-x^*))\|_2\|_{u,\infty,[0,1]}\|D_{*0}^{\alpha}g(x^*+t(x_1-x^*))\|_{t,L_q([0,1])}+$$

$$\|g(x_0+u(x^*-x_0))\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{\alpha}f(x_0+J(x^*-x_0))\right\|_2\right\|_{J,L_q([0,1])}$$

$$\left.+\|\|f(x_0+u(x^*-x_0))\|_2\|_{u,\infty,[0,1]}\left\|D_{1-}^{\alpha}g(x_0+J(x^*-x_0))\right\|_{J,L_q([0,1])}\right\}, \tag{55}$$

proving (iii). ■

### 3.2 About Grüss inequalities

We make

**Remark 13** *(to Theorem 11) Let* $0 < \alpha \leq 1$*, i.e.* $m = 1$*, and assume that* $x^* \in \overline{x_0 x_1}$*. Then, no initial conditions are needed, that is* $x^*$ *could be a variable over* $\overline{x_0 x_1}$*.*

*We can write* $x^* = x_0 + u(x_1 - x_0)$*,*

$$\theta(f,g)(x^*) = \theta(f,g)(x_0 + u(x_1 - x_0)), \quad \text{for } u \in [0,1].$$

*Therefore it holds*

$$\int_{x_0}^{x_1} \theta(f,g)(x^*)\,dx^* = \int_0^1 \theta(f,g)(x_0 + u(x_1 - x_0))\,du \overset{(25)}{=} \tag{56}$$

$$\left(2\int_{x_0}^{x_1} f(x)\,g(x)\,dx\right)\int_0^1 du - \left(\int_0^1 f(x_0 + u(x_1 - x_0))\,du\right)\int_{x_0}^{x_1} g(x)\,dx$$

$$-\left(\int_0^1 g(x_0 + u(x_1 - x_0))\,du\right)\int_{x_0}^{x_1} f(x)\,dx =$$

$$2\left[\int_{x_0}^{x_1} f(x)\,g(x)\,dx - \left(\int_{x_0}^{x_1} f(x)\,dx\right)\left(\int_{x_0}^{x_1} g(x)\,dx\right)\right].$$

*That is*

$$\int_{x_0}^{x_1} \theta(f,g)(x^*)\,dx^* = 2\left[\int_{x_0}^{x_1} f(x)\,g(x)\,dx - \left(\int_{x_0}^{x_1} f(x)\,dx\right)\left(\int_{x_0}^{x_1} g(x)\,dx\right)\right]. \tag{57}$$

*Denote by*

$$M(f,g) := \max\left\{\sup_{x^* \in \overline{x_0 x_1}} \|g(x^* + u(x_1 - x^*))\|_{u,\infty,[0,1]}, \right. \tag{58}$$

$$\sup_{x^* \in \overline{x_0 x_1}} \|\|\|f(x^* + u(x_1 - x^*))\|_2\|_{u,\infty,[0,1]},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \|g(x_0 + u(x^* - x_0))\|_{u,\infty,[0,1]},$$

$$\left.\sup_{x^* \in \overline{x_0 x_1}} \|\|\|f(x_0 + u(x^* - x_0))\|_2\|_{u,\infty,[0,1]}\right\} < \infty,$$

*and*

$$N_1(f,g) := \max\left\{\sup_{x^* \in \overline{x_0 x_1}} \|\|\|D_{*0}^{\alpha} f(x^* + t(x_1 - x^*))\|_2\|_{t,\infty,[0,1]}, \right. \tag{59}$$

$$\sup_{x^* \in \overline{x_0 x_1}} \|D_{*0}^{\alpha} g(x^* + t(x_1 - x^*))\|_{t,\infty,[0,1]},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D^{\alpha}_{1-} f\left(x_0 + J\left(x^* - x_0\right)\right) \right\|_2 \right\|_{J,\infty,[0,1]},$$

$$\left. \sup_{x^* \in \overline{x_0 x_1}} \left\| D^{\alpha}_{1-} g\left(x_0 + J\left(x^* - x_0\right)\right) \right\|_{J,\infty,[0,1]} \right\} < \infty.$$

*By (57) we have that*

$$\left\| \int_{x_0}^{x_1} f(x) g(x) \, dx - \left( \int_{x_0}^{x_1} f(x) \, dx \right) \left( \int_{x_0}^{x_1} g(x) \, dx \right) \right\|_2 =$$

$$\frac{1}{2} \left\| \int_{x_0}^{x_1} \theta(f,g)(x^*) \, dx^* \right\|_2 \leq \frac{1}{2} \int_{x_0}^{x_1} \left\| \theta(f,g)(x^*) \right\|_2 dx^* \tag{60}$$

$$\overset{(44)}{\leq} \frac{2M(f,g) N_1(f,g)}{\Gamma(\alpha+2)},$$

*proving a general fractional Grüss type inequality with respect to supremum norm.*

*Next, let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *with* $\frac{1}{q} < \alpha \leq 1$.

*Denote by*

$$N_2(f,g) := \max \left\{ \sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D^{\alpha}_{*0} f\left(x^* + t\left(x_1 - x^*\right)\right) \right\|_2 \right\|_{t, L_q([0,1])}, \right. \tag{61}$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D^{\alpha}_{*0} g\left(x^* + t\left(x_1 - x^*\right)\right) \right\|_{t, L_q([0,1])},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D^{\alpha}_{1-} f\left(x_0 + J\left(x^* - x_0\right)\right) \right\|_2 \right\|_{J, L_q([0,1])},$$

$$\left. \sup_{x^* \in \overline{x_0 x_1}} \left\| D^{\alpha}_{1-} g\left(x_0 + J\left(x^* - x_0\right)\right) \right\|_{J, L_q([0,1])} \right\} < \infty.$$

*We get now that*

$$\left\| \int_{x_0}^{x_1} f(x) g(x) \, dx - \left( \int_{x_0}^{x_1} f(x) \, dx \right) \left( \int_{x_0}^{x_1} g(x) \, dx \right) \right\|_2 \leq$$

$$\frac{1}{2} \int_{x_0}^{x_1} \left\| \theta(f,g)(x^*) \right\|_2 dx^* \overset{(46)}{\leq} \frac{2M(f,g) N_2(f,g)}{\left(p(\alpha-1)+1\right)^{\frac{1}{p}} \Gamma(\alpha) \left(\alpha + \frac{1}{p}\right)}, \tag{62}$$

*establishing a general fractional Grüss type inequality with respect to* $L_q([0,1])$ *norm.*

Since $M$ is compact an interesting application is when $x_0, x_1$ are endpoints of its diameter.

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