Sequential Fractional Calculus between Banach spaces and alternative Ostrowski and Grüss type inequalities

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Abstract

Here we present a sequential fractional calculus Caputo type for functions between Banach spaces. Left and right fractional derivatives are defined on a segment of a Banach space, then we develop our theory. We apply the above to new alternative Ostrowski and Grüss type inequalities at very abstract level.

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1 Introduction

The problem of estimating the difference of a value of a function from its average is a top one. The answer to it are the Ostrowski type inequalities. Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals. The problem of estimating the difference between the average of a product of functions from the product of their averages is also a very important one. The answer to it are the Grüss type inequalities. Grüss type inequalities are very useful among others in Probability for estimating expected values, etc. There exists a huge literature on Ostrowski and Grüss type inequalities to all possible directions. Mathematical community is very much interested to these inequalities due to their applications. So here we derive alternative very general fractional Ostrowski and Grüss type inequalities on a very

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abstract level. Our functions are between Banach spaces, and we develop first the related abstract sequential fractional calculus, which is based on a Banach space segment. Great sources to support our goal are the books [1], [4].

We are motivated by the following basic results:

Theorem 1 *(1938. Ostrowski [3]) Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *whose derivative* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b)*, i.e.,* $\|f'\|_{\infty}^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$*. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty}^{\sup}, \tag{1}$$

for any $x \in [a, b]$*. The constant* $\frac{1}{4}$ *is the best possible.*

Theorem 2 *(1935, Grüss [2]) Let* f, g *be integrable functions from* $[a, b]$ *into* $\mathbb{R}$*, that satisfy the conditions*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a, b],$$

where $m, M, n, N \in \mathbb{R}$*. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x)\, dx - \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\, dx \right) \right| \tag{2}$$

$$\leq \frac{1}{4} (M - m)(N - n).$$

2 Sequential Fractionality between a pair of Banach spaces

We make

Remark 3 *Throughout this article let* $(X, \|\cdot\|_1)$ *and* $(Y, \|\cdot\|_2)$ *be Banach spaces. Here* X^j *denotes the* j*-fold product space* $\underbrace{X \times X \times ... \times X}_{j}$ *endowed with the max-norm* $\|x\|_{X^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_1$*, where* $x := (x_1, ..., x_j) \in X^j$*.*

Let the space of $L_j := L_j\left(X^j, Y\right)$ *of all* j*-multilinear continuous maps* $h : X^j \to Y$*,* $j = 1, ..., m$*, which is a Banach space with norm*

$$\|h\| = \|h\|_{L_j} := \sup_{\left(\|x\|_{X^j} = 1\right)} \|h(x)\|_2 = \sup \frac{\|h(x)\|_2}{\|x_1\|_1 ... \|x_j\|_1}. \tag{3}$$

Let M *be a non-empty convex and compact set of* X *and* $x_0 \in M$ *is fixed.*

Let O be an open subset of $X : M \subset O$.

Let $f : O \to Y$ be a continuous function, whose Fréchet derivatives ([4], pp. 87-127) $f^{(j)} : O \to L_j = L_j\left(X^j, Y\right)$ exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in X^j$, $x \in M$.

We will work with $f|_M$ and here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We obtain

$$\left\|\left(f^{(m)}(x_0 + u(x - x_0))\right)(x - x_0)^m\right\|_2 \leq \left\|f^{(m)}(x_0 + u(x - x_0))\right\| \|x - x_0\|_1^m . \tag{4}$$

Above $(f|_M)^{(j)}$, $j = 0, 1, ..., m$, are norm bounded by continuity, and thus they are integrable.

Here

$$L(x_0, x_1) := \{\overline{x} | \overline{x} = \theta x_1 + (1 - \theta) x_0,\ 0 \leq \theta \leq 1\} \tag{5}$$

is the line segment joining the points x_0 and x_1 (notice that the last $\overline{x} = x_0 + \theta(x_1 - x_0)$).

Denote also $L(x_0, x_1) = \overline{x_0 x_1}$.

Here $(\cdot - x_0)^j$ maps M into X^j and it is continuous, also $f^{(j)}(x_0)$ maps X^j into Y and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into Y.

Let us restrict f on the line segment $\overline{x_0 x_1}$. Then for

$$\overline{x}(u) = u x_1 + (1 - u) x_0 = x_0 + u(x_1 - x_0),\ \ 0 \leq u \leq 1,$$

the abstract function

$$f(u) = f(\overline{x}(u)) = f(x_0 + u(x_1 - x_0))$$

will map $[0, 1]$ into an abstract arc in Y, which starts at $y_0 = f(x_0)$ and ends at $y_1 = f(x_1)$.

By [4], p. 124, we have that

$$f^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k , \tag{6}$$

for $k = 1, 2, ..., m$; $u \in [0, 1]$.

We need

Definition 4 *All as in Remark 3. We define the vector left Caputo-Fréchet fractional derivative of order $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), by*

$$D_{*0}^{\alpha}(f(x_0 + u(x_1 - x_0))) := J_0^{m-\alpha}\left(\left(f^{(m)}(x_0 + u(x_1 - x_0))\right)(x_1 - x_0)^m\right)$$

$$:= \frac{1}{\Gamma(m - \alpha)} \int_0^u (u - t)^{m-\alpha-1} \left(f^{(m)}(x_0 + t(x_1 - x_0))\right)(x_1 - x_0)^m dt, \tag{7}$$

all $0 \leq u \leq 1$,

defined via the vector left Riemann-Liouville fractional integral ([1], p. 2).

In particular when $0 < \alpha < 1$ we have

$$D_{*0}^{\alpha}\left(f\left(x_0 + u\left(x_1 - x_0\right)\right)\right) =$$

$$\frac{1}{\Gamma\left(1-\alpha\right)} \int_0^u \left(u-t\right)^{-\alpha} \left(f'\left(x_0 + t\left(x_1 - x_0\right)\right)\right)\left(x_1 - x_0\right) dt, \tag{8}$$

all $0 \leq u \leq 1$.

