Abstract Voronovskaya type asymptotic expansions for general sigmoid functions based quasi-interpolation neural network operators

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Abstract

Here we reexamine further the quasi-interpolation general sigmoid function based neural network operators of one hidden layer. Based on fractional calculus theory we derive fractional Voronovskaya type asymptotic expansions for the approximation of these operators to the unit operator, as we are studying the univariate case. We treat also analogously the multivariate case by using Fréchet derivatives. The functions under approximation are Banach space valued.

2020 AMS Mathematics Subject Classification: 26A33, 41A25, 41A45, 41A60.

Keywords and Phrases: Neural Network Fractional Approximation, Multivariate Neural Network Approximation, Voronovskaya Asymptotic Expansions, fractional derivative, general sigmoid activation function.

1 Background

We are motivated by [8], and [11], Ch. 10 and we generalize their results about expansions for arbitrary sigmoid functions.

We are also inspired by [14] - [16].

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and striclty concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

RGMIA Res. Rep. Coll. **26** (2023), Art. 4, 19 pp. Received 25/01/23

We consider the activation function

$$\psi(x) := \frac{1}{4}(h(x+1) - h(x-1)), \quad x \in \mathbb{R}, \tag{1}$$

As in [10], p. 285, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{2}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}(h'(x+1) - h'(x-1)) < 0,$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0, +\infty)$.

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi(x) = \frac{1}{4}(h(+\infty) - h(+\infty)) = 0, \tag{3}$$

and

$$\lim_{x \to -\infty} \psi(x) = \frac{1}{4}(h(-\infty) - h(-\infty)) = 0. \tag{4}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

Theorem 1 *We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x - i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Proof. As exactly the same as in [10], p. 286 is omitted. ■

Theorem 2 *It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{6}$$

Proof. Similar to [10], p. 287. It is omitted. ■

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We mention

Theorem 3 *([12]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\substack{\left\{ \begin{array}{l} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array} \right.}}^{\infty} \psi\left(nx-k\right) < \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} =: c\left(h,\alpha,n\right). \tag{7}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} = 0.$$

We further mention

Theorem 4 *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right)} < \frac{1}{\psi\left(1\right)} \left(=: \alpha^*\right), \quad \forall\, x \in \left[a,b\right]. \tag{8}$$

Proof. As similar to [10], p. 289 is omitted. ■

Remark 5 *([10], pp. 290-291)*

i) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right) \neq 1, \tag{9}$$

for at least some $x \in [a,b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 1, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right) \leq 1. \tag{10}$$

We introduce

$$Z\left(x_1,...,x_N\right) := Z\left(x\right) := \prod_{i=1}^{N} \psi\left(x_i\right), \quad x = \left(x_1,...,x_N\right) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{11}$$

It has the properties:

(i) $Z\left(x\right) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z\left(x-k\right) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z\left(x_1-k_1,...,x_N-k_N\right) = 1, \tag{12}$$

where $k := \left(k_1,...,k_n\right) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$, hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \tag{13}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\,dx = 1, \tag{14}$$

that is Z is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{gathered} \lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor), \end{gathered} \tag{15}$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \psi(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \psi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i) \right). \tag{16}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\begin{cases} k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k). \tag{17}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) As in [9], pp. 379-380, we derive that

$$\sum_{\begin{cases} k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) \overset{(7)}{<} \frac{1 - h\left(n^{1-\beta} - 2\right)}{2}, \quad 0 < \beta < 1, \tag{18}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{(\psi(1))^N} =: \gamma(N), \tag{19}$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z(nx-k) < \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} =: c(h, \beta, n), \tag{20}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1, \tag{21}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

The next integrals are of Bochner type ([19]).

We need

Definition 6 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{22}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one [22]), and also set $D_{*a}^{0} f := f$.

By [10], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [10], $D_{*a}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{*a}^{\alpha} f \right\| \in C([a,b])$.

