q-Deformed and λ-parametrized hyperbolic tangent induced Banach space valued ordinary and fractional neural network approximations

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Abstract

Here we study the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative of fractional derivatives. Our operators are defined by using a density function generated by a q-deformed and λ-parametrized hyperbolic tangent function, which is a sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

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1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types,

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by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

Again the author inspired by [17], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [9], [13].

Let h be a general sigmoid function with $h(0) = 0$, and $y = \pm 1$ the horizontal asymptotes. Of course h is strictly increasing over $\mathbb{R}$. Let the parameter $0 < r < 1$ and $x > 0$. Then clearly $-x < x$ and $-x < -rx < rx < x$, furthermore it holds $h(-x) < h(-rx) < h(rx) < h(x)$. Consequently the sigmoid $y = h(rx)$ has a graph inside the graph of $y = h(x)$, of course with the same asymptotes $y = \pm 1$. Therefore $h(rx)$ has derivatives (gradients) at more points x than $h(x)$ has different than zero or not as close to zero, thus killing less number of neurons! And of course $h(rx)$ is more distant from $y = \pm 1$, than $h(x)$ it is. A highly desired fact in Neural Networks theory.

Also brain asymmetry has been observed in animals and humans in terms of structure, function and behaviour. This lateralization is thought to reflect evolutionary, hereditary, developmental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this paper is a specific study under this philosophy of approaching reality as close as possible.

Consequently the author here performs q-deformed and λ-parametrized hyperbolic tangent function activated neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with valued to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we describe important properties of the basic density function defining our operators which is induced by a q-deformed and λ-parametrized hyperbolic tangent function, which is a sigmoid function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{=0}^{n} c_j \sigma \left(\langle a_j \cdot x \rangle + b_j \right), \quad x \in \mathbb{R}^s, \;\; s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [19], [21], [23].

2 About q-deformed and $\lambda-$parametrized hyperbolic tangent function $g_{q,\lambda}$

Here all this background comes from [16].

We use $g_{q,\lambda}$, see (1), and exhibit that it is a sigmoid function and we will present several of its properties related to the approximation by neural network operators.

So, let us consider the function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \; x \in \mathbb{R}. \tag{1}$$

We have that

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}.$$

We notice also that

$$g_{q,\lambda}(-x) = \frac{e^{-\lambda x} - q e^{\lambda x}}{e^{-\lambda x} + q e^{\lambda x}} = \frac{\frac{1}{q} e^{-\lambda x} - e^{\lambda x}}{\frac{1}{q} e^{-\lambda x} + e^{\lambda x}} = -\frac{\left(e^{\lambda x} - \frac{1}{q} e^{-\lambda x} \right)}{e^{\lambda x} + \frac{1}{q} e^{-\lambda x}} = -g_{\frac{1}{q},\lambda}(x). \tag{2}$$

That is

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \;\; \forall \, x \in \mathbb{R}, \tag{3}$$

and

$$g_{\frac{1}{q},\lambda}(x) = -g_{q,\lambda}(-x),$$

hence

$$g'_{\frac{1}{q},\lambda}(x) = g'_{q,\lambda}(-x). \tag{4}$$

It is

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} = \frac{1 - \frac{q}{e^{2lx}}}{1 + \frac{q}{e^{2\lambda x}}} \underset{(x \to +\infty)}{\longrightarrow} 1,$$

i.e.

$$g_{q,\lambda}(+\infty) = 1, \tag{5}$$

Furthermore

$$g_{q,\lambda}(x)=\frac{e^{2\lambda x}-q}{e^{2\lambda x}+q}\underset{(x\to-\infty)}{\to}\frac{-q}{q}=-1,$$

i.e.

$$g_{q,\lambda}(-\infty)=-1. \tag{6}$$

We find that

$$g'_{q,\lambda}(x)=\frac{4q\lambda e^{2\lambda x}}{\left(e^{2\lambda x}+q\right)^2}>0, \tag{7}$$

therefore $g_{q,\lambda}$ is striclty increasing.