If $\alpha \in \mathbb{N}$, we set $D_{*0}^{\alpha} f := f^{(\alpha)}$ the ordinary Y-valued derivative, and also set $D_{*0}^{0} f := f$.

We make

Remark 5 *We are again working on the segment $\overline{x_0 x_1}$, where $x_0, x_1 \in M$; $0 \leq u \leq 1$, with $\overline{x} = \overline{x}(u) := x_0 + u\left(x_1 - x_0\right)$, and $f(u) = f\left(\overline{x}(u)\right) = f\left(x_0 + u\left(x_1 - x_0\right)\right)$, i.e. $f : [0,1] \to Y$.*

When $u = 0$, $f(0) = f(x_0)$, and when $u = 1$, $f(1) = f(x_1)$.

We have (by [1], pp. 121-122) the abstract Riemann integral:

$$\int_{x_0}^{x_1} f\left(\overline{x}\right) d\overline{x} = \int_0^1 f(u)\, du. \tag{9}$$

We also have that ([1], p. 122, and p. 124)

$$f'(u) = f'\left(x_0 + u\left(x_1 - x_0\right)\right)\left(x_1 - x_0\right),$$

and

$$f^{(k)}(u) = f^{(k)}\left(x_0 + u\left(x_1 - x_0\right)\right)\left(x_1 - x_0\right)^k, \tag{10}$$

$k = 1, 2, ..., m$.

Let $\alpha > 0$, such that $\lceil \alpha \rceil = m$.

We consider the vector valued right hand side Riemann-Liouville fractional integral ([1], p. 34),

$$J_{1-}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^1 \left(J - u\right)^{\alpha - 1} f(J)\, dJ, \tag{11}$$

We define the vector right Caputo-Fréchet fractional derivative of order α, by

$$D_{1-}^{\alpha} f\left(x_0 + u\left(x_1 - x_0\right)\right) =$$

$$\frac{(-1)^m}{\Gamma\left(m-\alpha\right)} \int_u^1 \left(t-u\right)^{m-\alpha-1} f^{(m)}\left(x_0 + t\left(x_1 - x_0\right)\right)\left(x_1 - x_0\right)^m dt, \tag{12}$$

all $0 \leq u \leq 1$ (see also [1], p. 42-43).

In particular when $0 < \alpha < 1$ we have

$$D_{1-}^{\alpha}\left(f\left(x_0 + u\left(x_1 - x_0\right)\right)\right) =$$

$$\frac{-1}{\Gamma(1-\alpha)}\int_u^1 (t-u)^{-\alpha} f'(x_0+t(x_1-x_0))(x_1-x_0)\,dt, \tag{13}$$

all $0 \le u \le 1$.

We set $D_{1-}^m f = (-1)^m f^{(m)}$, for $m \in \mathbb{N}$, and $D_{1-}^0 f = f$.

Denote by

$$\int_{\langle x_0 x^* x_1\rangle} f(x)\,dx := \int_{x_0}^{x^*} f(x)\,dx + \int_{x^*}^{x_1} f(x)\,dx, \tag{14}$$

where $x_0, x^*, x_1 \in M$ and x^* not necessarily on the line segment $\overline{x_0x_1}$.

Denote the sequential Caputo-Bochner left and right fractional derivatives ($\alpha > 0$)

$$D_{*a}^{n\alpha} := D_{*a}^{\alpha} D_{*a}^{\alpha} ... D_{*a}^{\alpha} \quad (n\text{-times}),\ n \in \mathbb{N} \tag{15}$$

and

$$D_{b-}^{n\alpha} := D_{b-}^{\alpha} D_{b-}^{\alpha} ... D_{b}^{\alpha} \quad (n\text{-times}). \tag{16}$$

For the detailed definitions of D_{*a}^{α}, D_{b-}^{α} see [1], pp. 128-129. In this article we consider $[a,b] = [0,1]$.

We mention the following left alternative fractional Taylor's formula

Theorem 6 *([1], p. 129, Theorem 4.43) Let Y a Banach space and $0 < \alpha \le 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], Y)$.*

For $k = 1, ..., n$, we assume that $D_{*a}^{k\alpha} f \in C^1([a,b], Y)$ and $D_{*a}^{(n+1)\alpha} f \in C([a,b], Y)$. Then

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{*a}^{i\alpha} f\right)(a) + \tag{17}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\,dt,$$

$\forall\ x \in [a,b]$.

We also mention the following right alternative fractional Taylor's formula.

Theorem 7 *([1], p. 129, Theorem 4.44) Let Y a Banach space and $0 < \alpha \le 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], Y)$. For $k = 1, ..., n$, we assume that $D_{b-}^{k\alpha} f \in C^1([a,b], Y)$ and $D_{b-}^{(n+1)\alpha} f \in C([a,b], Y)$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) + \tag{18}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (t-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f\right)(t)\,dt,$$

$\forall\ x \in [a,b]$.

We give the corresponding left and right fractional alternative Taylor's formulae on the segment $\overline{x_0 x_1}$.

Theorem 8 *Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$.*

For $k = 1, ..., n$, we assume that $D_{*0}^{k\alpha} f(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$ and $D_{*0}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \in C([0,1], Y)$. Then

$$f(x_0 + u(x_1 - x_0)) = \sum_{i=0}^{n} \frac{u^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{*0}^{i\alpha} (f(x_0 + u(x_1 - x_0)))\right)(0) + \tag{19}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_0^u (u - t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha} (f(x_0 + u(x_1 - x_0)))\right)(t)\, dt,$$

$\forall\ u \in [0,1]$.

Proof. By Theorem 6. ■

Theorem 9 *Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$.*

For $k = 1, ..., n$, we assume that $D_{1-}^{k\alpha} f(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$ and $D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \in C([0,1], Y)$. Then

$$f(x_0 + u(x_1 - x_0)) = \sum_{i=0}^{n} \frac{(1-u)^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{1-}^{i\alpha} f(x_0 + u(x_1 - x_0))\right)(1) + \tag{20}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_u^1 (t - u)^{(n+1)\alpha - 1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0))\right)(t)\, dt,$$

$\forall\ u \in [0,1]$.