We mention

Lemma 7 *([10]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b], X)$ and $f^{(m)} \in L_\infty([a,b], X)$. Then $D_{*a}^{\alpha} f(a) = 0$.*

We mention

Definition 8 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\ x \in [a,b]. \tag{23}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We need

Lemma 9 *([10]) Let $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_\infty([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.*

We mention the left fractional vector Taylor formula

Theorem 10 *([10]) Let $m \in \mathbb{N}$ and $f \in C^m([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\alpha > 0 : m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D_{*a}^{\alpha} f\right)(z)\, dz, \tag{24}$$

$\forall\ x \in [a,b]$.

We also mention the right fractional vector Taylor formula

Theorem 11 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^m([a,b], X)$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(z)\, dz, \tag{25}$$

$\forall\ x \in [a,b]$.

Convention 12 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{26}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{27}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 13 *([10]) Let $f \in C^n([a,b],X)$, $n=\lceil \nu \rceil$, $\nu>0$. Then $D_{*a}^{\nu}f(x)$ is continuous in $x\in[a,b]$.*

Proposition 14 *([10]) Let $f \in C^m([a,b],X)$, $m=\lceil \alpha \rceil$, $\alpha>0$. Then $D_{b-}^{\nu}f(x)$ is continuous in $x\in[a,b]$.*

We also mention

Proposition 15 *([10]) Let $f\in C^{m-1}([a,b],X)$, $f^{(m)}\in L_\infty([a,b],X)$, $m=\lceil\alpha\rceil$, $\alpha>0$ and*

$$D_{*x_0}^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{x_0}^{x}(x-t)^{m-\alpha-1}f^{(m)}(t)\,dt, \tag{28}$$

for all $x,x_0\in[a,b]: x\geq x_0$.

Then $D_{*x_0}^{\alpha}f(x)$ is continuous in x_0.

Proposition 16 *([10]) Let $f\in C^{m-1}([a,b],X)$, $f^{(m)}\in L_\infty([a,b],X)$, $m=\lceil\alpha\rceil$, $\alpha>0$ and*

$$D_{x_0-}^{\alpha}f(x)=\frac{(-1)^m}{\Gamma(m-\alpha)}\int_{x}^{x_0}(\zeta-x)^{m-\alpha-1}f^{(m)}(\zeta)\,d\zeta, \tag{29}$$

for all $x,x_0\in[a,b]: x_0\geq x$.

Then $D_{x_0-}^{\alpha}f(x)$ is continuous in x_0.

Corollary 17 *([10]) Let $f\in C^m([a,b],X)$, $m=\lceil\alpha\rceil$, $\alpha>0$, $x,x_0\in[a,b]$. Then $D_{*x_0}^{a}f(x)$, $D_{x_0-}^{a}f(x)$ are jointly continuous functions in (x,x_0) from $[a,b]^2$ into X, X is a Banach space.*

We make

Remark 18 *([10], pp. 263-266) Let $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $N\in\mathbb{N}$; where $\|\cdot\|_p$ is the L_p-norm, $1\leq p\leq\infty$. $\mathbb{R}^N$ is a Banach space, and $\left(\mathbb{R}^N\right)^{j_*}$ denotes the j_*-fold product space $\mathbb{R}^N\times\ldots\times\mathbb{R}^N$ endowed with the max-norm $\|x\|_{(\mathbb{R}^N)^{j_*}}:=\max_{1\leq\lambda\leq j_*}\|x_\lambda\|_p$, where $x:=(x_1,\ldots,x_{j_*})\in\left(\mathbb{R}^N\right)^{j_*}$.*

Let $\left(X,\|\cdot\|_\gamma\right)$ be a general Banach space. Then the space $L_{j_*}:=L_{j_*}\left(\left(\mathbb{R}^N\right)^{j_*},X\right)$ of all j_*-multilinear continuous maps $g:\left(\mathbb{R}^N\right)^{j_*}\to X$, $j_*=1,\ldots,\overline{m}$, is a Banach space with norm

$$\|g\|:=\|g\|_{L_{j_*}}:=\sup_{\left(\|x\|_{(\mathbb{R}^N)^{j_*}}=1\right)}\|g(x)\|_\gamma=\sup\frac{\|g(x)\|_\gamma}{\|x_1\|_p\ldots\|x_{j_*}\|_p}. \tag{30}$$

Let M be a non-empty convex and compact subset of $\mathbb{R}^N$ and $x_0 \in M$ is fixed.

