LOWER AND UPPER BOUNDS FOR THE Φ-y-NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H, K be two Hilbert spaces. For $A \in \mathcal{B}(H)$, $A > 0$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, a linear positive and normalized functional, and $y \in K$ with $\|y\| = 1$ we define the Φ-y-*normalized determinant* by

$$\Delta_{\Phi,y}(A) := \exp \langle \Phi(\ln A) y, y \rangle .$$

In this paper we show, among others, that if $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $y \in K$ with $\|y\| = 1$, then for $A \in \mathcal{B}(H)$ with $0 < m \le A \le M$,

$$\begin{aligned} 1 &\le K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left\langle \Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right)y,y\right\rangle} \\ &\le \frac{\Delta_{\Phi,y}(A)}{m^{\frac{M-\langle\Phi(A)y,y\rangle}{M-m}} M^{\frac{\langle\Phi(A)y,y\rangle-m}{M-m}}} \le \frac{\Delta_y(\Phi(A))}{m^{\frac{M-\langle\Phi(A)y,y\rangle}{M-m}} M^{\frac{\langle\Phi(A)y,y\rangle-m}{M-m}}} \\ &\le K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\left\langle\left|\Phi(A)-\frac{1}{2}(m+M)\right|y,y\right\rangle} \le K\left(\frac{M}{m}\right), \end{aligned}$$

where $K(\cdot)$ is Kantorovich constant. Some applications for Hadamard product of operators are also given.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [3], [4], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [3].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive operators, Normalized determinants, Inequalities, Hadamard product, Tensorial product.

RGMIA Res. Rep. Coll. **26** (2023), Art. 2, 12 pp. Received 19/01/23

(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [3] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [4], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \le \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \le S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \le \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \le S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu}\leq\right)S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq\left(1-\nu\right)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.6}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.6) is due to Tominaga [10] while the first one is due to Furuichi [5].

We consider the *Kantorovich's constant* defined by

$$K\left(h\right):=\frac{\left(h+1\right)^{2}}{4h},\ \ h>0. \tag{1.7}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K\left(h\right)\geq 1$ for any $h>0$ and $K\left(h\right)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu}\leq\right)K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq\left(1-\nu\right)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.8) was obtained by Zou et al. in [11] while the second by Liao et al. [8].

Let H, K be complex Hilbert spaces. Following [1] (see also [6, p. 18]) we can introduce the following definition:

Definition 1. *A map* $\Phi:\mathcal{B}\left(H\right)\rightarrow\mathcal{B}\left(K\right)$ *is linear if it is additive and homogeneous, namely*

$$\Phi\left(\lambda A+\mu B\right)=\lambda\Phi\left(A\right)+\mu\Phi\left(B\right)$$

for any λ, $\mu\in\mathbb{C}$ *and* A, $B\in\mathcal{B}\left(H\right)$. *The linear map* $\Phi:\mathcal{B}\left(H\right)\rightarrow\mathcal{B}\left(K\right)$ *is positive if it preserves the operator order, i.e. if* $A\in\mathcal{B}^{+}\left(H\right)$ *then* $\Phi\left(A\right)\in\mathcal{B}^{+}\left(K\right)$. *We write* $\Phi\in\mathfrak{P}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$. *The linear map* $\Phi:\mathcal{B}\left(H\right)\rightarrow\mathcal{B}\left(K\right)$ *is normalized if it preserves the identity operator, i.e.* $\Phi\left(1_{H}\right)=1_{K}$. *We write* $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A\leq B\text{ implies }\Phi\left(A\right)\leq\Phi\left(B\right)$$

and preserves the adjoint operation $\Phi\left(A^{*}\right)=\Phi\left(A\right)^{*}$. If $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$ and $\alpha 1_{H}\leq A\leq\beta 1_{H}$, then $\alpha 1_{K}\leq\Phi\left(A\right)\leq\beta 1_{K}$.