Next we obtain $(x\in\mathbb{R})$

$$g''_{q,\lambda}(x)=8q\lambda^2e^{2\lambda x}\left(\frac{q-e^{2\lambda x}}{\left(e^{2\lambda x}+q\right)^3}\right)\in C(\mathbb{R}). \tag{8}$$

We observe that

$$q-e^{2\lambda x}\gtreqless 0\Leftrightarrow q\gtreqless e^{2\lambda x}\Leftrightarrow \ln q\gtreqless 2\lambda x\Leftrightarrow x\lesseqgtr\frac{\ln q}{2\lambda}.$$

So, in case of $x<\frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=0$.

And in case of $x>\frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0)=\frac{1-q}{1+q}$, and $g_{q,\lambda}(-x)=-g_{q^{-1},\lambda}(x)$, (a semi-odd function), see also [14].

By $1>-1$, $x+1>x-1$, we consider the activation function

$$M_{q.\lambda}(x):=\frac{1}{4}\left(g_{q,\lambda}(x+1)-g_{q,\lambda}(x-1)\right)>0, \tag{9}$$

$\forall x\in\mathbb{R};\ q,\lambda>0$. Notice that $M_{q,\lambda}(\pm\infty)=0$, so the x-axis is horizontal asymptote.

We have that

$$\begin{gathered}M_{q,\lambda}(-x)=\frac{1}{4}\left(g_{q,\lambda}(-x+1)-g_{q,\lambda}(-x-1)\right)=\\ \frac{1}{4}\left(g_{q,\lambda}(-(x-1))-g_{q,\lambda}(-(x+1))\right)=\\ \frac{1}{4}\left(-g_{\frac{1}{q},\lambda}(x-1)+g_{\frac{1}{q},\lambda}(x+1)\right)=\\ \frac{1}{4}\left(g_{\frac{1}{q},\lambda}(x+1)-g_{\frac{1}{q},\lambda}(x-1)\right)=M_{\frac{1}{q},\lambda}(x),\quad\forall\, x\in\mathbb{R}.\end{gathered} \tag{10}$$

Thus

$$M_{q,\lambda}(-x)=M_{\frac{1}{q},\lambda}(x),\ \ \forall\, x\in\mathbb{R};\ q,\lambda>0, \tag{11}$$

a deformed symmetry.

Next, we have that

$$M'_{q,\lambda}(x)=\frac{1}{4}\left(g'_{q,\lambda}(x+1)-g'_{q,\lambda}(x-1)\right),\quad \forall\, x\in\mathbb{R}. \tag{12}$$

Let $x<\frac{\ln q}{2\lambda}-1$, then $x-1<x+1<\frac{\ln q}{2\lambda}$ and $g'_{q,\lambda}(x+1)>g'_{q,\lambda}(x-1)$ (by $g_{q,\lambda}$ being strictly concave up for $x<\frac{\ln q}{2\lambda}$), that is $M'_{q,\lambda}(x)>0$. Hence $M_{q,\lambda}$ is stricIty increasing over $\left(-\infty,\frac{\ln q}{2\lambda}-1\right)$.

Let now $x-1>\frac{\ln q}{2\lambda}$, then $x+1>x-1>\frac{\ln q}{2\lambda}$, and $g'_{q,\lambda}(x+1)<g'_{q,\lambda}(x-1)$, that is $M'_{q,\lambda}(x)<0$.

Therefore $M_{q,\lambda}$ is strictly decreasing over $\left(\frac{\ln q}{2\lambda}+1,+\infty\right)$.

Let us next consider, $\frac{\ln q}{2\lambda}-1\leq x\leq\frac{\ln q}{2\lambda}+1$. We have that

$$M''_{q,\lambda}(x)=\frac{1}{4}\left(g''_{q,\lambda}(x+1)-g''_{q,\lambda}(x-1)\right)=$$

$$2q\lambda^2\left[e^{2\lambda(x+1)}\left(\frac{q-e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)}+q\right)^3}\right)-e^{2\lambda(x-1)}\left(\frac{q-e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)}+q\right)^3}\right)\right]. \tag{13}$$

By $\frac{\ln q}{2\lambda}-1\leq x\Leftrightarrow\frac{\ln q}{2\lambda}\leq x+1\Leftrightarrow\ln q\leq 2\lambda(x+1)\Leftrightarrow q\leq e^{2\lambda(x+1)}\Leftrightarrow q-e^{2\lambda(x+1)}\leq 0$.