Let O be an open subset of $\mathbb{R}^N : M \subset O$. Let $f : O \to X$ be a continuous function, whose Fréchet derivatives (see [21]) $f^{(j_*)} : O \to L_{j_*} = L_{j_*}\left(\left(\mathbb{R}^N\right)^{j_*}; X\right)$ exist and are continuous for $1 \leq j_* \leq \overline{m}$, $\overline{m} \in \mathbb{N}$.

Call $(x - x_0)^{j_*} := (x - x_0, ..., x - x_0) \in \left(R^N\right)^{j_*}$, $x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula ([13], [21], p. 124), we get

$$f(x) = \sum_{j_*=0}^{\overline{m}-1} \frac{f^{(j_*)}(x_0)(x - x_0)^{j_*}}{j_*!} + R_{\overline{m}}(x, x_0), \quad \text{all } x \in M, \tag{31}$$

where the remainder is the Riemann integral

$$R_{\overline{m}}(x, x_0) := \int_0^1 \frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!} f^{(\overline{m})}(x_0 + u(x - x_0))(x - x_0)^{\overline{m}} du, \tag{32}$$

here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We obtain

$$\left\| f^{(\overline{m})}(x_0 + u(x - x_0))(x - x_0)^{\overline{m}} \right\|_\gamma \leq$$

$$\left\| f^{(\overline{m})}(x_0 + u(x - x_0)) \right\| \|x - x_0\|_p^{\overline{m}} \leq \left\| \left\| f^{(\overline{m})} \right\| \right\|_\infty \|x - x_0\|_p^{\overline{m}}, \tag{33}$$

and

$$\left\| R_{\overline{m}}(x, x_0) \right\|_\gamma \leq \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_\infty}{m!} \|x - x_0\|_p^{\overline{m}}. \tag{34}$$

Let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space.

We will study the following neural network operators.

Definition 19 *Let $f \in C([a, b], X)$, $n \in \mathbb{N}$. We set*

$$A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k)}, \quad \forall\ x \in [a.b]. \tag{35}$$

These are univariate neural network operators.

Definition 20 *Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. We will study the following multivariate linear neural network operators ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i]\right)$)*

$$H_n(f, x) := H_n(f, x_1, ..., x_N) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} = \tag{36}$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi\left(nx_i-k_i\right)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi\left(nx_i-k_i\right)\right)}.$$

For large enough n we always obtain $\lceil na_i\rceil \leq \lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i\rceil \leq k_i \leq \lfloor nb_i\rfloor$, $i=1,...,N$.

In this article first we find fractional Voronovskaya type asymptotic expansion for $A_n(f,x)$, $x\in[a,b]$, then we find multivariate Voronovskaya type asymptotic expansion for $H_n(f,x)$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$; $n\in\mathbb{N}$.

Our considered neural networks here are of one hidden layer.

For earlier motivational neural networks related work, see [1] - [10]. For neural networks in general, read [17], [18] and [20].

2 Main Results

We present our first univariate main result, as Voronovskaya type asymptotic expansion.

Theorem 21 *Let $\left(X,\|\cdot\|_\gamma\right)$ be a Banach space, $0<\beta<\frac{1}{2}$ and $0<\alpha\leq\frac{1-\beta}{\beta}$, $N\in\mathbb{N}: N=\lceil\alpha\rceil$, $f\in C^N([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}$ large enough : $n^{1-\beta}>2$. Assume that $\left\|\|D_{x-}^{\alpha}f\|_\gamma\right\|_{\infty,[a,x]}$, $\left\|\|D_{*x}^{\alpha}f\|_\gamma\right\|_{\infty,[x,b]}\leq M$, $M>0$. Then*

$$A_n(f,x)-f(x)=\sum_{j_*=1}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}A_n\left((\cdot-x)^{j_*}\right)(x)+o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \quad (37)$$

where $0<\varepsilon\leq\alpha$.