If the map $\Psi:\mathcal{B}\left(H\right)\rightarrow\mathcal{B}\left(K\right)$ is linear, positive and $\Psi\left(1_{H}\right)\in\mathcal{B}^{++}\left(K\right)$ then by putting $\Phi=\Psi^{-1/2}\left(1_{H}\right)\Psi\Psi^{-1/2}\left(1_{H}\right)$ we get that $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$, namely it is also normalized.

The following Jensen's type result is well known [1]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let* $f:I\rightarrow\mathbb{R}$ *be an operator convex function on the interval* I *and* $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$, *then for any selfadjoint operator* A *whose spectrum is contained in* I *we have* ln

$$f\left(\Phi\left(A\right)\right)\leq\Phi\left(f\left(A\right)\right). \tag{1.9}$$

For $A\in\mathcal{B}\left(H\right)$, $A>0$, $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$ and $y\in K$ with $\|y\|=1$ we define the Φ-*y-normalized determinant* by

$$\Delta_{\Phi,y}\left(A\right):=\exp\left\langle\Phi\left(\ln A\right)y,y\right\rangle.$$

We observe that for $A\in\mathcal{B}(H)$, $A>0$, $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$,

$$\Delta_{\Phi,y}(A^t)=\Delta_{\Phi,y}(A)^t,\ t>0;$$

$$\Delta_{\Phi,y}(tA)=t\Delta_{\Phi,y}(A),\ \Delta_{\Phi,y}(tI)=t,\ t>0$$

and, by the operator monotonicity of Φ and ln

$$0<A\le B \text{ implies that } \Delta_{\Phi,y}(A)\le\Delta_{\Phi,y}(B).$$

Also

$$\Delta_{\Phi,y}(AB)=\Delta_{\Phi,y}(A)\Delta_{\Phi,y}(B) \text{ for commuting } A \text{ and } B$$

and by the operator concavity of ln and by (1.9) we obtain

$$\Delta_{\Phi,y}(A)\le\Delta_y(\Phi(A)) \tag{1.10}$$

for $A\in\mathcal{B}(H)$, $A>0$, $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$.

We observe that if $K=H$ and $\Phi(A)=A$, $A\in\mathcal{B}(H)$ then $\Delta_{\Phi,y}(A)=\Delta_y(A)$, $y\in H$ with $\|y\|=1$.

Recall that the *Hadamard product* of A and B in $\mathcal{B}(H)$ is defined to be the operator $A\circ B\in\mathcal{B}(H)$ satisfying

$$\langle(A\circ B)e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [2], if "$\otimes$" denotes the tensorial product, then we have the representation

$$A\circ B=\mathcal{U}^*(A\otimes B)\mathcal{U} \tag{1.11}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

We can define the map $\Phi_{\mathcal{U}}:\mathcal{B}(H\otimes H)\to\mathcal{B}(H)$,

$$\Phi_{\mathcal{U}}(A\otimes B):=\mathcal{U}^*(A\otimes B)\mathcal{U}=A\circ B, \tag{1.12}$$

which is linear, positive and preserves the unity operator.

Motivated by the above results, in this paper we show, among others, that if $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $y\in K$ with $\|y\|=1$, then for $A\in\mathcal{B}(H)$ with $0<m\le A\le M$,

$$\begin{aligned}1&\le K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left\langle\Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right)y,y\right\rangle}\\&\le\frac{\Delta_{\Phi,y}(A)}{m^{\frac{M-\langle\Phi(A)y,y\rangle}{M-m}}M^{\frac{\langle\Phi(A)y,y\rangle-m}{M-m}}}\le\frac{\Delta_y(\Phi(A))}{m^{\frac{M-\langle\Phi(A)y,y\rangle}{M-m}}M^{\frac{\langle\Phi(A)y,y\rangle-m}{M-m}}}\\&\le K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\left\langle\left|\Phi(A)-\frac{1}{2}(m+M)\right|y,y\right\rangle}\le K\left(\frac{M}{m}\right),\end{aligned}$$

where $K(\cdot)$ is Kantorovich constant. Some applications for Hadamard product of operators are also given.

