SOME BASIC RESULTS FOR THE Φ-y-NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H, K be two Hilbert spaces. For $A \in \mathcal{B}(H)$, $A > 0$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, a linear positive and normalized functional, and $y \in K$ with $\|y\| = 1$ we define the Φ-y-*normalized determinant* by

$$\Delta_{\Phi,y}(A) := \exp \langle \Phi(\ln A) y, y \rangle .$$

In this paper we show, among other that

$$\begin{aligned} 1 &\leq \frac{\langle \Phi(A) y, y \rangle}{\Delta_{\Phi,y}(A)} \leq \frac{\langle \Phi(A) y, y \rangle}{\Delta_y(\Phi(A))} \\ &\leq \exp\left(\langle \Phi(A) y, y \rangle \left\langle (\Phi(A))^{-1} y, y \right\rangle - 1 \right) \end{aligned}$$

and

$$\begin{aligned} 1 &\leq \frac{\Delta_y(\Phi(A))}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \leq \frac{\Delta_{\Phi,y}(A)}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \\ &\leq \exp\left(\left\langle (\Phi(A))^{-1} y, y \right\rangle \langle \Phi(A) y, y \rangle - 1 \right), \end{aligned}$$

where $A, B \in \mathcal{B}(H)$ with $A, B > 0$ and $y \in K$ with $\|y\| = 1$. Some applications for Hadamard product of operators are also given.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [3], [4], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [3].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x \right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x \rangle$;

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(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha) A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [3] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [8]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [4], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \le \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \le S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \le \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \le S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu} \leq\right) S\left(\left(\frac{a}{b}\right)^{r}\right) a^{1-\nu}b^{\nu} \leq (1-\nu)\, a+\nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu}, \tag{1.6}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

The second inequality in (1.6) is due to Tominaga [9] while the first one is due to Furuichi [5].

Let $H,\ K$ be complex Hilbert spaces. Following [1] (see also [6, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A+\mu B)=\lambda\Phi(A)+\mu\Phi(B)$$

for any $\lambda,\ \mu\in\mathbb{C}$ and $A,\ B\in\mathcal{B}(H)$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A\in\mathcal{B}^{+}(H)$ then $\Phi(A)\in\mathcal{B}^{+}(K)$. We write $\Phi\in\mathfrak{P}[\mathcal{B}(H),\mathcal{B}(K)]$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is normalized if it preserves the identity operator, i.e. $\Phi(1_H)=1_K$. We write $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A\leq B \text{ implies } \Phi(A)\leq\Phi(B)$$

and preserves the adjoint operation $\Phi(A^{*})=\Phi(A)^{*}$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $\alpha 1_H\leq A\leq\beta 1_H$, then $\alpha 1_K\leq\Phi(A)\leq\beta 1_K$.

If the map $\Psi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear, positive and $\Psi(1_H)\in\mathcal{B}^{++}(K)$ then by putting $\Phi=\Psi^{-1/2}(1_H)\,\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, namely it is also normalized.

The following Jensen's type result is well known [1]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have* ln

$$f(\Phi(A))\leq\Phi(f(A)). \tag{1.7}$$

For $A\in\mathcal{B}(H),\ A>0,\ \Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$ we define the Φ-y-*normalized determinant* by

$$\Delta_{\Phi,y}(A):=\exp\langle\Phi(\ln A)\,y,y\rangle.$$

We observe that for $A\in\mathcal{B}(H),\ A>0,\ \Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$,

$$\Delta_{\Phi,y}(A^{t})=\Delta_{\Phi,y}(A)^{t},\ t>0;$$
$$\Delta_{\Phi,y}(tA)=t\Delta_{\Phi,y}(A),\ \Delta_{\Phi,y}(tI)=t,\ t>0$$

and, by the operator monotonicity of Φ and ln

$$0<A\leq B \text{ implies that } \Delta_{\Phi,y}(A)\leq\Delta_{\Phi,y}(B).$$

Also

$$\Delta_{\Phi,y}(AB)=\Delta_{\Phi,y}(A)\Delta_{\Phi,y}(B) \text{ for commuting } A \text{ and } B$$

and by the operator concavity of ln and by (1.7) we obtain

$$\Delta_{\Phi,y}(A)\leq\Delta_{y}(\Phi(A)) \tag{1.8}$$

for $A\in\mathcal{B}(H),\ A>0,\ \Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$.