If $N=1$, the sum in (37) collapses.

The last (37) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[A_n(f,x)-f(x)-\sum_{j_*=1}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}A_n\left((\cdot-x)^{j_*}\right)(x)\right]\to 0, \quad (38)$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$.

When $N=1$, or $f^{(j_*)}(x)=0$, $j_*=1,...,N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)}\left[A_n(f,x)-f(x)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$. Of great interest is the case of $\alpha=\frac{1}{2}$.

Proof. From Theorem 10 (24), we get by the left Caputo fractional vector Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\left(\frac{k}{n}-x\right)^{j_*}+\frac{1}{\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ, \tag{39}$$

for all $x\leq\frac{k}{n}\leq b$.

Also from Theorem 11 (25), using the right Caputo fractional vector Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\left(\frac{k}{n}-x\right)^{j_*}+\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ, \tag{40}$$

for all $a\leq\frac{k}{n}\leq x$.

We call

$$W(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k). \tag{41}$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right)\psi(nx-k)}{W(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\frac{\psi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j_*}+ \tag{42}$$

$$\frac{\psi(nx-k)}{W(x)\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ,$$

all $x\leq\frac{k}{n}\leq b$, iff $\lceil nx\rceil\leq k\leq\lfloor nb\rfloor$, and

$$\frac{f\left(\frac{k}{n}\right)\psi(nx-k)}{W(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\frac{\psi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j_*}+ \tag{43}$$

$$\frac{\psi(nx-k)}{W(x)\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ,$$

for all $a\leq\frac{k}{n}\leq x$, iff $\lceil na\rceil\leq k\leq\lfloor nx\rfloor$.

We have that $\lceil nx\rceil\leq\lfloor nx\rfloor+1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{f\left(\frac{k}{n}\right)\psi(nx-k)}{W(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{\psi(nx-k)\left(\frac{k}{n}-x\right)^{j_*}}{W(x)}+ \tag{44}$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi(nx-k)}{W(x)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right),$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor} f\left(\frac{k}{n}\right)\frac{\psi(nx-k)}{W(x)} = \sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\psi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j_*} +$$

$$\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\psi(nx-k)}{W(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right). \tag{45}$$

Adding the last two equalities (44) and (45) we obtain

$$A_n(f,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\frac{\psi(nx-k)}{W(x)} = \tag{46}$$

$$\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\psi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j_*} +$$

$$\frac{1}{\Gamma(\alpha)W(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ+\right.$$

$$\left.\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)\right)dJ\right\}.$$

So we have derived

$$\theta(x) := A_n(f,x) - f(x) - \sum_{j_*=1}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}A_n\left((\cdot-x)^{j_*}\right)(x) = \theta_n^*(x), \tag{47}$$

where

$$\theta_n^*(x) := \frac{1}{\Gamma(\alpha)W(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right.$$

$$\left.+\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right\}. \tag{48}$$

We set

$$\theta_{1n}^*(x) := \frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi(nx-k)}{W(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right), \tag{49}$$

and

$$\theta_{2n}^{*} := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \psi(nx-k)}{W(x)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right), \quad (50)$$

i.e.

$$\theta_{n}^{*}(x) = \theta_{1n}^{*}(x) + \theta_{2n}^{*}(x). \quad (51)$$

We assume $b-a > \frac{1}{n^{\beta}}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left\| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right\|_{\gamma} \leq \quad (52)$$

$$\int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left\| D_{x-}^{\alpha} f(J) \right\|_{\gamma} dJ$$

$$\leq \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n} \right)^{\alpha}}{\alpha} \leq \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \quad (53)$$

That is

$$\gamma_{1k} \leq \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \quad (54)$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left\| D_{x-}^{\alpha} f(J) \right\|_{\gamma} dJ \quad (55)$$

$$\leq \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n} \right)^{\alpha}}{\alpha} \leq \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{1}{n^{\alpha\beta}\alpha}.$$