2. Main Results

We start to the following result:

Theorem 2. *Assume that* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* $y \in K$ *with* $\|y\| = 1$, *then for* $A \in \mathcal{B}(H)$ *with* $0 < m \leq A \leq M$,

$$1 \leq \Delta_{\Phi,y}\left(S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)\right|}\right)\right) \tag{2.1}$$
$$\leq \frac{\Delta_{\Phi,y}(A)}{m^{\frac{M-\langle\Phi(A)y,y\rangle}{M-m}} M^{\frac{\langle\Phi(A)y,y\rangle-m}{M-m}}} \leq \frac{\Delta_y(\Phi(A))}{m^{\frac{M-\langle\Phi(A)y,y\rangle}{M-m}} M^{\frac{\langle\Phi(A)y,y\rangle-m}{M-m}}} \leq S\left(\frac{M}{m}\right).$$

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0,1]$. Then

$$\min\{1-\nu, \nu\} = \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right|$$
$$= \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t$$

and

$$m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}.$$

By using the inequality (1.6) we deduce

$$m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|}\right) m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \tag{2.2}$$
$$\leq t \leq S\left(\frac{M}{m}\right) m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}$$

for $t \in [m, M]$.

By taking the log in (2.2) we get

$$\frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \tag{2.3}$$
$$\leq \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|}\right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M$$
$$\leq \ln t \leq \ln S\left(\frac{M}{m}\right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M,$$

for $t \in [m, M]$.

If $0 < m \leq T \leq M$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$\ln m\frac{M-T}{M-m} + \ln M\frac{T-m}{M-m} \tag{2.4}$$
$$\leq \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|T-\frac{1}{2}(m+M)\right|}\right) + \ln m\frac{M-T}{M-m} + \ln M\frac{T-m}{M-m}$$
$$\leq \ln T \leq \ln S\left(\frac{M}{m}\right) + \ln m\frac{M-T}{M-m} + \ln M\frac{T-m}{M-m}.$$

If $0 < m \le A \le M$, then $0 < m \le \Phi(A) \le M$ and by (2.4) for $T = \Phi(A)$ we get

$$\begin{aligned}
(2.5)\qquad & \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m} \\
& \le \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|\Phi(A)-\frac{1}{2}(m+M)\right|}\right) \\
& + \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m} \\
& \le \ln \Phi(A) \le \ln S\left(\frac{M}{m}\right) + \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m}.
\end{aligned}$$

Also, if we write (2.4) for $T = A$ and then take the functional Φ, then we get

$$\begin{aligned}
(2.6)\qquad & \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m} \\
& \le \Phi\left[\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)\right|}\right)\right] \\
& + \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m} \\
& \le \Phi(\ln A) \le \ln S\left(\frac{M}{m}\right) + \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m}.
\end{aligned}$$

Since by operator concavity of ln and Jensen's operator inequality

$$\Phi(\ln A) \le \ln \Phi(A),$$

then we get the chain of inequalities

$$\begin{aligned}
(2.7)\qquad & \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m} \\
& \le \Phi\left[\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)\right|}\right)\right] \\
& + \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m} \\
& \le \Phi(\ln A) \le \ln \Phi(A) \\
& \le \ln S\left(\frac{M}{m}\right) + \ln m \frac{M-\Phi(A)}{M-m} + \ln M \frac{\Phi(A)-m}{M-m}.
\end{aligned}$$