We observe that if $K=H$ and $\Phi(A)=A,\ A\in\mathcal{B}(H)$ then $\Delta_{\Phi,y}(A)=\Delta_{y}(A)$, $y\in H$ with $\|y\|=1$.

Recall that the *Hadamard product* of A and B in $\mathcal{B}(H)$ is defined to be the operator $A \circ B \in \mathcal{B}(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle Ae_j, e_j \rangle \langle Be_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [2], if "$\otimes$" denotes the tensorial product, then we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.9}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

We can define the map $\Phi_{\mathcal{U}} : \mathcal{B}(H \otimes H) \to \mathcal{B}(H)$,

$$\Phi_{\mathcal{U}} (A \otimes B) := \mathcal{U}^* (A \otimes B) \mathcal{U} = A \circ B, \tag{1.10}$$

which is linear, positive and preserves the unity operator.

Motivated by the above results, in this paper we show, among other that

$$\begin{aligned} 1 &\le \frac{\langle \Phi(A) y, y\rangle}{\Delta_{\Phi,y}(A)} \le \frac{\langle \Phi(A) y, y\rangle}{\Delta_y(\Phi(A))} \\ &\le \exp\left(\langle \Phi(A) y, y\rangle \left\langle (\Phi(A))^{-1} y, y \right\rangle - 1 \right) \end{aligned}$$

and

$$\begin{aligned} 1 &\le \frac{\Delta_y(\Phi(A))}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \le \frac{\Delta_{\Phi,y}(A)}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \\ &\le \exp\left(\left\langle (\Phi(A))^{-1} y, y \right\rangle \langle \Phi(A) y, y\rangle - 1 \right) \end{aligned}$$

where $A, B \in \mathcal{B}(H)$ with $A, B > 0$ and $y \in K$ with $\|y\| = 1$. Some applications for Hadamard product of operators are also given.

2. Main Results

We start to the following log-concavity property:

Proposition 1. *For $A,\ B \in \mathcal{B}(H)$, $A, B > 0$, $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$ and $y \in K$ with $\|y\| = 1$, then*

$$\Delta_{\Phi,y}((1-t)A + tB) \ge [\Delta_{\Phi,y}(A)]^{1-t} [\Delta_{\Phi,y}(B)]^t \tag{2.1}$$

for all $t \in [0, 1]$.

Proof. By the operator concavity of ln on $(0, \infty)$ we have

$$\ln((1-t)A + tB) \ge (1-t)\ln A + t \ln B$$

for all $A,\ B > 0$ and $t \in [0, 1]$.

If we apply Φ to this inequality we get

$$\Phi[\ln((1-t)A + tB)] \ge (1-t)\Phi(\ln A) + t\Phi(\ln B)$$

and by taking the inner product over $y \in K$ with $\|y\| = 1$, we obtain

$$\langle \Phi[\ln((1-t)A + tB)] y, y\rangle \ge (1-t)\langle \Phi(\ln A) y, y\rangle + t\langle \Phi(\ln B) y, y\rangle.$$

By taking the exponential we derive that

$$\begin{aligned}\Delta_{\Phi,y}\left((1-t)A+tB\right) &= \exp\left\langle \Phi\left[\ln\left((1-t)A+tB\right)\right]y,y\right\rangle \\ &\geq \exp\left[(1-t)\left\langle \Phi(\ln A)y,y\right\rangle + t\left\langle \Phi(\ln B)y,y\right\rangle\right] \\ &= \left[\exp\left\langle \Phi(\ln A)y,y\right\rangle\right]^{1-t}\left[\exp\left\langle \Phi(\ln B)y,y\right\rangle\right]^{t} \\ &= \left[\Delta_{\Phi,y}(A)\right]^{1-t}\left[\Delta_{\Phi,y}(B)\right]^{t}\end{aligned}$$

and the proposition is proved. □

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 1. *For* $A,\ B \in \mathcal{B}(H),\ A, B > 0,\ \Phi \in \mathfrak{P}_N\left[\mathcal{B}(H), \mathcal{B}(K)\right]$ *and* $y \in K$ *with* $\|y\| = 1$,

$$\int_0^1 \Delta_{\Phi,y}((1-t)A+tB)dt \geq L\left(\Delta_{\Phi,y}(A), \Delta_{\Phi,y}(B)\right). \tag{2.2}$$