So that, when $\left| x - \frac{k}{n} \right| \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{1k} \leq \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{a\beta}}. \quad (56)$$

Therefore

$$\left\| \theta_{1n}^{*}(x) \right\|_{\gamma} \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \psi(nx-k)}{W(x)} \gamma_{1k} \right) = \frac{1}{\Gamma(\alpha)}.$$

$$\left\{ \frac{\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}}}^{\lfloor nx\rfloor} \psi(nx-k)}{W(x)} \gamma_{1k} + \frac{\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}}}^{\lfloor nx\rfloor} \psi(nx-k)}{W(x)} \gamma_{1k} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}}}^{\lfloor nx\rfloor} \psi(nx-k)}{W(x)} \right) \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{W(x)} \left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}}}^{\lfloor nx\rfloor} \psi(nx-k) \right) \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha} \right\} \quad (57)$$

(by (7), (8))

$$\leq \frac{\left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \alpha^{*} c(h,\beta,n)(x-a)^{\alpha} \right\}.$$

Therefore we proved

$$\left\| \theta_{1n}^{*}(x) \right\|_{\gamma} \leq \frac{\left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \alpha^{*} c(h,\beta,n)(x-a)^{\alpha} \right\}. \quad (58)$$

But for large enough $n \in \mathbb{N}$ we get

$$\left\| \theta_{1n}^{*}(x) \right\|_{\gamma} \leq \frac{2 \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \quad (59)$$

Similarly, we have that

$$\gamma_{2k} := \left\| \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} D_{*x}^{\alpha} f(J) \, dJ \right\|_{\gamma} \leq$$

$$\int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \left\| D_{*x}^{\alpha} f(J) \right\|_{\gamma} dJ \leq$$

$$\left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{\left(\frac{k}{n} - x \right)^{\alpha}}{\alpha} \leq \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \quad (60)$$

That is

$$\gamma_{2k} \leq \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}, \tag{61}$$

for $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{2k} \leq \frac{\left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}. \tag{62}$$

Consequently it holds

$$\left\| \theta_{2n}^{*}(x) \right\|_{\gamma} \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} \psi(nx-k)}{W(x)} \gamma_{2k} \right) =$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k)}{W(x)} \right) \frac{\left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{W(x)} \left(\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k) \right) \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha} \right\} \leq$$

$$\frac{\left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \alpha^{*} c(h,\beta,n)(b-x)^{\alpha} \right\}. \tag{63}$$

That is

$$\left\| \theta_{2n}^{*}(x) \right\|_{\gamma} \leq \frac{\left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \alpha^{*} c(h,\beta,n)(b-x)^{\alpha} \right\}. \tag{64}$$

But for large enough $n \in \mathbb{N}$ we get

$$\left\| \theta_{2n}^{*}(x) \right\|_{\gamma} \leq \frac{2 \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{65}$$

Since $\left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}, \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \leq M$, $M > 0$, we derive

$$\left\| \theta_{n}^{*}(x) \right\|_{\gamma} \leq \left\| \theta_{1n}^{*}(x) \right\|_{\gamma} + \left\| \theta_{2n}^{*}(x) \right\|_{\gamma} \overset{\text{(by (59), (65))}}{\leq} \frac{4M}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{66}$$

That is for large enough $n \in \mathbb{N}$ we get

$$\|\theta(x)\|_{\gamma} = \|\theta_n^*(x)\|_{\gamma} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\alpha\beta}}\right), \tag{67}$$

resulting to

$$\|\theta(x)\|_{\gamma} = O\left(\frac{1}{n^{\alpha\beta}}\right), \tag{68}$$

and

$$\|\theta(x)\|_{\gamma} = o(1). \tag{69}$$

And, letting $0 < \varepsilon \leq \alpha$, we derive

$$\frac{\|\theta(x)\|_{\gamma}}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\beta\varepsilon}}\right) \to 0, \tag{70}$$

as $n \to \infty$.

I.e.

$$\|\theta(x)\|_{\gamma} = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{71}$$

proving the claim. ■

It follows a multivariate Voronovskaya type asymptotic expansion.