Now, if we take the inner product over $y \in K$ with $\|y\| = 1$ we get

$$\begin{aligned}
&\ln m \frac{M - \langle \Phi(A) y, y\rangle}{M-m} + \ln M \frac{\langle \Phi(A) y, y\rangle - m}{M-m} \\
&\leq \left\langle \Phi\left[\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|A - \frac{1}{2}(m+M)\right|}\right)\right] y, y\right\rangle \\
&+ \ln m \frac{M - \langle \Phi(A) y, y\rangle}{M-m} + \ln M \frac{\langle \Phi(A) y, y\rangle - m}{M-m} \\
&\leq \langle \Phi(\ln A) y, y\rangle \leq \langle \ln \Phi(A) y, y\rangle \\
&\leq \ln S\left(\frac{M}{m}\right) + \ln m \frac{M - \langle \Phi(A) y, y\rangle}{M-m} + \ln M \frac{\langle \Phi(A) y, y\rangle - m}{M-m},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left(m^{\frac{M - \langle \Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle \Phi(A) y, y\rangle - m}{M-m}}\right) \\
&\leq \left\langle \Phi\left[\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|A - \frac{1}{2}(m+M)\right|}\right)\right] y, y\right\rangle \\
&+ \ln\left(m^{\frac{M - \langle \Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle \Phi(A) y, y\rangle - m}{M-m}}\right) \\
&\leq \langle \Phi(\ln A) y, y\rangle \leq \langle \ln \Phi(A) y, y\rangle \\
&\leq \ln\left[S\left(\frac{M}{m}\right)\left(m^{\frac{M - \langle \Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle \Phi(A) y, y\rangle - m}{M-m}}\right)\right].
\end{aligned} \tag{2.8}$$

Finally, if we take the exponential in (2.8) we obtain (2.1). □

Theorem 3. *Assume that* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* $y \in K$ *with* $\|y\| = 1$, *then for* $A \in \mathcal{B}(H)$ *with* $0 < m \leq A \leq M$,

$$\begin{aligned}
1 &\leq K\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left\langle \Phi\left(\left|A - \frac{1}{2}(m+M)\right|\right) y, y\right\rangle} \\
&\leq \frac{\Delta_{\Phi,y}(A)}{m^{\frac{M - \langle \Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle \Phi(A) y, y\rangle - m}{M-m}}} \leq \frac{\Delta_y(\Phi(A))}{m^{\frac{M - \langle \Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle \Phi(A) y, y\rangle - m}{M-m}}} \\
&\leq K\left(\frac{M}{m}\right)^{\frac{1}{2} + \frac{1}{M-m}\left\langle \left|\Phi(A) - \frac{1}{2}(m+M)\right| y, y\right\rangle} \leq K\left(\frac{M}{m}\right).
\end{aligned} \tag{2.9}$$

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0, 1]$. Then

$$\begin{aligned}
\max\{1 - \nu, \nu\} &= \frac{1}{2} + \left|\nu - \frac{1}{2}\right| = \frac{1}{2} + \left|\frac{t-m}{M-m} - \frac{1}{2}\right| \\
&= \frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,
\end{aligned}$$

and by using (1.8) we get

$$\begin{aligned}
m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} &\leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \\
&\leq t \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}
\end{aligned} \tag{2.10}$$

for $t \in [m, M]$.

By taking the log in (2.10) we obtain

$$
\begin{aligned}
&\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\le\left[\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
&+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\le\ln t\le\left[\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
&+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\le\ln K\left(\frac{M}{m}\right)+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M
\end{aligned}
\tag{2.11}
$$

for $t \in [m, M]$.

If $0 < m \le T \le M$, then by using the continuous functional calculus for selfadjoint operators we get from (2.11) that

$$
\begin{aligned}
&\ln m\frac{M-T}{M-m}+\ln M\frac{T-m}{M-m} \\
&\le\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\left|T-\frac{1}{2}(m+M)\right|\right] \\
&+\ln m\frac{M-T}{M-m}+\ln M\frac{T-m}{M-m} \\
&\le\ln T\le\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left|T-\frac{1}{2}(m+M)\right|\right] \\
&+\ln m\frac{M-T}{M-m}+\ln M\frac{T-m}{M-m} \\
&\le\ln K\left(\frac{M}{m}\right)+\ln m\frac{M-T}{M-m}+\ln M\frac{T-m}{M-m}.
\end{aligned}
\tag{2.12}
$$