Proof. By Theorem 7. ■

3 Main Results

3.1 About general alternative mixed fractional Ostrowski type inequalities

We present

Theorem 10 *Let X, Y be Banach spaces, $0 < \alpha \leq 1$, $n \in \mathbb{N}$. Here $x_0, x^*, x_1 \in X$ and are not necessarily colinear; $0 \leq u \leq 1$. Let $f : X \to Y$ and $g : X \to \mathbb{R}$ such that $f(x^* + u(x_1 - x^*)) \in C^1([0,1], Y)$, $g(x^* + u(x_1 - x^*)) \in C^1([0,1], \mathbb{R})$; for $k = 1, ..., n$ we assume that $D_{*0}^{k\alpha} f(x^* + u(x_1 - x^*)) \in C^1([0,1], Y)$ and $D_{*0}^{(n+1)\alpha} f(x^* + u(x_1 - x^*)) \in C([0,1], Y)$, and $D_{*0}^{k\alpha} g(x^* + u(x_1 - x^*)) \in$*

$C^1([0,1],\mathbb{R})$, $D_{*0}^{(n+1)\alpha} g(x^* + u(x_1 - x^*)) \in C([0,1],\mathbb{R})$. *Furthermore we suppose that* $f(x_0 + u(x^* - x_0)) \in C^1([0,1],Y)$, $g(x_0 + u(x^* - x_0)) \in C^1([0,1],\mathbb{R})$; *and for* $k = 1,...,n$ *we assume that* $D_{1-}^{k\alpha} f(x_0 + u(x^* - x_0)) \in C^1([0,1],Y)$ *and* $D_{1-}^{(n+1)\alpha} f(x_0 + u(x^* - x_0)) \in C([0,1],Y)$, *and* $D_{1-}^{k\alpha} g(x_0 + u(x^* - x_0)) \in C^1([0,1],\mathbb{R})$, $D_{1-}^{(n+1)\alpha} g(x_0 + u(x^* - x_0)) \in C([0,1],\mathbb{R})$.

Finally, we assume that $D_{*0}^{i\alpha}(f(x^* + u(x_1 - x^*)))(0) = 0$, $D_{*0}^{i\alpha}(g(x^* + u(x_1 - x^*)))(0) = 0$, *for all* $i = 1,...,n$, *along with* $D_{1-}^{i\alpha}(f(x_0 + u(x^* - x_0)))(1) = 0$, $D_{1-}^{i\alpha}(g(x_0 + u(x^* - x_0)))(1) = 0$, *for all* $i = 1,...,n$.

Then

$$K(f,g)(x^*) := 2\int_{\langle x_0 x^* x_1\rangle} f(x) g(x) dx - f(x^*) \int_{\langle x_0 x^* x_1\rangle} g(x) dx$$

$$-g(x^*) \int_{\langle x_0 x^* x_1\rangle} f(x) dx = \frac{1}{\Gamma((n+1)\alpha)}$$

$$\Big\{ \Big[\int_0^1 \Big[g(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha}(f(x^* + t(x_1 - x^*)))\right)(t) dt \tag{21}$$

$$+f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha}(g(x^* + t(x_1 - x^*)))\right)(t) dt \Big] du \Big] +$$

$$\Big[\int_0^1 \Big[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha - 1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0))\right)(t) dt$$

$$+f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha - 1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0))\right)(t) dt \Big] du \Big] \Big\}.$$

Proof. Along with $f : X \to Y$ we consider $g : X \to \mathbb{R}$ with the same properties as f. We also we have the line segments $\overline{x_0 x^*}$ and $\overline{x^* x_1}$, with not necessarily colinear $x_0, x^*, x_1 \in M$. Assume that $D_{*0}^{i\alpha}(f(x^* + u(x_1 - x^*)))(0) = 0$, and $D_{*0}^{i\alpha}(g(x^* + u(x_1 - x^*)))(0) = 0$, $i = 1,..,n$.

We apply Theorem 8.

Then (on $\overline{x^* x_1}$)

$$f(x^* + u(x_1 - x^*)) - f(x^*) = \tag{22}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha}(f(x^* + t(x_1 - x^*)))\right)(t) dt,$$

$\forall\ u \in [0,1]$.

And, similarly it holds

$$g(x^* + u(x_1 - x^*)) - g(x^*) = \tag{23}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha}(g(x^* + t(x_1 - x^*)))\right)(t) dt,$$

$\forall\ u \in [0,1]$.

Assume also (on $\overline{x_0 x^*}$) $\left(D_{1-}^{i\alpha}\left(f\left(x_0+u\left(x^*-x_0\right)\right)\right)\right)(1)=0$, $D_{1-}^{i\alpha}\left(g\left(x_0+u\left(x^*-x_0\right)\right)\right)(1)=0$, $i=1,..,n$.

We apply Theorem 9.

Then

$$f\left(x_0+u\left(x^*-x_0\right)\right)-f\left(x^*\right)= \tag{24}$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}\left(f\left(x_0+u\left(x^*-x_0\right)\right)\right)\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

Similarly, it holds

$$g\left(x_0+u\left(x^*-x_0\right)\right)-g\left(x^*\right)= \tag{25}$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}g\left(x_0+u\left(x^*-x_0\right)\right)\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

We have that

$$g\left(x^*+u\left(x_1-x^*\right)\right)f\left(x^*+u\left(x_1-x^*\right)\right)-g\left(x^*+u\left(x_1-x^*\right)\right)f\left(x^*\right)= \tag{26}$$

$$\frac{g\left(x^*+u\left(x_1-x^*\right)\right)}{\Gamma\left((n+1)\alpha\right)}\int_0^u (u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(f\left(x^*+t\left(x_1-x^*\right)\right)\right)\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