Theorem 22 *Let $\left(X, \|\cdot\|_{\gamma}\right)$ be a Banach space, $\overline{m} \in \mathbb{N}$ such that $\overline{m} \leq \frac{1-\beta}{\beta}$, where $0 < \beta < \frac{1}{2}$. Let $f \in C^{\overline{m}}\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$ ($\overline{m}$-times continuously Fréchet differentiable functions), $x \in \prod_{i=1}^{N}[a_i, b_i]$, and $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

$$H_n(f, x) - f(x) =$$

$$\sum_{j_*=1}^{\overline{m}-1} \frac{1}{j_*!} H_n\left(f^{(j_*)}(x)(\cdot - x)^{j_*}, x\right) + o\left(\frac{1}{n^{\beta(\overline{m}-\varepsilon)}}\right), \tag{72}$$

where $0 < \varepsilon \leq \overline{m}$.

If $\overline{m} = 1$, the sum in (72) collapses.

The last (72) implies that

$$n^{\beta(\overline{m}-\varepsilon)}\left[H_n(f, x) - f(x) - \sum_{j_*=1}^{\overline{m}-1} \frac{1}{j_*!} H_n\left(f^{(j_*)}(x)(\cdot - x)^{j_*}, x\right)\right] \to 0, \tag{73}$$

as $n \to \infty$, $0 < \varepsilon \leq \overline{m}$.

When $\overline{m} = 1$, or $f^{(j_*)}(x) = 0$, $j_* = 1, ..., \overline{m} - 1$, then we derive that

$$n^{\beta(\overline{m}-\varepsilon)}\left[H_n(f, x) - f(x)\right] \to 0, \tag{74}$$

as $n \to \infty$, $0 < \varepsilon \leq \overline{m}$.

Proof. We have that

$$f\left(\frac{k}{n}\right)-f(x)=\sum_{j_*=1}^{\overline{m}-1}\frac{f^{(j_*)}(x)}{j_*!}\left(\frac{k}{n}-x\right)^{j_*}+R_{\overline{m}}\left(\frac{k}{n},x\right), \tag{75}$$

where

$$R_{\overline{m}}\left(\frac{k}{n},x\right):=\int_0^1\frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!}f^{(\overline{m})}\left(x+u\left(\frac{k}{n}-x\right)\right)\left(\frac{k}{n}-x\right)^{\overline{m}}du, \tag{76}$$

here we set $f^{(0)}(x)\left(\frac{k}{n}-x\right)^0=f(x)$.

By (34) we get that

$$\left\|R_{\overline{m}}\left(\frac{k}{n},x\right)\right\|_{\gamma}\leq\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_{\infty}}{\overline{m}!}\left\|\frac{k}{n}-x\right\|_{\infty}^{\overline{m}} \tag{77}$$

$$\leq\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_{\infty}}{\overline{m}!}\|b-a\|_{\infty}^{\overline{m}}.$$

Call

$$V(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k). \tag{78}$$

Hence, we have

$$U_n(x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)R_{\overline{m}}\left(\frac{k}{n},x\right)}{V(x)}=$$

$$\frac{\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z(nx-k)R_{\overline{m}}\left(\frac{k}{n},x\right)}{V(x)}+ \tag{79}$$

$$\frac{\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z(nx-k)R_{\overline{m}}\left(\frac{k}{n},x\right)}{V(x)}.$$

Therefore, we obtain

$$\|U_n(x)\|_{\gamma}\leq\left(\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}\frac{Z(nx-k)}{V(x)}\right)\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_{\infty}}{\overline{m}!}\frac{1}{n^{\overline{m}\beta}}+$$

$$\left(\sum_{\substack{k=\lceil na\rceil \\ :\left\| \frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} \frac{Z(nx-k)}{V(x)} \right) \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left\| b-a \right\|_{\infty}^{\overline{m}} \tag{80}$$