If $0 < m \le A \le M$, then $0 < m \le \Phi(A) \le M$ and by (2.12) for $T = \Phi(A)$ we get

$$
\begin{aligned}
&\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\le\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\left|\Phi(A)-\frac{1}{2}(m+M)\right|\right] \\
&+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\le\ln\Phi(A)\le\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left|\Phi(A)-\frac{1}{2}(m+M)\right|\right] \\
&+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\le\ln K\left(\frac{M}{m}\right)+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m}.
\end{aligned}
\tag{2.13}
$$

Also, if we write (2.12) for $T = A$ and then take the functional Φ, then we get

$$
\begin{aligned}
&\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right)\right] \\
&+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\leq \Phi(\ln A)\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right)\right] \\
&+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m}.
\end{aligned}
\tag{2.14}
$$

Since by operator concavity of ln and Jensen's operator inequality

$$\Phi(\ln A)\leq \ln \Phi(A),$$

then we get the chain of inequalities

$$
\begin{aligned}
&\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right)\right] \\
&+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\leq \Phi(\ln A)\leq \ln\Phi(A) \\
&\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left|\Phi(A)-\frac{1}{2}(m+M)\right|\right] \\
&+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)+\ln m\frac{M-\Phi(A)}{M-m}+\ln M\frac{\Phi(A)-m}{M-m}.
\end{aligned}
\tag{2.15}
$$

Now, if we take the inner product over $y\in K$ with $\|y\|=1$ we get

$$
\begin{aligned}
&\ln m\frac{M-\langle\Phi(A)y,y\rangle}{M-m}+\ln M\frac{\langle\Phi(A)y,y\rangle-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right)y,y\right\rangle\right] \\
&+\ln m\frac{M-\langle\Phi(A)y,y\rangle}{M-m}+\ln M\frac{\langle\Phi(A)y,y\rangle-m}{M-m}
\end{aligned}
$$

$$\leq \langle \Phi(\ln A) y, y\rangle \leq \langle \ln \Phi(A) y, y\rangle$$
$$\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|\Phi(A)-\frac{1}{2}(m+M)\right| y, y\right\rangle\right]$$
$$+\ln m \frac{M-\langle\Phi(A) y, y\rangle}{M-m}+\ln M \frac{\langle\Phi(A) y, y\rangle-m}{M-m}$$
$$\leq \ln K\left(\frac{M}{m}\right)+\ln m \frac{M-\langle\Phi(A) y, y\rangle}{M-m}+\ln M \frac{\langle\Phi(A) y, y\rangle-m}{M-m},$$

which is equivalent to:

$$\ln \left(m^{\frac{M-\langle\Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle\Phi(A) y, y\rangle-m}{M-m}}\right) \tag{2.16}$$
$$\leq \ln \left(m^{\frac{M-\langle\Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle\Phi(A) y, y\rangle-m}{M-m}} K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\Phi\left(\left|A-\frac{1}{2}(m+M)\right|\right) y, y\right\rangle\right]}\right)$$
$$\leq \langle \Phi(\ln A) y, y\rangle \leq \langle \ln \Phi(A) y, y\rangle$$
$$\leq \ln \left(m^{\frac{M-\langle\Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle\Phi(A) y, y\rangle-m}{M-m}} K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|\Phi(A)-\frac{1}{2}(m+M)\right| y, y\right\rangle\right]}\right)$$
$$\leq \ln \left[K\left(\frac{M}{m}\right)\left(m^{\frac{M-\langle\Phi(A) y, y\rangle}{M-m}} M^{\frac{\langle\Phi(A) y, y\rangle-m}{M-m}}\right)\right].$$

By taking the exponential in (2.16) we obtain (2.9). □

3. Applications for Hadamard Product

We observe that for the functional $\Phi_{\mathcal{U}}$,

$$\Delta_{\Phi_{\mathcal{U}}, x}(A \otimes B)=\exp \langle\mathcal{U}^*(\ln (A \otimes B)) \mathcal{U} x, x\rangle$$

for $A,\ B \in \mathcal{B}(H)$ with $A,\ B>0$ and $x \in H$ with $\|x\|=1$.