By $x\leq\frac{\ln q}{2\lambda}+1\Leftrightarrow x-1\leq\frac{\ln q}{2\lambda}\Leftrightarrow 2\lambda(x-1)\leq\ln q\Leftrightarrow e^{2\lambda(x-1)}\leq q\Leftrightarrow q-e^{2\lambda\beta(x-1)}\geq 0$.

Clearly by (13) we get that $M''_{q,\lambda}(x)\leq 0$, for $x\in\left[\frac{\ln q}{2\lambda}-1,\frac{\ln q}{2\lambda}+1\right]$.

More precisely $M_{q,\lambda}$ is concave down over $\left[\frac{\ln q}{2\lambda}-1,\frac{\ln q}{2\lambda}+1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda}-1,\frac{\ln q}{2\lambda}+1\right)$.

Consequently $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$.

Of course it holds $M''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)<0$.

At $x=\frac{\ln q}{2\lambda}$, we have

$$M'_{q,\lambda}(x)=\frac{1}{4}\left(g'_{q,\lambda}(x+1)-g'_{q,\lambda}(x-1)\right)=$$

$$q\lambda\left(\frac{e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)}+q\right)^2}-\frac{e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)}+q\right)^2}\right). \tag{14}$$

Thus

$$M'_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=q\lambda\left(\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}+q\right)^2}-\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}+q\right)^2}\right)=$$

$$q\lambda\left(\frac{qe^{2\lambda}}{\left(qe^{2\lambda}+q\right)^{2}}-\frac{qe^{-2\lambda}}{\left(qe^{-2\lambda}+q\right)^{2}}\right)=$$

$$\lambda\left(\frac{e^{2\lambda}}{\left(e^{2\lambda}+1\right)^{2}}-\frac{e^{-2\lambda}}{\left(e^{-2\lambda}+1\right)^{2}}\right)= \tag{15}$$

$$\lambda\left(\frac{e^{2\lambda}\left(e^{-2\lambda}+1\right)^{2}-e^{-2\lambda}\left(e^{2\lambda}+1\right)^{2}}{\left(e^{2\lambda}+1\right)^{2}\left(e^{-2\lambda}+1\right)^{2}}\right)=0.$$

That is, $\frac{\ln q}{2\lambda}$ is the only critical number of $M_{q,\lambda}$ over $\mathbb{R}$. Hence at $x=\frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=\frac{1}{4}\left[g_{q,\lambda}\left(\frac{\ln q}{2\lambda}+1\right)-g_{q,\lambda}\left(\frac{\ln q}{2\lambda}-1\right)\right]=$$

$$\frac{1}{4}\left[\left(\frac{e^{\lambda\left(\frac{\ln q}{2\lambda}+1\right)}-qe^{-\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{e^{\lambda\left(\frac{\ln q}{2\lambda}+1\right)}+qe^{-\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}\right)-\left(\frac{e^{\lambda\left(\frac{\ln q}{2\lambda}-1\right)}-qe^{-\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{e^{\lambda\left(\frac{\ln q}{2\lambda}-1\right)}+qe^{-\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}\right)\right]=$$

$$\frac{1}{4}\left[\left(\frac{\sqrt{q}e^{\lambda}-qq^{-\frac{1}{2}}e^{-\lambda}}{\sqrt{q}e^{\lambda}+qq^{-\frac{1}{2}}e^{-\lambda}}\right)-\left(\frac{\sqrt{q}e^{-\lambda}-qq^{-\frac{1}{2}}e^{\lambda}}{\sqrt{q}e^{-\lambda}+qq^{-\frac{1}{2}}e^{\lambda}}\right)\right]= \tag{17}$$

$$\frac{1}{4}\left[\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right)-\left(\frac{e^{-\lambda}-e^{\lambda}}{e^{-\lambda}+e^{\lambda}}\right)\right]=$$

$$\frac{1}{4}\left[\frac{2\left(e^{\lambda}-e^{-\lambda}\right)}{e^{\lambda}+e^{-\lambda}}\right]=\frac{1}{2}\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right)=\frac{\tanh(\lambda)}{2}.$$

Conclusion: The maximum value of $M_{q,\lambda}$ is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=\frac{\tanh(\lambda)}{2},\quad \lambda>0. \tag{18}$$

We mention

Theorem 1 *([16]) We have that*

$$\sum_{i=-\infty}^{\infty}M_{q,\lambda}(x-i)=1,\ \ \forall\ x\in\mathbb{R},\ \forall\ \lambda,q>0. \tag{19}$$

Thus

$$\sum_{i=-\infty}^{\infty}M_{q,\lambda}(nx-i)=1,\ \forall\ n\in\mathbb{N},\ \ \forall\ x\in\mathbb{R}. \tag{20}$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty}M_{\frac{1}{q},\lambda}(x-i)=1,\ \forall\ x\in\mathbb{R}. \tag{21}$$

But $M_{\frac{1}{q},\lambda}(x-i) \overset{(11)}{=} M_{q,\lambda}(i-x)$, $\forall\, x \in \mathbb{R}$.

Hence

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(i-x) = 1,\ \forall\, x \in \mathbb{R}, \tag{22}$$

and

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(i+x) = 1,\ \forall\, x \in \mathbb{R}. \tag{23}$$

It follows

Theorem 2 *([16]) It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \quad \lambda, q > 0. \tag{24}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

We need the following result

Theorem 3 *([16]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}}, \tag{25}$$

where $T := \max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

Theorem 4 *([16]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, we consider the number $\lambda_q > z_0 > 0$ with $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{M_{q,\lambda}(\lambda_q)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta(q). \tag{26}$$

We also mention

Remark 5 *([16]) (i) We have that*

$$\lim_{n \to +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \neq 1, \quad \textit{for at least some } x \in [a,b], \tag{27}$$

where $\lambda, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \leq 1. \tag{28}$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 6 *Let $f \in C([a,b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators*

$$H_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}, \quad x \in [a,b]\,;\, q>0,\ q \neq 1. \tag{29}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. The same H_n is used for real valued functions. We study here the pointwise and uniform convergence of $H_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k), \tag{30}$$

(the same H_n^* can be defined for real valued functions) that is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}. \tag{31}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} - f(x) = \tag{32}$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}.$$

Consequently, we derive that

$$\|H_n(f,x) - f(x)\| \leq \Delta(q) \left\| H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\right)\right\| =$$

$$\Delta(q)\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)M_{q,\lambda}(nx-k)\right\|, \quad (33)$$

where $\Delta(q)$ as in (26).

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a,b], X)$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]\\ |x-y|\le\delta}} \|f(x)-f(y)\|, \quad \delta > 0. \quad (34)$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X-valued), and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta\to 0}\omega_1(f,\delta) = 0$, see [11].

We make

Definition 7 *When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define*

$$\overline{H_n}(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)M_{q,\lambda}(nx-k), \quad (35)$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$, the X-valued quasi-interpolation neural network operator.

We give

Remark 8 *We have that*

$$\left\|f\left(\frac{k}{n}\right)\right\| \le \|f\|_{\infty,\mathbb{R}} < +\infty,$$

and

$$\left\|f\left(\frac{k}{n}\right)\right\| M_{q,\lambda}(nx-k) \le \|f\|_{\infty,\mathbb{R}} M_{q,\lambda}(nx-k) \quad (36)$$

and

$$\sum_{k=-\lambda}^{\lambda}\left\|f\left(\frac{k}{n}\right)\right\| M_{q,\lambda}(nx-k) \le \|f\|_{\infty,\mathbb{R}}\left(\sum_{k=-\lambda}^{\lambda} M_{q,\lambda}(nx-k)\right),$$

and finally

$$\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)\right\| M_{q,\lambda}(nx-k) \le \|f\|_{\infty,\mathbb{R}}, \quad (37)$$

a convergent series in $\mathbb{R}$.

So, the series $\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)\right\| M_{q,\lambda}(nx-k)$ is absolutely convergent in X, hence it is convergent in X and $\overline{H_n}(f,x) \in X$. We denote by $\|f\|_{\infty} := \sup_{x\in[a,b]}\|f(x)\|$, for $f \in C([a,b], X)$, similarly it is defined for $f \in C_B(\mathbb{R}, X)$.

3 Main Results

We present a set of X-valued neural network approximations to a function given with rates.

Theorem 9 *Let $f \in C([a,b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $q > 0$, $q \neq 1$, $x \in [a,b]$. Then*

i)

$$\|H_n(f,x) - f(x)\| \leq \Delta(q)\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty T e^{-2\lambda n^{(1-\alpha)}}\right] =: \tau, \quad (38)$$

where T as in (25),

and

ii)

$$\|H_n(f) - f\|_\infty \leq \tau. \quad (39)$$

We get that $\lim_{n\to\infty} H_n(f) = f$, pointwise and uniformly.

Proof. We see that

$$\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right) M_{q,\lambda}(nx-k)\right\| \leq$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right) - f(x)\right\| M_{q,\lambda}(nx-k) =$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}\end{cases}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right) - f(x)\right\| M_{q,\lambda}(nx-k) +$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}\end{cases}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right) - f(x)\right\| M_{q,\lambda}(nx-k) \leq \quad (40)$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}\end{cases}}^{\lfloor nb\rfloor}\omega_1\left(f, \left|\frac{k}{n}-x\right|\right) M_{q,\lambda}(nx-k) +$$

$$2\|f\|_\infty \sum_{\begin{cases} k=\lceil na\rceil \\ |k-nx| > n^{1-\alpha}\end{cases}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k) \leq$$

$$\omega_1\left(f,\frac{1}{n^\alpha}\right)\sum_{\begin{cases}k=-\infty\\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}\end{cases}}^{\infty} M_{q,\lambda}(nx-k)+$$

$$2\|f\|_\infty \sum_{\begin{cases}k=-\infty\\ |k-nx|>n^{1-\alpha}\end{cases}}^{\infty} M_{q,\lambda}(nx-k) \underset{\text{(by Theorem 3)}}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^\alpha}\right)+2\|f\|_\infty Te^{-2\lambda n^{(1-\alpha)}} \tag{41}$$

That is

$$\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)M_{q,\lambda}(nx-k)\right\|\leq$$

$$\omega_1\left(f,\frac{1}{n^\alpha}\right)+2\|f\|_\infty Te^{-2\lambda n^{(1-\alpha)}}. \tag{42}$$

Using the last equality we derive (38). ■

Next we give

Theorem 10 *Let $f\in C_B(\mathbb{R},X)$, $0<\alpha<1$, $q>0$, $q\neq1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in\mathbb{R}$. Then*

i)

$$\left\|\overline{H}_n(f,x)-f(x)\right\|\leq\omega_1\left(f,\frac{1}{n^\alpha}\right)+2\|f\|_\infty Te^{-2\lambda n^{(1-\alpha)}}=:\gamma, \tag{43}$$

and

ii)

$$\left\|\overline{H}_n(f)-f\right\|_\infty\leq\gamma. \tag{44}$$

For $f\in C_{uB}(\mathbb{R},X)$ we get $\lim_{n\to\infty}\overline{H}_n(f)=f$, pointwise and uniformly.

Proof. We observe that

$$\left\|\overline{H}_n(f,x)-f(x)\right\|\overset{(19)}{=}\left\|\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)M_{q,\lambda}(nx-k)-f(x)\sum_{k=-\infty}^{\infty}M_{q,\lambda}(nx-k)\right\|=$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right)-f(x)\right)M_{q,\lambda}(nx-k)\right\|\leq$$

$$\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|M_{q,\lambda}(nx-k)=$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx-k) \leq \tag{45}$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) M_{q,\lambda}(nx-k) +$$

$$2\|f\|_{\infty} \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\infty} M_{q,\lambda}(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} M_{q,\lambda}(nx-k) + 2\|f\|_{\infty} T e^{-2\lambda n^{(1-\alpha)}} \leq$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 2\|f\|_{\infty} T e^{-2\lambda n^{(1-\alpha)}}, \tag{46}$$

proving the claim. ■

We need the X-valued Taylor's formula in an appropiate form:

Theorem 11 *([10], [12]) Let $N \in \mathbb{N}$, and $f \in C^N([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space. Let any $x, y \in [a,b]$. Then*

$$f(x) = \sum_{i=0}^{N} \frac{(x-y)^i}{i!} f^{(i)}(y) + \frac{1}{(N-1)!} \int_y^x (x-t)^{N-1} \left(f^{(N)}(t) - f^{(N)}(y)\right) dt. \tag{47}$$

The derivatives $f^{(i)}$, $i \in \mathbb{N}$, are defined like the numerical ones, see [24], p. 83. The integral $\int_y^x$ in (47) is of Bochner type, see [22].

By [12], [20] we have that: if $f \in C([a,b], X)$, then $f \in L_{\infty}([a,b], X)$ and $f \in L_1([a,b], X)$.

In the next we discuss high order neural network X-valued approximation by using the smoothness of f.

Theorem 12 *Let $f \in C^N([a,b],X)$, $n,N \in \mathbb{N}$, $q>0$, $q \neq 1$, $0<\alpha<1$, $x \in [a,b]$ and $n^{1-\alpha}>2$. Then*

i)

$$\|H_n(f,x)-f(x)\| \leq \Delta(q)\left\{\sum_{j=1}^{N}\frac{\|f^{(j)}(x)\|}{j!}\left[\frac{1}{n^{\alpha j}}+(b-a)^j Te^{-2\lambda n^{(1-\alpha)}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\|f^{(N)}\|_\infty(b-a)^N}{N!}Te^{-2\lambda n^{(1-\alpha)}}\right]\right\}, \tag{48}$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$, for some $x_0 \in [a,b]$, it holds

$$\|H_n(f,x_0)-f(x_0)\| \leq \Delta(q)\cdot$$

$$\left\{\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\|f^{(N)}\|_\infty(b-a)^N}{N!}Te^{-2\lambda n^{(1-\alpha)}}\right\}, \tag{49}$$

and

iii)

$$\|H_n(f)-f\|_\infty \leq \Delta(q)\left\{\sum_{j=1}^{N}\frac{\|f^{(j)}\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}}+(b-a)^j Te^{-2\lambda n^{(1-\alpha)}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+2\left\|f^{(N)}\right\|_\infty(b-a)^N Te^{-2\lambda n^{(1-\alpha)}}\right]\right\}. \tag{50}$$

Again we obtain $\lim_{n\to\infty}H_n(f)=f$, pointwise and uniformly.

Proof. It is lengthy, and as similar to [15] is omitted. ■

All integrals from now on are of Bochner type [22].

We need

Definition 13 *([12]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha>0$; $m=\lceil\alpha\rceil \in \mathbb{N}$, ($\lceil\cdot\rceil$ is the ceiling of the number), $f:[a,b]\to X$. We assume that $f^{(m)} \in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$(D_{*a}^{\alpha}f)(x) := \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}f^{(m)}(t)\,dt, \quad \forall\, x \in [a,b]. \tag{51}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha}f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [24], p. 83), and also set $D_{*a}^{0}f := f$.

By [12], $(D_{*a}^{\alpha}f)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha}f \in L_1([a,b],X)$.

If $\|f^{(m)}\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha}f \in C([a,b],X)$, hence $\|D_{*a}^{\alpha}f\| \in C([a,b])$.

We mention

Definition 14 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\,dz, \quad \forall\, x \in [a,b]. \tag{52}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We make

Remark 15 *([11]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then*

$$\left\| D_{*a}^{\nu} f(x) \right\| \leq \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{53}$$

Thus we observe

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left\| D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y) \right\| \leq \tag{54}$$

$$\sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}.$$

Consequently

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) \leq \frac{2\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{55}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{56}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{57}$$

and

$$\sup_{x_0\in[a,b]} \omega_1\left(D^{\alpha}_{x_0-}f,\delta\right)_{[a,x_0]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \qquad (58)$$

By [12] we get that $D^{\alpha}_{*x_0}f \in C([x_0,b],X)$, and by [10] we obtain that $D^{\alpha}_{x_0-}f \in C([a,x_0],X)$.

We present the following X-valued fractional approximation result by neural networks.

Theorem 16 *Let $\alpha>0$, $q>0$, $q\neq 1$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C^N([a,b],X)$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}: n^{1-\beta}>2$. Then*

i)

$$\left\|H(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot-x)^j\right)(x)-f(x)\right\| \leq$$

$$\frac{\Delta(q)}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.Te^{-2\lambda n^{(1-\beta)}}\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^\alpha+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}, \qquad (59)$$

ii) if $f^{(j)}(x)=0$, for $j=1,...,N-1$, we have

$$\|H_n(f,x)-f(x)\| \leq \frac{\Delta(q)}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.Te^{-2\lambda n^{(1-\beta)}}\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^\alpha+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}, \qquad (60)$$

iii)

$$\|H_n(f,x)-f(x)\| \leq \Delta(q)$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\,Te^{-2\lambda n^{(1-\beta)}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.Te^{-2\lambda n^{(1-\beta)}}\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^\alpha+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}\right\}, \qquad (61)$$

$\forall\, x\in[a,b]$,

and

iv)

$$\left\| H_n f - f \right\|_\infty \leq \Delta(q)$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j T e^{-2\lambda n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. T e^{-2\lambda n^{(1-\beta)}} (b-a)^\alpha \left(\sup_{x \in [a,b]} \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \right) \right\} \right\}. \quad (62)$$

Above, when $N = 1$ *the sum* $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X*-valued fractionally type pointwise and uniform convergence with rates of* $H_n \to I$ *the unit operator, as* $n \to \infty$.

Proof. The proof is very lengthly and similar to [15], as such is omitted. ■

Next we apply Theorem 16 for $N = 1$.

Theorem 17 *Let* $0 < \alpha, \beta < 1$, $q > 0$, $q \neq 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\left\| H_n(f,x) - f(x) \right\| \leq$$

$$\frac{\Delta(q)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. T e^{-2\lambda n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (63)$$

and

ii)

$$\left\| H_n f - f \right\|_\infty \leq \frac{\Delta(q)}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. (b-a)^\alpha T e^{-2\lambda n^{(1-\beta)}} \left(\sup_{x \in [a,b]} \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \right) \right\}. \quad (64)$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 18 *Let $0 < \beta < 1$, $q > 0$, $q \neq 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\|H_n(f,x) - f(x)\| \leq$$

$$\frac{2\Delta(q)}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. Te^{-2\lambda n^{(1-\beta)}} \left(\left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \quad (65)$$

and

ii)

$$\|H_n f - f\|_\infty \leq \frac{2\Delta(q)}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \sqrt{(b-a)} Te^{-2\lambda n^{(1-\beta)}} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty. \quad (66)$$

We make

Remark 19 *Some convergence analysis follows based on Corollary 18.*

Let $0 < \beta < 1$, $\lambda > 0$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. We elaborate on (66). Assume that

$$\omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} \leq \frac{R_1}{n^\beta}, \quad (67)$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \leq \frac{R_2}{n^\beta}, \quad (68)$$

$\forall\, x \in [a,b]$, $\forall\, n \in \mathbb{N}$, where $R_1, R_2 > 0$.

Then it holds

$$\frac{\left[\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \right]}{n^{\frac{\beta}{2}}} \leq$$

$$\frac{\frac{(R_1+R_2)}{n^{\beta}}}{n^{\frac{\beta}{2}}} = \frac{(R_1+R_2)}{n^{\frac{3\beta}{2}}} = \frac{R}{n^{\frac{3\beta}{2}}}, \tag{69}$$

where $R := R_1 + R_2 > 0$.

The other summand of the right hand side of (66), for large enough n*, converges to zero at the speed* $e^{-2\lambda n^{(1-\beta)}}$*, so it is about* $Ae^{-2\lambda n^{(1-\beta)}}$*, where* $A > 0$ *is a constant.*

Then, for large enough $n \in \mathbb{N}$*, by (66), (69) and the above comment, we obtain that*

$$\|H_n f - f\|_{\infty} \leq \frac{B}{n^{\frac{3\beta}{2}}}, \tag{70}$$

where $B > 0$*, converging to zero at the high speed of* $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 9, for $f \in C([a,b], X)$ *and for large enough* $n \in \mathbb{N}$*, the speed is* $\frac{1}{n^{\beta}}$*. So by (70),* $\|H_n f - f\|_{\infty}$ *converges much faster to zero. The last comes because we assumed differentiability of* f*. Notice that in Corollary 18 no initial condition is assumed.*

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