(by (19), (20))

$$\leq \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left[\frac{1}{n^{\beta\overline{m}}} + \gamma(N)\, c(h,\beta,n) \left\| b-a \right\|_{\infty}^{\overline{m}} \right].$$

Consequently, we get that

$$\left\| U_n(x) \right\|_{\gamma} \leq \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left[\frac{1}{n^{\beta\overline{m}}} + \gamma(N)\, c(h,\beta,n) \left\| b-a \right\|_{\infty}^{\overline{m}} \right]. \tag{81}$$

For large enough $n \in \mathbb{N}$, we get

$$\left\| U_n(x) \right\|_{\gamma} \leq \frac{2 \left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left(\frac{1}{n^{\beta\overline{m}}} \right). \tag{82}$$

That is

$$\left\| U_n(x) \right\|_{\gamma} = O\left(\frac{1}{n^{\beta\overline{m}}} \right), \tag{83}$$

and

$$\left\| U_n(x) \right\|_{\gamma} = o(1). \tag{84}$$

And, letting $0 < \varepsilon \leq \overline{m}$, we derive

$$\frac{\left\| U_n(x) \right\|_{\gamma}}{\left(\frac{1}{n^{\beta(\overline{m}-\varepsilon)}} \right)} \leq \left(\frac{2 \left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \right) \frac{1}{n^{\beta\varepsilon}} \to 0, \tag{85}$$

as $n \to \infty$.

I.e.

$$\left\| U_n(x) \right\|_{\gamma} = o\left(\frac{1}{n^{\beta(\overline{m}-\varepsilon)}} \right). \tag{86}$$

By (75) we observe that

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{V(x)} - f(x) = \tag{87}$$

$$\sum_{j_*=1}^{\overline{m}-1} \left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left(f^{(j_*)}(x) \left(\frac{k}{n}-x\right)^{j_*} \right) Z(nx-k)}{j_*! V(x)} \right) +$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\, R_{\overline{m}}\left(\frac{k}{n}, x\right)}{V(x)}.$$

The last says

$$H_n(f,x) - f(x) - \sum_{j_*=1}^{\overline{m}-1} \frac{1}{j_*!} H_n\left(f^{(j_*)}(x)(\cdot - x)^{j_*}, x\right) = U_n(x). \tag{88}$$

The proof of the theorem now is complete. ■

References

[1] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appli. 212 (1997), 237-262.

[2] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.

[3] G.A. Anastassiou, *Univariate hyperbolic tangent neural network approximation*, Mathematics and Computer Modelling, 53(2011), 1111-1132.

[4] G.A. Anastassiou, *Multivariate hyperbolic tangent neural network approximation*, Computers and Mathematics 61(2011), 809-821.

[5] G.A. Anastassiou, *Multivariate sigmoidal neural network approximation*, Neural Networks 24(2011), 378-386.

[6] G.A. Anastassiou, *Univariate sigmoidal neural network approximation*, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.

[7] G.A. Anastassiou, *Fractional neural network approximation*, Computers and Mathematics with Applications, 64 (2012), 1655-1676.

[8] G.A. Anastassiou, *Voronovskaya type asymptotic expansions for error function based quasi-interpolation neural network operators*, Revista Colombiana De Matematicas, 49 (1), 2015, 171-192.

[9] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.

[10] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[11] G.A. Anastassiou, *Banach Space Valued Neural Network*, Springer, Heidelberg, New York, 2023.

[12] G.A. Anastassiou, *General sigmoid based Banach space valued neural network approximation*, J. Computational Analysis and Applications, 31 (4) (2023), 520-534.

[13] H. Cartan, *Differential Calculus*, Hermann, Paris, 1971.

[14] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.

[15] D. Costarelli, R. Spigler, *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks 44 (2013), 101-106.

[16] D. Costarelli, R. Spigler, *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks 48 (2013), 72-77.

[17] I.S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[18] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.

[19] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[20] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[21] L.B. Rall, *Computational Solution of Nonlinear Operator Equations*, John Wiley & Sons, New York, 1969.

[22] G.E. Shilov, *Elementary Functional Analysis*, Dover, New York, 1996.