Since

$$A \otimes B=(A \otimes 1)(1 \otimes B)$$

and the operators $A \otimes 1,\ 1 \otimes B$ commute, then

$$\ln (A \otimes B)=\ln (A \otimes 1)+\ln (1 \otimes B).$$

Also, by the continuous functional calculus for tensorial products of selfadjoint operators we have

$$\ln (A \otimes 1)=(\ln A) \otimes 1,\ \ \ln (1 \otimes B)=1 \otimes \ln B.$$

Therefore

$$\begin{aligned}
\langle\mathcal{U}^*(\ln (A \otimes B)) \mathcal{U} x, x\rangle &= \langle\mathcal{U}^*((\ln A) \otimes 1+1 \otimes \ln B) \mathcal{U} x, x\rangle \\
&= \langle[\mathcal{U}^*((\ln A) \otimes 1) \mathcal{U}+\mathcal{U}^*(1 \otimes \ln B) \mathcal{U}] x, x\rangle \\
&= \langle\mathcal{U}^*((\ln A) \otimes 1) \mathcal{U} x, x\rangle+\langle\mathcal{U}^*(1 \otimes \ln B) \mathcal{U} x, x\rangle \\
&= \langle((\ln A) \circ 1) x, x\rangle+\langle(1 \circ(\ln B)) x, x\rangle \\
&= \langle((\ln A) \circ 1) x, x\rangle+\langle((\ln B) \circ 1) x, x\rangle,
\end{aligned}$$

which shows that

$$\Delta_{\Phi_{\mathcal{U}}, x}(A \otimes B)=\exp \langle((\ln A) \circ 1) x, x\rangle \exp \langle((\ln B) \circ 1) x, x\rangle$$

for $A,\ B \in \mathcal{B}(H)$ with $A,\ B>0$ and $x \in H$ with $\|x\|=1$.

Proposition 1. *Assume that* $0 < m_1 \le A \le M_1$, $0 < m_2 \le B \le M_2$, *for some constants* m_i, M_i $(i = 1, 2)$. *Define* $m := m_1 m_2$, $M := M_1 M_2$, *then*

$$1 \le \Delta_{\Phi_{\mathcal{U}},x}\left(S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A\otimes B-\frac{1}{2}(m+M)\right|}\right)\right) \tag{3.1}$$
$$\le \frac{\exp\left\langle\left(\left(\ln A\right)\circ 1\right)x,x\right\rangle \exp\left\langle\left(\left(\ln B\right)\circ 1\right)x,x\right\rangle}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}}$$
$$\le \frac{\Delta_x(A\circ B)}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}} \le S\left(\frac{M}{m}\right)$$

and

$$1 \le K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left\langle\mathcal{U}^*\left(\left|A\otimes B-\frac{1}{2}(m+M)\right|\right)\mathcal{U}x,x\right\rangle} \tag{3.2}$$
$$\le \frac{\exp\left\langle\left(\left(\ln A\right)\circ 1\right)x,x\right\rangle \exp\left\langle\left(\left(\ln B\right)\circ 1\right)x,x\right\rangle}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}}$$
$$\le \frac{\Delta_x(A\circ B)}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}}$$
$$\le K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A\circ B-\frac{1}{2}(m+M)\right|x,x\right\rangle} \le K\left(\frac{M}{m}\right)$$

for $x \in H$ *with* $\|x\| = 1$.