And, similarly, we get that

$$f\left(x^*+u\left(x_1-x^*\right)\right)g\left(x^*+u\left(x_1-x^*\right)\right)-f\left(x^*+u\left(x_1-x^*\right)\right)g\left(x^*\right)= \tag{27}$$

$$\frac{f\left(x^*+u\left(x_1-x^*\right)\right)}{\Gamma\left((n+1)\alpha\right)}\int_0^u (u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(g\left(x^*+t\left(x_1-x^*\right)\right)\right)\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

Furthermore we have

$$g\left(x_0+u\left(x^*-x_0\right)\right)f\left(x_0+u\left(x^*-x_0\right)\right)-g\left(x_0+u\left(x^*-x_0\right)\right)f\left(x^*\right)= \tag{28}$$

$$\frac{g\left(x_0+u\left(x^*-x_0\right)\right)}{\Gamma\left((n+1)\alpha\right)}\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}f\left(x_0+t\left(x^*-x_0\right)\right)\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

And, next we have

$$f\left(x_0+u\left(x^*-x_0\right)\right)g\left(x_0+u\left(x^*-x_0\right)\right)-f\left(x_0+u\left(x^*-x_0\right)\right)g\left(x^*\right)= \tag{29}$$

$$\frac{f\left(x_0+u\left(x^*-x_0\right)\right)}{\Gamma\left((n+1)\alpha\right)}\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}g\left(x_0+t\left(x^*-x_0\right)\right)\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

Adding (26) and (27) we get

$$2f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)-g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)f\left(x^{*}\right)-$$

$$f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)g\left(x^{*}\right)=\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}$$

$$\left[g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}\left(u-t\right)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(f\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt\right.$$

$$\left.+f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}\left(u-t\right)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt\right], \tag{30}$$

$\forall\ u \in [0,1]$.

Adding (28) and (29) we obtain

$$2g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)-g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)f\left(x^{*}\right)$$

$$-f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)g\left(x^{*}\right)=\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}$$

$$\left[g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}\left(t-u\right)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}f\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\,dt\right.$$

$$\left.+f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}\left(t-u\right)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}g\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\,dt\right], \tag{31}$$

$\forall\ u \in [0,1]$.

Next we integrate (30) and (31) to obtain:

$$2\int_{x^{*}}^{x_{1}}f(x)\,g(x)\,dx-f\left(x^{*}\right)\int_{x^{*}}^{x_{1}}g(x)\,dx-g\left(x^{*}\right)\int_{x^{*}}^{x_{1}}f(x)\,dx=\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}$$

$$\left[\int_{0}^{1}\left[g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}\left(u-t\right)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(f\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt\right.\right.$$

$$\left.\left.+f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}\left(u-t\right)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt\right]du\right], \tag{32}$$

and

$$2\int_{x_{0}}^{x^{*}}f(x)\,g(x)\,dx-f\left(x^{*}\right)\int_{x_{0}}^{x^{*}}g(x)\,dx-g\left(x^{*}\right)\int_{x_{0}}^{x^{*}}f(x)\,dx=\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}$$

$$\left[\int_{0}^{1}\left[g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}\left(t-u\right)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}f\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\,dt\right.\right.$$

$$\left.\left.+f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}\left(t-u\right)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}g\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\,dt\right]du\right]. \tag{33}$$

Adding (32) and (33) we derive

$$K(f,g)(x^*) :=$$

$$2\int_{\langle x_0x^*x_1\rangle} f(x)\,g(x)\,dx - f(x^*)\int_{\langle x_0x^*x_1\rangle} g(x)\,dx - g(x^*)\int_{\langle x_0x^*x_1\rangle} f(x)\,dx$$

$$= \frac{1}{\Gamma((n+1)\alpha)}$$

$$\Bigg\{\Bigg[\int_0^1 \Bigg[g(x^*+u(x_1-x^*))\int_0^u (u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}(f(x^*+t(x_1-x^*)))\right)(t)\,dt$$

$$+f(x^*+u(x_1-x^*))\int_0^u (u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}(g(x^*+t(x_1-x^*)))\right)(t)\,dt\Bigg]du+ \tag{34}$$

$$\Bigg[\int_0^1 \Bigg[g(x_0+u(x^*-x_0))\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}f(x_0+t(x^*-x_0))\right)(t)\,dt$$

$$+f(x_0+u(x^*-x_0))\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}g(x_0+t(x^*-x_0))\right)(t)\,dt\Bigg]du\Bigg]\Bigg\},$$

proving the claim. ■

We present the following alternative general fractional Ostrowski type inequalities:

Theorem 11 *Here all as in Theorem 10. Then*

i)

$$\|K(f,g)(x^*)\|_2 \le \frac{1}{\Gamma((n+1)\alpha+2)}$$

$$\Bigg\{\|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]}\left\|\left\|D_{*0}^{(n+1)\alpha}(f(x^*+t(x_1-x^*)))(t)\right\|_2\right\|_{t,\infty,[0,1]}$$

$$+\left\|\|f(x^*+u(x_1-x^*))\|_2\right\|_{u,\infty,[0,1]}\left\|D_{*0}^{(n+1)\alpha}(g(x^*+t(x_1-x^*)))(t)\right\|_{t,\infty,[0,1]}+$$

$$\|g(x_0+u(x^*-x_0))\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{(n+1)\alpha}f(x_0+t(x^*-x_0))(t)\right\|_2\right\|_{t,\infty,[0,1]}$$

$$+\left\|\|f(x_0+u(x^*-x_0))\|_2\right\|_{u,\infty,[0,1]}\left\|D_{1-}^{(n+1)\alpha}g(x_0+t(x^*-x_0))(t)\right\|_{t,\infty,[0,1]}\Bigg\}, \tag{35}$$

ii) if $\alpha \ge \frac{1}{n+1}$, we obtain

$$\|K(f,g)(x^*)\|_2 \le \frac{1}{\Gamma((n+1)\alpha+1)}$$

$$\Bigg\{\|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]}\left\|\left\|D_{*0}^{(n+1)\alpha}(f(x^*+t(x_1-x^*)))(t)\right\|_2\right\|_{t,L_1([0,1])}$$

$$+\left\|\left\|f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right\|_{2}\right\|_{u,\infty,[0,1]}\left\|D_{*0}^{(n+1)\alpha}\left(g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)(t)\right\|_{t,L_{1}([0,1])}+$$

$$\left\|g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{(n+1)\alpha}\left(f\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\right\|_{2}\right\|_{t,L_{1}([0,1])}$$

$$\left.+\left\|\left\|f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right\|_{2}\right\|_{u,\infty,[0,1]}\left\|D_{1-}^{(n+1)\alpha}\left(g\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\right\|_{t,L_{1}([0,1])}\right\}, \tag{36}$$

iii) if $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $\alpha>\frac{1}{(n+1)q}$, *we derive*

$$\left\|K(f,g)\left(x^{*}\right)\right\|_{2}\leq\frac{1}{\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}\Gamma\left((n+1)\alpha\right)\left((n+1)\alpha+\frac{1}{p}\right)}$$

$$\left\{\left\|g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{*0}^{(n+1)\alpha}f\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)(t)\right\|_{2}\right\|_{t,L_{q}([0,1])}\right.$$

$$+\left\|\left\|f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right\|_{2}\right\|_{u,\infty,[0,1]}\left\|D_{*0}^{(n+1)\alpha}g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)(t)\right\|_{t,L_{q}([0,1])}+$$

$$\left\|g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right\|_{u,\infty,[0,1]}\left\|\left\|D_{1-}^{(n+1)\alpha}f\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)(t)\right\|_{2}\right\|_{t,L_{q}([0,1])}$$

$$\left.+\left\|\left\|f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right\|_{2}\right\|_{u,\infty,[0,1]}\left\|D_{1-}^{(n+1)\alpha}g\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)(t)\right\|_{t,L_{q}([0,1])}\right\}. \tag{37}$$

Proof. We have that

$$\left\|K(f,g)\left(x^{*}\right)\right\|_{2}=$$

$$\left\|2\int_{\langle x_{0}x^{*}x_{1}\rangle}f(x)g(x)\,dx-f\left(x^{*}\right)\int_{\langle x_{0}x^{*}x_{1}\rangle}g(x)\,dx-g\left(x^{*}\right)\int_{\langle x_{0}x^{*}x_{1}\rangle}f(x)\,dx\right\|_{2}$$

$$\overset{(21)}{\leq}\frac{1}{\Gamma\left((n+1)\alpha\right)}$$

$$\left\{\left\|\left[\int_{0}^{1}\left[g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}(u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(f\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt\right.\right.\right.\right. \tag{38}$$

$$\left.\left.\left.+f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\int_{0}^{u}(u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}\left(g\left(x^{*}+t\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt\right]du\right]\right\|_{2}+$$

$$\left\|\left[\int_{0}^{1}\left[g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}(t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}f\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\,dt\right.\right.\right.$$

$$\left.\left.\left.\left.+f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\int_{u}^{1}(t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}g\left(x_{0}+t\left(x^{*}-x_{0}\right)\right)\right)(t)\,dt\right]du\right]\right\|_{2}\right\}$$

$$\leq\frac{1}{\Gamma\left((n+1)\alpha\right)}$$

$$\left\{ \left\| \int_0^1 \left[g\left(x^* + u\left(x_1 - x^*\right)\right) \int_0^u (u-t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha} \left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)\right)(t)\, dt \right] du \right\|_2 \right.$$

$$+ \left\| \int_0^1 \left[f\left(x^* + u\left(x_1 - x^*\right)\right) \int_0^u (u-t)^{(n+1)\alpha - 1} \left(D_{*0}^{(n+1)\alpha} \left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)\right)(t)\, dt \right] du \right\|_2 + \tag{39}$$

$$\left\| \int_0^1 \left[g\left(x_0 + u\left(x^* - x_0\right)\right) \int_u^1 (t-u)^{(n+1)\alpha - 1} \left(D_{1-}^{(n+1)\alpha} f\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t)\, dt \right] du \right\|_2$$

$$\left. + \left\| \int_0^1 \left[f\left(x_0 + u\left(x^* - x_0\right)\right) \int_u^1 (t-u)^{(n+1)\alpha - 1} \left(D_{1-}^{(n+1)\alpha} \left(g\left(x_0 + t\left(x^* - x_0\right)\right)\right)\right)(t)\, dt \right] du \right\|_2 \right\}$$

$$\leq \frac{1}{\Gamma\left((n+1)\alpha\right)}$$

$$\left\{ \int_0^1 \left[\left| g\left(x^* + u\left(x_1 - x^*\right)\right)\right| \int_0^u (u-t)^{(n+1)\alpha - 1} \left\| D_{*0}^{(n+1)\alpha} \left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 dt \right] du \right.$$

$$+ \int_0^1 \left[\left\| f\left(x^* + u\left(x_1 - x^*\right)\right)\right\|_2 \int_0^u (u-t)^{(n+1)\alpha - 1} \left| D_{*0}^{(n+1)\alpha} \left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right| dt \right] du +$$

$$\int_0^1 \left[\left| g\left(x_0 + u\left(x^* - x_0\right)\right)\right| \int_u^1 (t-u)^{(n+1)\alpha - 1} \left\| D_{1-}^{(n+1)\alpha} f\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_2 dt \right] du$$

$$\left. + \int_0^1 \left[\left\| f\left(x_0 + u\left(x^* - x_0\right)\right)\right\|_2 \int_u^1 (t-u)^{(n+1)\alpha - 1} \left| D_{1-}^{(n+1)\alpha} g\left(\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right| dt \right] du \right\} \tag{40}$$

$$=: (\psi).$$

i) We observe that

$$(\psi) \leq \frac{1}{\Gamma\left((n+1)\alpha\right)}$$

$$\left\{ \int_0^1 \left| g\left(x^* + u\left(x_1 - x^*\right)\right)\right| \frac{u^{(n+1)\alpha}}{(n+1)\alpha} \left\| \left\| D_{*0}^{(n+1)\alpha} \left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 \right\|_{t,\infty,[0,1]} du \right.$$

$$+ \int_0^1 \left\| f\left(x^* + u\left(x_1 - x^*\right)\right)\right\|_2 \frac{u^{(n+1)\alpha}}{(n+1)\alpha} \left\| D_{*0}^{(n+1)\alpha} \left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_{t,\infty,[0,1]} du + \tag{41}$$

$$\int_0^1 \left| g\left(x_0 + u\left(x^* - x_0\right)\right)\right| \frac{(1-u)^{(n+1)\alpha}}{(n+1)\alpha} \left\| \left\| D_{1-}^{(n+1)\alpha} f\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_2 \right\|_{t,\infty,[0,1]} du$$

$$\left. + \int_0^1 \left\| f\left(x_0 + u\left(x^* - x_0\right)\right)\right\|_2 \frac{(1-u)^{(n+1)\alpha}}{(n+1)\alpha} \left\| D_{1-}^{(n+1)\alpha} g\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_{t,\infty,[0,1]} du \right\}$$

$$\leq \frac{1}{\Gamma\left((n+1)\alpha + 2\right)}$$

$$\left\{ \left\| g\left(x^* + u\left(x_1 - x^*\right)\right)\right\|_{u,\infty,[0,1]} \left\| \left\| D_{*0}^{(n+1)\alpha} \left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 \right\|_{t,\infty,[0,1]} \right.$$

$$+ \left\| \left\| f\left(x^* + u\left(x_1 - x^*\right)\right) \right\|_2 \right\|_{u,\infty,[0,1]} \left\| D_{*0}^{(n+1)\alpha}\left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_{t,\infty,[0,1]} +$$

$$\left\| g\left(x_0 + u\left(x^* - x_0\right)\right) \right\|_{u,\infty,[0,1]} \left\| \left\| D_{1-}^{(n+1)\alpha} f\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_2 \right\|_{t,\infty,[0,1]}$$

$$+ \left\| \left\| f\left(x_0 + u\left(x^* - x_0\right)\right) \right\|_2 \right\|_{u,\infty,[0,1]} \left\| D_{1-}^{(n+1)\alpha} g\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_{t,\infty,[0,1]} \Bigg\}, \quad (42)$$

proving (i).

ii) Let $(n+1)\alpha \geq 1$ or $\alpha \geq \frac{1}{n+1}$, then

$$(\psi) \leq \frac{1}{\Gamma\left((n+1)\alpha\right)}$$

$$\Bigg\{ \int_0^1 \left| g\left(x^* + u\left(x_1 - x^*\right)\right) \right| u^{(n+1)\alpha - 1} \left\| \left\| D_{*0}^{(n+1)\alpha}\left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 \right\|_{t,L_1([0,1])} du$$

$$+ \int_0^1 \left\| f\left(x^* + u\left(x_1 - x^*\right)\right) \right\|_2 u^{(n+1)\alpha - 1} \left\| D_{*0}^{(n+1)\alpha}\left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_{t,L_1([0,1])} du +$$

$$\int_0^1 \left| g\left(x_0 + u\left(x^* - x_0\right)\right) \right| (1-u)^{(n+1)\alpha - 1} \left\| \left\| D_{1-}^{(n+1)\alpha}\left(f\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right\|_2 \right\|_{t,L_1([0,1])} du \quad (43)$$

$$+ \int_0^1 \left\| f\left(x_0 + u\left(x^* - x_0\right)\right) \right\|_2 (1-u)^{(n+1)\alpha - 1} \left\| D_{1-}^{(n+1)\alpha}\left(g\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right\|_{t,L_1([0,1])} du \Bigg\}$$

$$\leq \frac{1}{\Gamma\left((n+1)\alpha + 1\right)}$$

$$\Bigg\{ \left\| g\left(x^* + u\left(x_1 - x^*\right)\right) \right\|_{u,\infty,[0,1]} \left\| \left\| D_{*0}^{(n+1)\alpha}\left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 \right\|_{t,L_1([0,1])}$$

$$+ \left\| \left\| f\left(x^* + u\left(x_1 - x^*\right)\right) \right\|_2 \right\|_{u,\infty,[0,1]} \left\| D_{*0}^{(n+1)\alpha}\left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_{t,L_1([0,1])} +$$

$$\left\| g\left(x_0 + u\left(x^* - x_0\right)\right) \right\|_{u,\infty,[0,1]} \left\| \left\| D_{1-}^{(n+1)\alpha}\left(f\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right\|_2 \right\|_{t,L_1([0,1])} \quad (44)$$

$$+ \left\| \left\| f\left(x_0 + u\left(x^* - x_0\right)\right) \right\|_2 \right\|_{u,\infty,[0,1]} \left\| D_{1-}^{(n+1)\alpha}\left(g\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right\|_{t,L_1([0,1])} \Bigg\},$$

proving (ii).

iii) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $(n+1)\alpha > \frac{1}{q}$ or $\alpha > \frac{1}{(n+1)q}$. Then (by applying Hölder's inequality) we obtain:

$$(\psi) \leq \frac{1}{\Gamma\left((n+1)\alpha\right)}$$

$$\Bigg\{ \int_0^1 \left| g\left(x^* + u\left(x_1 - x^*\right)\right) \right| \frac{u^{(n+1)\alpha - 1 + \frac{1}{p}}}{\left(p\left((n+1)\alpha - 1\right) + 1\right)^{\frac{1}{p}}}$$

$$
\left\|\left\|\left(D_{*0}^{(n+1)\alpha} f\left(x^*+t\left(x_1-x^*\right)\right)\right)(t)\right\|_2\right\|_{t,L_q([0,1])} du
$$

$$
+\int_0^1 \left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2 \frac{u^{(n+1)\alpha-1+\frac{1}{p}}}{\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}} \tag{45}
$$

$$
\left\|\left(D_{*0}^{(n+1)\alpha} g\left(x^*+t\left(x_1-x^*\right)\right)\right)(t)\right\|_{t,L_q([0,1])} du+
$$

$$
\int_0^1 \left|g\left(x_0+u\left(x^*-x_0\right)\right)\right| \frac{(1-u)^{(n+1)\alpha-1+\frac{1}{p}}}{\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}}
$$

$$
\left\|\left\|\left(D_{1-}^{(n+1)\alpha} f\left(x_0+t\left(x^*-x_0\right)\right)\right)(t)\right\|_2\right\|_{t,L_q([0,1])} du
$$

$$
+\int_0^1 \left\|f\left(x_0+u\left(x^*-x_0\right)\right)\right\|_2 \frac{(1-u)^{(n+1)\alpha-1+\frac{1}{p}}}{\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}}
$$

$$
\left.\left\|\left(D_{1-}^{(n+1)\alpha} g\left(x_0+t\left(x^*-x_0\right)\right)\right)(t)\right\|_{t,L_q([0,1])} du\right\}
$$

$$
\leq \frac{1}{\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}} \Gamma\left((n+1)\alpha\right)\left((n+1)\alpha+\frac{1}{p}\right)}
$$

$$
\left\{\left\|g\left(x^*+u\left(x_1-x^*\right)\right)\right\|_{u,\infty,[0,1]} \left\|\left\|\left(D_{*0}^{(n+1)\alpha} f\left(x^*+t\left(x_1-x^*\right)\right)\right)(t)\right\|_2\right\|_{t,L_q([0,1])}\right.
$$

$$
+\left\|\left\|f\left(x^*+u\left(x_1-x^*\right)\right)\right\|_2\right\|_{u,\infty,[0,1]} \left\|\left(D_{*0}^{(n+1)\alpha} g\left(x^*+t\left(x_1-x^*\right)\right)\right)(t)\right\|_{t,L_q([0,1])}+
$$

$$
\left\|g\left(x_0+u\left(x^*-x_0\right)\right)\right\|_{u,\infty,[0,1]} \left\|\left\|\left(D_{1-}^{(n+1)\alpha} f\left(x_0+t\left(x^*-x_0\right)\right)\right)(t)\right\|_2\right\|_{t,L_q([0,1])}
$$

$$
\left.+\left\|\left\|f\left(x_0+u\left(x^*-x_0\right)\right)\right\|_2\right\|_{u,\infty,[0,1]} \left\|\left(D_{1-}^{(n+1)\alpha} g\left(x_0+t\left(x^*-x_0\right)\right)\right)(t)\right\|_{t,L_q([0,1])}\right\}, \tag{46}
$$

proving (iii). ■

We make

Remark 12 *We notice that (*$0<\alpha<1$*)*

$$
D_{*0}^{\alpha}\left(f\left(x^*+u\left(x_1-x^*\right)\right)\right) \overset{(8)}{=} \tag{47}
$$

$$
\frac{1}{\Gamma(1-\alpha)} \int_0^u (u-t)^{-\alpha} \left(f'\left(x^*+u\left(x_1-x^*\right)\right)\right)\left(x_1-x^*\right) dt,
$$

and

$$
\left\|D_{*0}^{\alpha}\left(f\left(x^*+u\left(x_1-x^*\right)\right)\right)\right\|_2 \leq
$$

$$
\frac{1}{\Gamma(1-\alpha)} \int_0^u (u-t)^{-\alpha} \left\|f'\left(x^*+u\left(x_1-x^*\right)\right)\left(x_1-x^*\right)\right\|_2 dt \leq
$$

$$\frac{u^{1-\alpha}}{\Gamma(2-\alpha)}\left\|\left\|f'\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\left(x_{1}-x^{*}\right)\right\|_{2}\right\|_{t,\infty,[0,1]}. \tag{48}$$

Clearly we have that

$$D_{*0}^{\alpha}\left(f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right)(0)=0, \tag{49}$$

and, similarly, it holds

$$D_{*0}^{\alpha}\left(g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right)(0)=0. \tag{50}$$

Next, we have $(0<\alpha<1)$

$$D_{1-}^{\alpha}\left(f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right)\overset{(13)}{=} \tag{51}$$

$$\frac{-1}{\Gamma(1-\alpha)}\int_{u}^{1}(t-u)^{-\alpha}f'\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\left(x^{*}-x_{0}\right)dt,$$

and

$$\left\|D_{1-}^{\alpha}\left(f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right)\right\|_{2}\leq$$

$$\frac{1}{\Gamma(1-\alpha)}\int_{u}^{1}(t-u)^{-\alpha}\left\|f'\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\left(x^{*}-x_{0}\right)\right\|_{2}dt\leq$$

$$\frac{(1-u)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|\left\|f'\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\left(x^{*}-x_{0}\right)\right\|_{2}\right\|_{t,\infty,[0,1]}. \tag{52}$$

Hence we get

$$D_{1-}^{\alpha}\left(f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right)(1)=0, \tag{53}$$

and, similarly,

$$D_{1-}^{\alpha}\left(g\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)\right)(1)=0. \tag{54}$$

The above (49), (50), (53), (54) are valid for any $x_0, x^*, x_1 \in M$.

When $n=1$*, by Theorems 8 and 9, we obtain directly and without any initial conditions the following* $(0<\alpha<1)$

$$f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)-f\left(x^{*}\right)=$$

$$\frac{1}{\Gamma(2\alpha)}\int_{0}^{u}(u-t)^{2\alpha-1}\left(D_{*0}^{2\alpha}\left(f\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt,$$

and

$$g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)-g\left(x^{*}\right)= \tag{55}$$

$$\frac{1}{\Gamma(2\alpha)}\int_{0}^{u}(u-t)^{2\alpha-1}\left(D_{*0}^{2\alpha}\left(g\left(x^{*}+u\left(x_{1}-x^{*}\right)\right)\right)\right)(t)\,dt,$$

$\forall\ u\in[0,1]$.

Similarly, we derive that

$$f\left(x_{0}+u\left(x^{*}-x_{0}\right)\right)-f\left(x^{*}\right)=$$

$$\frac{1}{\Gamma(2\alpha)}\int_u^1 (t-u)^{2\alpha-1}\left(D_{1-}^{2\alpha} f(x_0+u(x^*-x_0))\right)(t)\,dt,$$

and

$$g(x_0+u(x^*-x_0)) - g(x^*) = \tag{56}$$

$$\frac{1}{\Gamma(2\alpha)}\int_u^1 (t-u)^{2\alpha-1}\left(D_{1-}^{2\alpha} g(x_0+u(x^*-x_0))\right)(t)\,dt,$$

$\forall\ u \in [0,1]$.

So Theorems 10, 11 are valid, without any initial conditions, when $n=1$, $0<\alpha<1$.

3.2 About alternative general mixed fractional Grüss type inequalities

We make

Remark 13 *(to Theorem 10) Case of* $n=1$, $0<\alpha<1$, *and from now on* $x^* \in \overline{x_0x_1}$. *We have that*

$$K(f,g)(x^*) = K(f,g)(x_0+u(x_1-x_0)), \quad \textit{for some } u\in[0,1].$$

Furthermore, it holds

$$\int_0^1 K(f,g)(x_0+u(x_1-x_0))\,du = \tag{57}$$

$$2\left[\int_{x_0}^{x_1} f(x)\,g(x)\,dx - \left(\int_{x_0}^{x_1} f(x)\,dx\right)\left(\int_{x_0}^{x_1} g(x)\,dx\right)\right],$$

by (21) being valid for any $x^* \in \overline{x_0x_1}$.

That is

$$\Delta(f,g;x_0,x_1) := \left\|\int_{x_0}^{x_1} f(x)\,g(x)\,dx - \left(\int_{x_0}^{x_1} f(x)\,dx\right)\left(\int_{x_0}^{x_1} g(x)\,dx\right)\right\|_2$$

$$= \frac{1}{2}\left\|\int_0^1 K(f,g)(x_0+u(x_1-x_0))\,du\right\|_2 \le \frac{1}{2}\int_0^1 \|K(f,g)(x_0+u(x_1-x_0))\|_2\,du. \tag{58}$$

We denote by

$$M(f,g) := \max\Bigg\{\sup_{x^*\in\overline{x_0x_1}} \|g(x^*+u(x_1-x^*))\|_{u,\infty,[0,1]},$$

$$\sup_{x^*\in\overline{x_0x_1}} \|\|f(x^*+u(x_1-x^*))\|_2\|_{u,\infty,[0,1]},$$

$$\sup_{x^*\in\overline{x_0x_1}} \|g(x_0+u(x^*-x_0))\|_{u,\infty,[0,1]}, \tag{59}$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| f\left(x_0 + u\left(x^* - x_0\right)\right) \right\|_2 \right\|_{u,\infty,[0,1]} \Bigg\} < \infty,$$

and call

$$N_1(f,g) := \max \Bigg\{ \sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} \left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 \right\|_{t,\infty,[0,1]},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D_{*0}^{2\alpha} \left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_{t,\infty,[0,1]},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} f\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_2 \right\|_{t,\infty,[0,1]},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D_{1-}^{2\alpha} g\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_{t,\infty,[0,1]} \Bigg\} \tag{60}$$

assumed to be finite,

and call

$$N_2(f,g) := \max \Bigg\{ \sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} \left(f\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_2 \right\|_{t,L_1([0,1])},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D_{*0}^{2\alpha} \left(g\left(x^* + t\left(x_1 - x^*\right)\right)\right)(t) \right\|_{t,L_1([0,1])}, \tag{61}$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} \left(f\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right\|_2 \right\|_{t,L_1([0,1])},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D_{1-}^{2\alpha} \left(g\left(x_0 + t\left(x^* - x_0\right)\right)\right)(t) \right\|_{t,L_1([0,1])} \Bigg\}$$

assumed to be finite,

and call

$$N_3(f,g) := \max \Bigg\{ \sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} f\left(x^* + t\left(x_1 - x^*\right)\right)(t) \right\|_2 \right\|_{t,L_q([0,1])},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D_{*0}^{2\alpha} g\left(x^* + t\left(x_1 - x^*\right)\right)(t) \right\|_{t,L_q([0,1])},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} f\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_2 \right\|_{t,L_q([0,1])},$$

$$\sup_{x^* \in \overline{x_0 x_1}} \left\| D_{1-}^{2\alpha} g\left(x_0 + t\left(x^* - x_0\right)\right)(t) \right\|_{t,L_q([0,1])} \Bigg\} \tag{62}$$

assumed to be finite.

We present the following Grüss type inequalities:

Theorem 14 *Following the assumptions of Theorem 10 for* $n = 1$, $0 < \alpha < 1$ *and* $x^* \in \overline{x_0 x_1}$; $x_0, x^*, x_1 \in M$, *but without any initial conditions. All the rest as in Remark 13. Then*

i)

$$\Delta(f, g; x_0, x_1) \leq \frac{2M(f, g) N_1(f, g)}{\Gamma(2(\alpha + 1))}, \tag{63}$$

ii) when $\alpha \geq \frac{1}{2}$, *we have*

$$\Delta(f, g; x_0, x_1) \leq \frac{2M(f, g) N_2(f, g)}{\Gamma(2\alpha + 1)}, \tag{64}$$

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; \alpha > \frac{1}{2q}$. *we have*

$$\Delta(f, g; x_0, x_1) \leq \frac{2M(f, g) N_3(f, g)}{(p(2\alpha - 1) + 1)^{\frac{1}{p}} \Gamma(2\alpha) \left(2\alpha + \frac{1}{p}\right)}. \tag{65}$$

Proof. By Theorem 11, see also Remark 12. ■

Since M is compact an interesting application can be when x_0, x_1 are the endpoints of its diameter.

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