Proof. If we take the integral over $t \in [0,1]$ in (2.1), then we get

$$\begin{aligned}\int_0^1 \Delta_{\Phi,y}((1-t)A+tB)dt &\geq \int_0^1 \left[\Delta_{\Phi,y}(A)\right]^{1-t}\left[\Delta_{\Phi,y}(B)\right]^{t}dt \\ &= L\left(\Delta_{\Phi,y}(A), \Delta_{\Phi,y}(B)\right),\end{aligned}$$

which proves (2.2). □

Corollary 2. *For* $A,\ B \in \mathcal{B}(H),\ A, B > 0,\ \Phi \in \mathfrak{P}_N\left[\mathcal{B}(H), \mathcal{B}(K)\right]$ *and* $y \in K$ *with* $\|y\| = 1$,

$$\Delta_{\Phi,y}\left(\frac{A+B}{2}\right) \geq \int_0^1 \left[\Delta_{\Phi,y}\left((1-t)A+tB\right)\right]^{1/2}\left[\Delta_{\Phi,y}\left(tA+(1-t)B\right)\right]^{1/2}dt. \tag{2.3}$$

Proof. We get from (2.1) for $t = 1/2$ that

$$\Delta_{\Phi,y}\left(\frac{A+B}{2}\right) \geq \left[\Delta_{\Phi,y}(A)\right]^{1/2}\left[\Delta_{\Phi,y}(B)\right]^{1/2}.$$

If we replace A by $(1-t)A+tB$ and B by $tA+(1-t)B$ we obtain

$$\Delta_{\Phi,y}\left(\frac{A+B}{2}\right) \geq \left[\Delta_{\Phi,y}\left((1-t)A+tB\right)\right]^{1/2}\left[\Delta_{\Phi,y}\left(tA+(1-t)B\right)\right]^{1/2}.$$

By taking the integral, we derive the desired result (2.3). □

We have the following upper and lower bounds for the

Theorem 2. *Assume that* $A \in \mathcal{B}(H),\ A > 0,\ \Phi \in \mathfrak{P}_N\left[\mathcal{B}(H), \mathcal{B}(K)\right]$ *and* $y \in K$ *with* $\|y\| = 1$, *then*

$$\begin{aligned}a\exp\left(1 - a\left\langle (\Phi(A))^{-1}y,y\right\rangle\right) &\leq \Delta_y(\Phi(A)) \\ &\leq \Delta_{\Phi,y}(A) \leq a\exp\left(a^{-1}\left\langle \Phi(A)y,y\right\rangle - 1\right)\end{aligned} \tag{2.4}$$

for all $a > 0$.

In particular,

$$\exp\left(1-\langle\Phi(A)y,y\rangle\left\langle(\Phi(A))^{-1}y,y\right\rangle\right)\leq\frac{\Delta_y(\Phi(A))}{\langle\Phi(A)y,y\rangle}\leq\frac{\Delta_{\Phi,y}(A)}{\langle\Phi(A)y,y\rangle}\leq 1. \tag{2.5}$$

The last inequality is the best possible one can get from the second inequality in (2.4).

Also,

$$1\leq\frac{\Delta_y(\Phi(A))}{\left\langle(\Phi(A))^{-1}y,y\right\rangle^{-1}}\leq\frac{\Delta_{\Phi,y}(A)}{\left\langle(\Phi(A))^{-1}y,y\right\rangle^{-1}}\leq\exp\left(\left\langle(\Phi(A))^{-1}y,y\right\rangle\langle\Phi(A)y,y\rangle-1\right). \tag{2.6}$$

The first inequality is the best possible one can get from the first inequality in (2.4).

Proof. It is well know that, if f is differentiable convex on an interval I, then for all u, $v\in I$ we have

$$f'(u)(u-v)\geq f(u)-f(v)\geq f'(v)(u-v).$$

If we write this inequality for $f(t)=-\ln t$, we get for all $u,v>0$ that

$$-\frac{1}{u}(u-v)\geq\ln v-\ln u\geq-\frac{1}{v}(u-v),$$

namely

$$\frac{1}{u}(u-v)\leq\ln u-\ln v\leq\frac{1}{v}(u-v) \tag{2.7}$$

for u, $v>0$.

If we use the continuous functional calculus for selfadjoint operators, we have from (2.7) that

$$1+\ln a-aV^{-1}\leq\ln V\leq a^{-1}V+\ln a-1 \tag{2.8}$$

for the operator $V>0$ and positive number a.

Now, if we take $V=\Phi(A)$, then we get

$$1+\ln a-a(\Phi(A))^{-1}\leq\ln\Phi(A)\leq a^{-1}\Phi(A)+\ln a-1 \tag{2.9}$$

for $a>0$.

Also, if we take $V=A$ and then apply Φ, then we get

$$1+\ln a-a\Phi\left(A^{-1}\right)\leq\Phi(\ln A)\leq a^{-1}\Phi(A)+\ln a-1 \tag{2.10}$$

for $a>0$.

Applying Jensen's operator inequality for the operator concave functions ln, we have $\ln\Phi(A)\leq\Phi(\ln A)$. Therefore by (2.9) and (2.10) we deduce the chain of inequalities

$$1+\ln a-a(\Phi(A))^{-1}\leq\ln\Phi(A)\leq\Phi(\ln A)\leq a^{-1}\Phi(A)+\ln a-1$$

for $A>0$ and $a>0$.

Now, if we take the inner product over $y\in K$ with $\|y\|=1$, then we get

$$1+\ln a-a\left\langle(\Phi(A))^{-1}y,y\right\rangle\leq\langle\ln\Phi(A)y,y\rangle\leq\langle\Phi(\ln A)y,y\rangle\leq a^{-1}\langle\Phi(A)y,y\rangle+\ln a-1 \tag{2.11}$$

for $A>0$ and $a>0$.

Finally, by applying the exponential in (2.11) we derive (2.4).

Now, consider the function $f(t)=t\exp\left[t^{-1}\langle\Phi(A)y,y\rangle-1\right]$, $t>0$, then

$$f'(t)=\exp\left[t^{-1}\langle\Phi(A)y,y\rangle-1\right]+t\exp\left[t^{-1}\langle\Phi(A)y,y\rangle-1\right]\left(-\frac{\langle\Phi(A)y,y\rangle}{t^2}\right)$$
$$=\exp\left[t^{-1}\langle\Phi(A)y,y\rangle-1\right]\left(1-\frac{\langle\Phi(A)y,y\rangle}{t}\right).$$

We have that $f'(t_0)=0$ for $t_0=\langle\Phi(A)y,y\rangle$ which shows that f is strictly decreasing on $(0,\langle\Phi(A)y,y\rangle)$ and strictly increasing on $(\langle\Phi(A)y,y\rangle,\infty)$. Therefore

$$\inf_{t\in(0,\infty)}f(t)=f(\langle\Phi(A)y,y\rangle)=\langle\Phi(A)y,y\rangle,$$

which proves that the last inequality in (2.5) is the best possible one can get from the second inequality in (2.4).

Also if we take the function $g(t)=t\exp\left[1-t\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]$, $t>0$, then

$$g'(t)=\exp\left[1-t\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]$$
$$-t\left\langle(\Phi(A))^{-1}y,y\right\rangle\exp\left[1-t\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]$$
$$=\exp\left[1-t\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]\left(1-t\left\langle(\Phi(A))^{-1}y,y\right\rangle\right),$$

which shows that g is strictly increasing on $\left(0,\left\langle(\Phi(A))^{-1}y,y\right\rangle^{-1}\right)$ and strictly decreasing on $\left(\left[\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]^{-1},\infty\right)$, therefore

$$\sup_{t\in(0,\infty)}g(t)=g\left(\left[\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]^{-1}\right)=\left[\left\langle(\Phi(A))^{-1}y,y\right\rangle\right]^{-1}.$$

This proves that, the first inequality in (2.6) is the best possible one can get from the first inequality in (2.4). □

Corollary 3. *With the assumptions of Theorem 2 we also have*

$$\int_0^1\Delta_{\Phi,y}((1-t)A+tB)\,dt\leq\left\langle\left(\frac{\Phi(A)+\Phi(B)}{2}\right)y,y\right\rangle. \tag{2.12}$$

Proof. From (2.5) we have

$$\Delta_{\Phi,y}((1-t)A+tB)\leq\langle\Phi((1-t)A+tB)y,y\rangle=\langle[(1-t)\Phi(A)+t\Phi(B)]y,y\rangle$$

for $t\in[0,1]$.

If we take the integral, we derive (2.12). □

Corollary 4. *Assume that $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$, then for $A\in\mathcal{B}(H)$ with $0<m\leq A\leq M$,*

$$1\leq\frac{\langle\Phi(A)y,y\rangle}{\Delta_{\Phi,y}(A)}\leq\frac{\langle\Phi(A)y,y\rangle}{\Delta_y(\Phi(A))} \tag{2.13}$$
$$\leq\exp\left(\langle\Phi(A)y,y\rangle\left\langle(\Phi(A))^{-1}y,y\right\rangle-1\right)\leq\exp\left[\frac{(M-m)^2}{4mM}\right]$$

and

$$1 \le \frac{\Delta_y(\Phi(A))}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \le \frac{\Delta_{\Phi,y}(A)}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \tag{2.14}$$
$$\le \exp\left(\left\langle (\Phi(A))^{-1} y, y \right\rangle \langle \Phi(A) y, y \rangle - 1 \right) \le \exp\left[\frac{(M-m)^2}{4mM} \right].$$

Proof. We use Kantorovich inequality

$$\langle Tx, x \rangle \left\langle T^{-1} x, x \right\rangle \le \frac{(M+m)^2}{4mM} \tag{2.15}$$

for $x \in H$, $\|x\| = 1$, where $0 < m \le T \le M$ for some constants $m, M > 0$.

If $0 < m \le A \le M$, then $0 < m \le \Phi(A) \le M$,

$$\langle \Phi(A) y, y \rangle \left\langle (\Phi(A))^{-1} y, y \right\rangle - 1 \le \frac{(M+m)^2}{4mM} - 1 = \frac{(M-m)^2}{4mM} \tag{2.16}$$

for $y \in H$, $\|y\| = 1$.

We then get

$$\exp\left(\langle \Phi(A) y, y \rangle \left\langle (\Phi(A))^{-1} y, y \right\rangle - 1 \right) \le \exp\left[\frac{(M-m)^2}{4mM} \right]$$

and by (2.5) and (2.6) we obtain the desired results. □

Corollary 5. *With the assumptions of Corollary 5,*

$$1 \le \frac{\langle \Phi(A) y, y \rangle}{\Delta_{\Phi,y}(A)} \le \frac{\langle \Phi(A) y, y \rangle}{\Delta_y(\Phi(A))} \tag{2.17}$$
$$\le \exp\left(\langle \Phi(A) y, y \rangle \left\langle (\Phi(A))^{-1} y, y \right\rangle - 1 \right)$$
$$\le \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{mM} \langle \Phi(A) y, y \rangle \right] \le \exp\left[\left(\sqrt{\frac{M}{m}} - 1 \right)^2 \right]$$

and

$$1 \le \frac{\Delta_y(\Phi(A))}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \le \frac{\Delta_{\Phi,y}(A)}{\left\langle (\Phi(A))^{-1} y, y \right\rangle^{-1}} \tag{2.18}$$
$$\le \exp\left(\left\langle (\Phi(A))^{-1} y, y \right\rangle \langle \Phi(A) y, y \rangle - 1 \right)$$
$$\le\le \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{mM} \langle \Phi(A) y, y \rangle \right] \le \exp\left[\left(\sqrt{\frac{M}{m}} - 1 \right)^2 \right].$$

Proof. We use the following reverse inequality, [6, p. 28]

$$\left\langle T^{-1} x, x \right\rangle - \langle Tx, x \rangle^{-1} \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{mM}$$

for $x \in H$, $\|x\| = 1$, where $0 < m \le T \le M$ for some constants $m, M > 0$.

By multiplying with $\langle Tx, x\rangle \geq 0$, we get

$$\left\langle T^{-1}x, x\right\rangle \langle Tx, x\rangle - 1 \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\langle Tx, x\rangle \leq \left(\sqrt{\frac{M}{m}}-1\right)^2 \tag{2.19}$$

If $0 < m \leq A \leq M$, then $0 < m \leq \Phi(A) \leq M$,

$$\langle \Phi(A)y, y\rangle \left\langle (\Phi(A))^{-1}y, y\right\rangle - 1 \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\langle \Phi(A)y, y\rangle \leq \left(\sqrt{\frac{M}{m}}-1\right)^2$$

for $y \in H$, $\|y\| = 1$.

By utilizing Theorem 2 we derive (2.17) and (2.18). □

3. Applications for Hadamard Product

We observe that for the functional $\Phi_{\mathcal{U}}$,

$$\Delta_{\Phi_{\mathcal{U}},x}(A\otimes B) = \exp\langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x, x\rangle$$

for A, $B \in \mathcal{B}(H)$ with A, $B > 0$ and $x \in H$ with $\|x\| = 1$.

Utilizing the inequality (1.8), we derive the determinant inequality

$$\Delta_{\Phi_{\mathcal{U}},x}(A\otimes B) \leq \Delta_x(A\circ B) \tag{3.1}$$

for A, $B > 0$ and $x \in H$ with $\|x\| = 1$.

Since

$$A\otimes B = (A\otimes 1)(1\otimes B)$$

and the operators $A\otimes 1$, $1\otimes B$ commute, then

$$\ln(A\otimes B) = \ln(A\otimes 1) + \ln(1\otimes B).$$

Also, by the continuous functional calculus for tensorial products of selfadjoint operators we have

$$\ln(A\otimes 1) = (\ln A)\otimes 1,\ \ \ln(1\otimes B) = 1\otimes \ln B.$$

Therefore

$$\begin{aligned}
\langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x, x\rangle &= \langle \mathcal{U}^*((\ln A)\otimes 1 + 1\otimes \ln B)\mathcal{U}x, x\rangle \\
&= \langle [\mathcal{U}^*((\ln A)\otimes 1)\mathcal{U} + \mathcal{U}^*(1\otimes \ln B)\mathcal{U}]x, x\rangle \\
&= \langle \mathcal{U}^*((\ln A)\otimes 1)\mathcal{U}x, x\rangle + \langle \mathcal{U}^*(1\otimes \ln B)\mathcal{U}x, x\rangle \\
&= \langle ((\ln A)\circ 1)x, x\rangle + \langle (1\circ(\ln B))x, x\rangle \\
&= \langle ((\ln A)\circ 1)x, x\rangle + \langle ((\ln B)\circ 1)x, x\rangle,
\end{aligned}$$

which shows that

$$\Delta_{\Phi_{\mathcal{U}},x}(A\otimes B) = \exp\langle ((\ln A)\circ 1)x, x\rangle \exp\langle ((\ln B)\circ 1)x, x\rangle$$

and by (3.1) we get:

Proposition 2. *For A, $B > 0$ and $x \in H$ with $\|x\| = 1$, we have*

$$\exp\langle ((\ln A)\circ 1)x, x\rangle \exp\langle ((\ln B)\circ 1)x, x\rangle \leq \Delta_x(A\circ B). \tag{3.2}$$

If $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H, then

$$\langle(\ln A\circ 1)\,e_j,e_j\rangle=\langle\ln Ae_j,e_j\rangle\,\langle 1e_j,e_j\rangle=\langle\ln Ae_j,e_j\rangle\,,$$

and

$$\langle(\ln B\circ 1)\,e_j,e_j\rangle=\langle\ln ABe_j,e_j\rangle\,\langle 1e_j,e_j\rangle=\langle\ln Be_j,e_j\rangle$$

and by (3.2) we derive

$$\Delta_{\Phi_{\mathcal{U}},e_j}\,(A\otimes B)=\exp\langle\ln Ae_j,e_j\rangle\exp\langle\ln Be_j,e_j\rangle\le\Delta_{e_j}(A\circ B).$$

Therefore we can state the following:

Corollary 6. *For A, $B>0$ and $\{e_j\}_{j\in\mathbb{N}}$ an orthonormal basis for the separable Hilbert space H, we have*

$$\Delta_{e_j}\,(A)\,\Delta_{e_j}\,(B)\le\Delta_{e_j}(A\circ B)\ \textit{for } j\in\mathbb{N}. \tag{3.3}$$

We can also state the results:

Proposition 3. *For A, $B>0$ and $x\in H$ with $\|x\|=1$, we have*

$$1\le\frac{\langle(A\circ B)\,x,x\rangle}{\Delta_{\Phi_{\mathcal{U}},x}\,(A\otimes B)}\le\frac{\langle(A\circ B)\,x,x\rangle}{\Delta_x((A\circ B))} \tag{3.4}$$
$$\le\exp\left(\langle(A\circ B)\,x,x\rangle\left\langle(A\circ B)^{-1}\,x,x\right\rangle-1\right)$$

and

$$1\le\frac{\Delta_x(A\circ B)}{\left\langle(A\circ B)^{-1}\,x,x\right\rangle^{-1}}\le\frac{\Delta_{\Phi_{\mathcal{U}},x}\,(A\otimes B)}{\left\langle(A\circ B)^{-1}\,x,x\right\rangle^{-1}} \tag{3.5}$$
$$\le\exp\left(\left\langle(A\circ B)^{-1}\,x,x\right\rangle\langle(A\circ B)\,x,x\rangle-1\right).$$

The proof follows by applying Theorem 2 for the functional

$$\Phi_{\mathcal{U}}\,(A\otimes B):=\mathcal{U}^*\,(A\otimes B)\,\mathcal{U}=A\circ B,$$

where A, $B>0$ and $x\in H$ with $\|x\|=1$.

Proposition 4. *Assume that $0<m_1\le A\le M_1$, $0<m_2\le B\le M_2$, for some constants m_i,M_i $(i=1,2)$. Then*

$$1\le\frac{\langle(A\circ B)\,x,x\rangle}{\Delta_{\Phi_{\mathcal{U}},x}\,(A\otimes B)}\le\frac{\langle(A\circ B)\,x,x\rangle}{\Delta_x((A\circ B))}\le\exp\left[\frac{(M_1M_2-m_1m_2)^2}{4m_1m_2M_1M_2}\right] \tag{3.6}$$

and

$$1\le\frac{\Delta_x(A\circ B)}{\left\langle(A\circ B)^{-1}\,x,x\right\rangle^{-1}}\le\frac{\Delta_{\Phi_{\mathcal{U}},x}\,(A\otimes B)}{\left\langle(A\circ B)^{-1}\,x,x\right\rangle^{-1}} \tag{3.7}$$
$$\le\exp\left[\frac{(M_1M_2-m_1m_2)^2}{4m_1m_2M_1M_2}\right].$$

Corollary 7. *With the assumptions of Proposition 4 and if $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for the separable Hilbert space H, then*

$$1\le\frac{\langle Ae_j,e_j\rangle\,\langle Be_j,e_j\rangle}{\Delta_{e_j}\,(A)\,\Delta_{e_j}\,(B)}\le\frac{\langle Ae_j,e_j\rangle\,\langle Be_j,e_j\rangle}{\Delta_{e_j}(A\circ B)}\le\exp\left[\frac{(M_1M_2-m_1m_2)^2}{4m_1m_2M_1M_2}\right]. \tag{3.8}$$

The proof follows by (3.6) for $x=e_j$, $j\in\mathbb{N}$.

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