Proof. From (2.1) for $m = m_1 m_2$, $M = M_1 M_2$, $\Phi_{\mathcal{U}}(A \otimes B) := \mathcal{U}^*(A\otimes B)\mathcal{U} = A \circ B$, $x \in H$ with $\|x\| = 1$ we get

$$1 \le \Delta_{\Phi_{\mathcal{U}},x}\left(S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A\otimes B-\frac{1}{2}(m+M)\right|}\right)\right)$$
$$\le \frac{\Delta_{\Phi_{\mathcal{U}},x}(A\otimes B)}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}} \le \frac{\Delta_x(A\circ B)}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}} \le S\left(\frac{M}{m}\right),$$

which gives (3.1).

From (2.9) we have

$$1 \le K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left\langle\mathcal{U}^*\left(\left|A\otimes B-\frac{1}{2}(m+M)\right|\right)\mathcal{U}x,x\right\rangle}$$
$$\le \frac{\Delta_{\Phi_{\mathcal{U}},x}(A\otimes B)}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}} \le \frac{\Delta_x(A\circ B)}{m^{\frac{M-\langle A\circ Bx,x\rangle}{M-m}}M^{\frac{\langle A\circ Bx,x\rangle-m}{M-m}}}$$
$$\le K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A\circ B-\frac{1}{2}(m+M)\right|x,x\right\rangle} \le K\left(\frac{M}{m}\right),$$

which is equivalent to (3.2) □

If $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H, then

$$\left\langle\left(\ln A\circ 1\right)e_j,e_j\right\rangle = \left\langle\ln Ae_j,e_j\right\rangle\left\langle 1e_j,e_j\right\rangle = \left\langle\ln Ae_j,e_j\right\rangle = \Delta_{e_j}(A)$$

and

$$\left\langle\left(\ln B\circ 1\right)e_j,e_j\right\rangle = \left\langle\ln ABe_j,e_j\right\rangle\left\langle 1e_j,e_j\right\rangle = \left\langle\ln Be_j,e_j\right\rangle = \Delta_{e_j}(B)$$

and by Proposition 1 we derive:

Corollary 1. *With the assumptions of Proposition 1 and if* $\{e_j\}_{j\in\mathbb{N}}$ *is an orthonormal basis for the separable Hilbert space* H*, then*

$$
\begin{aligned}
1 &\leq \Delta_{\Phi_{\mathcal{U}},e_j}\left(S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A\otimes B-\frac{1}{2}(m+M)\right|}\right)\right) \\
&\leq \frac{\Delta_{e_j}(A)\Delta_{e_j}(B)}{m^{\frac{M-\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle}{M-m}}M^{\frac{\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle-m}{M-m}}} \\
&\leq \frac{\Delta_{e_j}(A\circ B)}{m^{\frac{M-\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle}{M-m}}M^{\frac{\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle-m}{M-m}}} \leq S\left(\frac{M}{m}\right)
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
1 &\leq K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left\langle\mathcal{U}^*\left(\left|A\otimes B-\frac{1}{2}(m+M)\right|\right)\mathcal{U}e_j,e_j\right\rangle} \\
&\leq \frac{\Delta_{e_j}(A)\Delta_{e_j}(B)}{m^{\frac{M-\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle}{M-m}}M^{\frac{\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle-m}{M-m}}} \\
&\leq \frac{\Delta_{e_j}(A\circ B)}{m^{\frac{M-\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle}{M-m}}M^{\frac{\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle-m}{M-m}}} \\
&\leq K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A\circ B-\frac{1}{2}(m+M)\right|e_j,e_j\right\rangle} \leq K\left(\frac{M}{m}\right)
\end{aligned} \tag{3.4}
$$

for $x\in H$ *with* $\|x\|=1$.

References

[1] M. D. Choi, Positive linear maps on C*-algebras. *Canad. J. Math.* **24** (1972), 520–529.

[2] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[3] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[4] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.

[5] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[6] T. Furuta, J. Mičić-Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Croatia.

[7] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.

[8] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[9] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[10] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[11] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa